Optimal Division of a Dollar under Ordinal Reports
(Incomplete Version, do not distribute)

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Abstract

We study the problem of dividing a dollar when agents report rankings of the contributions of other people. We find optimal rules using the maximum absolute loss from the true profile for any number of agents. If budget-balance is required, optimal rules exist only for 3 and 4 agents. Budget balance rules that are nearly optimal are provided for 5 or more agents.

Keywords: *Fair division, Judgment aggregation, Ordinal reports, Maximum loss*

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1 Introduction

A dollar was earned by a group of agents and needs to be split among them based on the work performed by each member toward generating this dollar. The problem is how to evaluate each agent's work. We study sharing rules where the full dollar is split based on the reports of the agents about the work performed by their peers. We focus on the case where agents report ordinal rankings.

This problem has multiple applications. The canonical example is the division of the profit generated by partnerships of different firms or agents. For instance, the division of the end of year bonus by a group of employees; the division of the collective profit generated by a group of partners in a law firm; or the division of an unexpected amount of money to professors in a department.

We study the problem where each agent works and observes the work of other agents. They evaluate and report the ordinal contribution of the work performed by other agents but themselves.

The available information is the agents' opinions that show their subjective evaluation of his partners. It means that each member has to report his ranking about the other colleagues. Based on the agents' report, the division rule is constructed in the form of a function such that the input is the ranking of the agents and the output is the amounts they will receive.

This paper is the first to analyze the division of the dollar under ordinal evaluations. Cardinal evaluations have been discussed in previous papers, see De Clippel et al. (2008) or Juarez (2008). Ordinal evaluations are more appealing than cardinal evaluations in many scenarios, since agents might lack perfect exactness on the work performed by their peers, but could determine who worked more than who (their ordinal ranking). For instance, in views of agents i and j, agent k might work more than agent l but their relative ranking might differ (for instance i might think k worked twice as hard than l; but j might think k worked only 50% more than l). Therefore, consensuality, a property that respects the relative rankings of the agents, is easier to be satisfied under ordinal reporting than under cardinal reporting.

The main downside of ordinal evaluations is that typically the final payment to the agents do not coincide with the real ratio of true contributions, therefore there might be a loss. For instance, assume that there are three agents whose relative work is (50%, 40%, 10%). Then, under common knowledge, ideally agent 1 would report that 2 worked more than 3; agent 2 would report that 1 worked more than 3; and agent 3 would report than agent 1 worked more than 2. However, if the only available information is the relative rankings of the agents, then when the relative work of the agents is (35%, 33%, 32%), the agents would report the same ordinal ranking, and thus obtain the same shares of the dollar.

The loss of a rule is the absolute difference between the true profile and the final payment allocated to the agents. For instance, if the true vector of contributions is (50%, 40%, 10%) and the rule allocates (r_1, r_2, r_3), then the loss at that profile is: $|.5 - r_1| + |.4 - r_2| + |.1 - r_3|$. We look at the maximal loss that the rule generates for all potential vectors of true contributions. Then we choose the rule that minimize the maximum loss.

1.1 Overview of the results

In this paper, two types of rules are studied: unanimity rule, and scoring method. The former, unanimity rule, is a rule in which if all members of the group agree on the mutual ranking then the distribution is given by a fixed sharing rule, otherwise they would get zero. The latter, a scoring method, is a rule where every agent gets a score based on his position at the different rankings. The final payoff of an agent is the ratio between his individual score and sum of the scores of all the agents.

Among the unanimity rules, for any number of agents n the maximum loss of a division rule is any value between $(n-1)/n$ and 2 (Proposition 1). Then the optimal profiles are the ones that have the maximum loss equal to $(n-1)/n$. Moreover, the necessary condition for optimal profile is derived in Propositions 2. On the other hand, for any number of the agents, there exists an optimal unanimity rule. Theorem 1 characterizes such rules.

One of the downplays of unanimity rules is that they do not satisfy budget balance, e.g. when the agents do not agree on the ranking, it will not allocate the full amount of money.

On the other hand, scoring methods satisfy budget balance. The main result of the paper, Theorem 2, shows that for 3 and 4 agents there exist unique scoring rules that are optimal (that is, achieve the bound $(n-1)/n$, where n is the number of agents). Theorem 2 also shows that no optimal scoring rule exists for 5 or more agents; however, it provide rules that achieve a loss of $n/(n+1)$, which is a very good approximation to $(n-1)/n$.

Finally, Theorem 3 shows that studying unanimity and scoring rules is by no means a restriction. In particular, it shows that if a rule always has a Nash equilibrium, then its loss will not be smaller than a unanimity rule. Moreover, if such rule is budget balance then the loss can be bounded by a scoring rule.

1.2 Literature overview

According to Knoblauch (2008), many authors survey the literature pertaining to the resolution of conflicting claims over a resource. Thomson (2003) and Brams et al. (2006) showed a literature on cake cutting and pie cutting that focuses on fair division of a divisible good, parts of which are valued differently by different parties. Holzman (2010) presented the rule of voting to find the winner to award of a prize.

De Clippel et al. (2008) raised a similar question of dividing a dollar using a rule that depends only on agents' cardinal evaluations of their associates. This paper is the first one using the concept of ranking contribution to construct a model that can divide a dollar for a team of agents by their peer review. It also mentions the potential properties of division rule such as *exactness*, *impartiality* and *consent*. The model requires at least three agents. With exactly three, there is a unique impartial and consensual division rule. Their rule is anonymous and feasible. However, that rule distributes exactly the dollar only when the three reports are consistent; otherwise it distributes strictly less. Four or more agents, de Clippel et al (2008) propose many anonymous, impartial and consensual rules that always distribute the dollar exactly.

1.3 Organization of the paper

The structure of this paper is as follows. Section 2 introduces the model. Section 3 discusses unanimity rules and gives the set of optimal unanimity rules. Section 4 studies budget balance rules and presents the main result of the paper in optimal scoring rules. Section 5 states the main theorem on non-truthful rules. Section 6 concludes. All proofs are left to the appendix.

2 The Model

Let $N = \{1, 2, ..., n\}$ be the set of agents where $n \geq 3$. For all $i \in N$, define $\mathcal{R}_i$, the set of strict rankings[1] over $N \backslash \{i\}$. Let $R_i \in \mathcal{R}_i$. Define f, the sharing rule, to be $f : \mathcal{R}_1 \times \cdots \times \mathcal{R}_n \to \Delta^{n-1} \cup \{0\}$ where Δ^{n-1} is the $(n-1)$-dimensional simplex. Let $v^* \in \Delta^{n-1}$ be a truthful ranking of contributions.

Assume that there is common knowledge about the contribution of people toward the project. Let v be a vector of true contributions. A rule will elicit the rankings of the agents over other agents but himself. Given those rankings, the rule decides on a distribution of money. We focus in two types of rules: the unanimity rules that we discuss in section 3, and the scoring methods, which is presented in section 4.

3 Unanimity Rules

Consider a vector $u \in \Delta^{n-1}$ such that $u_1 \geq u_2 \geq \cdots \geq u_n$. The *unanimity rule* ξ^u is such that if all members of N agree on the mutual ranking of N, say $i_1 \succ i_2 \succ \cdots \succ i_n$ then $\xi^u_{i_k} = u_k$ for all k. If there is no common agreement then $\xi^u_{i_k} = 0$ for all k.

For instance, let $N = \{1, 2, 3\}$ and consider the unanimity rule for the vector u such that $u_1 \geq u_2 \geq u_3 \geq 0$. For all i, let $R_i \in \mathcal{R}_i$. Consider the ranking $2R_13$, $1R_23$ and $1R_32$. In this case, agents agree in the ranking $1 \succeq 2 \succeq 3$. Therefore, agent 1 gets a payoff u_1, agent 2 gets a payoff u_2 and agent 3 gets a payoff u_3.

On the other hand, consider the ranking $2R_13$, $1R_23$ and $1R_32$. Then, the agents do not agree in the ranking, therefore all agents get zero payoff.

[1] A strict ranking over a set is a complete, transitive and antisymmetric binary relation over that set.

3.1 The total loss of a rule

Consider the unanimity rule ξ^u. The loss of the unanimity rule ξ^u at the vector of contributions $v \in \Delta^{n-1}$, denoted $L^u(v)$, where $v_{i_1} \geq v_{i_2} \geq \cdots \geq v_{i_n}$, is:

$$L^u(v) = \sum_{k=1}^{n} |u_k - v_{i_k}| \, .$$

In a ideal world, the vector of contributions and the assigned amount would be the same. The loss of a rule measures the absolute difference between these two vectors, that is it aggregates over all the agents the errors in their assignements.

In practice, the vector of contributions is unknown, therefore we evaluate the loss of a rule as the maximum loss over all potential vector of contributions. This type of worse-case measures have been used in the literature before, see for instance Moulin and Shenker[2001], Juarez[2008].

The maximum loss of a unanimity rule is

$$L(\xi^u) = \max_{v_1 \geq \ldots \geq v_n} L^u(v) = \max_{v_1 \geq \cdots \geq v_n} |u_1 - v_1| + \ldots + |u_n - v_n|$$

A unanimity rule ξ^{u^*} is *optimal* if it has the smallest maximal loss over all unanimity rules. That is, $L(\xi^{u^*}) \leq L(\xi^u)$ for any other unanimity rule ξ^u. The profile u^* is an *optimal profile* if the rule ξ^{u^*} is optimal.

3.2 Results for unanimity rules

3.2.1 Bounds on the rule

Proposition 1 *For any unanimity rule ξ^u:*

$$\frac{n-1}{n} \leq L(\xi^u) \leq 2$$

For a group of three agents, the minimum of the maximum total loss is 2/3. For a group of 4 agents, it is 3/4. As we can see next, this loss will be binding.

The goal of the paper is to find rules that achieve the smallest loss $\frac{n-1}{n}$. We say that a unanimity rule ξ^u is optimal if $L(\xi^u) = \frac{n-1}{n}$.

3.2.2 Necessary and sufficient conditions for optimality

We next state the first main theorem of the paper. It states the necessary and sufficient conditions for optimality of a unanimity mechanism based on the utility profile.

Theorem 1 *A profile $(u_1, \ldots, u_n)$ is optimal if and only if*

- $u_1 = \frac{n+1}{2n}$,
- $u_i \leq \frac{1}{n}$ *for* $i \neq 1$,

- $2(u_2 + \dots u_k) \geq \frac{1}{n} + \frac{k-2}{k}$ *for* $k = 2, 3, \dots, n-1$.

Note that at the optimal profile the agent who is highest ranked gets $\frac{n+1}{2n}$, which tends to $\frac{1}{2}$ as n increases. On the other hand, all the other agents gets no more than $\frac{1}{n}$.

There are multiple profiles that generate an optimal unanimity mechanism.

Corollary 2 *The following profiles are optimal for the number of agents n.*

i. If n is odd:

$$u_i^* = \begin{cases} \dfrac{n+1}{2n} & \textit{if} \quad i = 1 \\[2ex] \dfrac{1}{n} & \textit{if} \quad 2 \leq i \leq \dfrac{n+1}{2} \\[2ex] 0 & \quad \textit{otherwise} \end{cases}$$

ii. If n is even:

$$u_i^* = \begin{cases} \dfrac{n+1}{2n} & \textit{if} \quad i = 1 \\[2ex] \dfrac{1}{n} & \textit{if} \quad 2 \leq i \leq \dfrac{n}{2} \\[2ex] \dfrac{1}{2n} & \textit{if} \quad i = \frac{n}{2} + 1 \\[2ex] 0 & \quad \textit{otherwise} \end{cases}$$

Finally, we describe below the class of optimal profiles for 3 and 4 agents.

Corollary 3

- *For* $n = 3$, $\Phi^3 = \{(\dfrac{2}{3}, u_2, u_3) \mid u_2 + u_3 = \dfrac{1}{3}$ *and* $u_2 \geqslant u_3\}$
- *For* $n = 4$, *consider* $u^1 = \{\frac{5}{8}, \frac{1}{4}, \frac{1}{8}, 0\}$, $u^2 = \{\frac{5}{8}, \frac{25}{120}, \frac{1}{12}, \frac{1}{12}\}$, $u^3 = \{\frac{5}{8}, \frac{1}{4}, \frac{1}{16}, \frac{1}{16}\}$, $u^4 = \{\frac{5}{8}, \frac{3}{16}, \frac{3}{16}, 0\}$ *and* $u^5 = \{\frac{5}{8}, \frac{7}{48}, \frac{7}{48}, \frac{1}{12}\}$.

  Then, Φ^4 *is the convex hull of* u^1, u^2, u^3, u^4 *and* u^5. *That is,*

  $$\Phi = \{\lambda_1 u^1 + \lambda_2 u^2 + \lambda_3 u^3 + \lambda_4 u^4 + \lambda_5 u^5 | (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \in \Delta^{n-1}\}.$$

4 Budget balance rules

A huge downplay of unanimity rule is that they do not distribute the full amount of money if agents do not coincide on the ranking of their peers. In fact, when there is no agreement, unanimity rules distribute no money at all. In particular, if one agent overlooks the contribution of another agent, then everyone gets punished.

In this section we study rules that always distribute the full amount of money even when the agents do not coincide on the general ranking.

Definition 3. A rule f is budget balance if $f : R^1 \times ... \times R^N \rightarrow \Delta^{n-1}$.

4.1 The scoring rules

Let $N = \{1, ..., n\}$ be a set of agents with $n \geq 3$. Each agent reports his ranking of the other partners. We denote by c_i^k the number of agents who would put agent i as their k^{th} highest ranking. We define $a = (a_1, ..., a_k, ..., a_n)$ as the scoring vector where a_k is the score assigned to an agent who has the position k. It is further assumed that $a_1 \geq ... \geq a_n$.[2]

The total score of agent i is defined as $s_i = \sum_{k=1}^{n} \left(c_i^k a_k\right)$. The final shares of the agents are proportional to their scores, e.g. $f_i = \frac{s_i}{\sum_{i=1}^{n} s_i}$

Remark 4 *Scoring rules are budget balance. To see this,* $\sum_{i=1}^{n} f_i = \sum_{i=1}^{n} \frac{s_i}{\sum_{i=1}^{n} s_i} = \frac{\sum_{i=1}^{n} s_i}{\sum_{i=1}^{n} s_i} = 1$

Remark 5 *Scoring rules are impartial, that is, the report of agent i does not affect his payoff. To see this,* $f_i = \frac{\sum_{k=1}^{n} \left(c_i^k a_k\right)}{\sum_{i=1}^{n} \sum_{k=1}^{n} \left(c_i^k a_k\right)} = \frac{\sum_{k=1}^{n} \left(c_i^k a_k\right)}{n \sum_{k=1}^{n} a_k}$.

Remark 6 *Scoring rule are not necessarily optimal for some scoring vectors. Let a be a Borda scoring vector, e.g. $a = (n-1, n-2, ...1, 0)$. Let $v^* = (v_1, ...,$ $v_n)$, and let all the agents agree on the ranking given by v^*. Then the share of agent 1 is given by*

$$f_1 = \frac{s_1}{\sum_{i=1}^{n} s_i} = \frac{\sum_{k=1}^{n} \left(c_1^k a_k\right)}{\sum_{i=1}^{n} \sum_{k=1}^{n} \left(c_i^k a_k\right)} = \frac{(n-1)(n-1)}{n^2 \left(\frac{n-1}{2}\right)} = \frac{2(n-1)}{n^2} \neq \frac{n+1}{2n} = u_1.$$

4.2 Main Theorem: Optimal scoring rule

Now we will find scoring vectors that would give us optimal rules.

Theorem 7 *a) For $n = 3$, there exist a unique scoring rule with the optimal shares $u^* = (\frac{2}{3}, \frac{1}{3}, 0)$ that achieves the optimal loss of $\frac{2}{3}$ with the scoring profile $a_1 = 1$, $a_2 = 0$.*

b) For $n = 4$, there exist a unique scoring rule with the optimal shares $u^* = (\frac{5}{8}, \frac{1}{4}, \frac{1}{16}, \frac{1}{16})$ that achieves the optimal loss of $\frac{3}{4}$ with the scoring profile $a_1 = \frac{5}{6}$, $a_2 = \frac{1}{12}$, $a_3 = \frac{1}{12}$.

c) For $n > 4$, there is no scoring rule that achieves the optimal loss of $\frac{n-1}{n}$. However, there exist scoring rules that achieve the loss of $\frac{n}{n+1}$.

[2] We have Borda scoring method when $a = (n-1, n-2, ..., 0)$.

5 More general rules

In this section, we see that focusing on Unanimity and Scoring rule is by no means a restriction, since they will always generate a loss not larger than any other rule that always has a Nash equilibrium.

Let $v^* \in \Delta^{n-1}$ be a truthful ranking of contributions. Let $r_i(v) \in R_i$ be the ranking over $N \backslash i$ generated by v.

Definition 1 *A rule truthfully implements the ranking of contribution v^* if reporting $(r_1(v^*), ..., r_n(v^*))$ is a Nash equilibrium.*

We simply refer to these rules as truthful.

Theorem 8 *For any rule G that always has a Nash equilibrium in Pure strategies:*

a) $L(G) \geq L(Un)$ for some unanimity rule Un.

b) If G is budget-balanced, then $L(G) \geq L(Sc)$ for some scoring rule Sc.

6 Conclusion

We provide a new framework for aggregating minimal information (ordinal rather than cardinal) of the contributions of agents towards a project, and dividing the rewards in an optimal way. When we allow rules that are not budget-balanced, unanimity rules are optimal. For rule that are budget-balance, we show that scoring rules are optimal. We compute the unique optimal scoring for 3 and 4 agents. No scoring rule would be optimal for more than 4 agents.

Multiple questions remain open. For instance, are there rules that are less demanding on information than unanimity rules while still truthful? the partial answer to this question is Yes, since any rule such that any pair of agent is reported by at least two agents would satisfy the properties above. However, less demanding scenarios are easily conceivable.

Finally, it is left open the study of other setting where truthful implementation is a meaningful property.

7 Appendix

Proof of Proposition 1. a) We first prove the left hand side of inequality: $\frac{n-1}{n} \leq \max\limits_{v_1 \geq \ldots \geq v_n} L$.

Consider two truthful contribution vectors $\{1, 0, ..., 0\}$ and $\{\frac{1}{n}, ..., \frac{1}{n}\}$. We want to show that

$$\max\{|u_1 - 1| + |u_2 - 0| + ... + |u_n - 0|, |u_1 - \frac{1}{n}|, ..., |u_n - \frac{1}{n}|\} \geq \frac{n-1}{n} \quad (1)$$

Suppose (1) is not true. So we have

$$|u_1 - 1| + |u_2 - 0| + ... + |u_n - 0|, |u_1 - \frac{1}{n}| \ < \frac{n-1}{n} \quad (2)$$

and

$$|u_1 - \frac{1}{n}| + ... + |u_n - \frac{1}{n}| \ < \frac{n-1}{n} \quad (3)$$

We have $\frac{1}{n} < u_k \leq 1$ and $0 < u_{k+1} \leq \frac{1}{n}$ for some $k \in \{1, ..., n\}$.

$1 - u_1 + \sum\limits_{j=2}^{n} u_j = 2 - 2u_1 < \frac{n-1}{n} \Rightarrow u_1 > \frac{n+1}{2n} > \frac{1}{n}$ then (2) $\Leftrightarrow u_1 - \frac{1}{n} + \sum\limits_{j=2}^{n} u_j - \frac{k-1}{n} + \frac{n-k}{n} - \sum\limits_{j=k+1}^{n} u_j = \sum\limits_{i=1}^{k} u_j - \sum\limits_{j=k+1}^{n} u_j - \frac{k-1}{n} - \frac{1}{n} + \frac{n-k}{n} = \sum\limits_{i=1}^{k} u_j - (1 - \sum\limits_{i=1}^{k} u_j) + \frac{n-2k}{n} = 2\sum\limits_{i=1}^{k} u_j - \frac{2k}{n} < \frac{n-1}{n} \Rightarrow \sum\limits_{i=1}^{k} u_j < \frac{n+2k-1}{2n}$

From (1) $=> u_1 > \frac{n+1}{2n}$ From assumption $\frac{1}{n} < u_k \leq 1 \Rightarrow \sum\limits_{i=2}^{k} u_j \geq \frac{k-1}{n}$ Then $u_1 + \sum\limits_{i=2}^{k} u_j = \sum\limits_{i=2}^{k} u_j > \frac{n+2k-1}{n}$ a contradiction. At least one of components of {L} must be greater than $\frac{n-1}{n}$ or max {L} $\geq \frac{n-1}{n}$. Then, If X* is a profile that has {L}$=\frac{n-1}{n}$ then X* is an optimal profile.

2) We prove the right hand side inequality: . $L = |\ u_1{}^* - v_1| + |\ u_2{}^* - v_2|$ +...+ $|\ u_n{}^* - v_n|$ $L = \sum\limits_{i=1}^{n} |u_i - v_i| < \sum\limits_{i=1}^{n} |u_i| + \sum\limits_{i=1}^{n} |-v_i| = \sum\limits_{i=1}^{n} u_i + \sum\limits_{i=1}^{n} v_i = 2$ Then $\max\limits_{v_1 \geq v_2} .... \geq v_n\{L\} < 2$

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Proof of Proposition 2. a) We show that if X* is an optimal profile than $u_1 = \frac{n+1}{n}$.

Assume that X* is an optimal then X* has to satisfies proposition 1. It means that:

$\max_{v_1 \geq v_2} .... \geq v_n\{|u_1{}^* - v_1| + | u_2{}^* - v_2| + ... + | u_n{}^* - v_n|\} = \frac{n-1}{n}$ for every $v = \{v_1, v_2, ..., v_n)$.

In the other word:

$\{| u_1{}^* - v_1| + |u_2{}^* - v_2| + ... + |u_n{}^* - v_n|\} \leq \frac{n-1}{n}$ for every $v = \{v_1, v_2, ..., v_t n)$

Because it true for every $v = \{v_t 1, v_2, ..., v_n)$then it true for v={1, 0, ..., 0). We have:

$|u_1{}^* - 1| + |u_2{}^* - 0| + ... + |u_n{}^* - 0| \leq \frac{n-1}{n}$

$\Leftrightarrow 1 - u_1 + (u_2 + u_3 + ... + u_n \leq \frac{n-1}{n})$

$\Leftrightarrow 1 - u_1 + 1 - u_t 1 \leq \frac{n-1}{n}$ because $u_t 1 + u_2 + u_3 + ... + u_n = 1$

$\Leftrightarrow 2(1 - u_1) \leq \frac{n-1}{n} \Leftrightarrow u_1 \geq \frac{n+1}{2n}$

Suppose $u_1 > \frac{n+1}{2n}$, consider profiles $u = \{u_1 > \frac{n+1}{2n}, u_{i \neq 1} = u\}$ and $v = (\frac{1}{n}, \frac{1}{n}, ... \frac{1}{n})$. We have:

$u_1 + u_2 + u_3 + ... + u_n = 1$

$\Leftrightarrow u_2 + u_3 + ... + u_n = (n-1)\bar(u) = 1 - -u_1 < 1 - \frac{n+1}{2n} = \frac{n-1}{2n}$ because $u_1 > \frac{n+1}{2n}$

$\Leftrightarrow u < \frac{n-1}{2n(n-1)} = \frac{1}{2n} < \frac{1}{n}$

$=> L = |u_1 - \frac{1}{n} + |u - \frac{1}{n}| + ... + |u - \frac{1}{n}| = u_1 - \frac{1}{n} + (n-1)(u - \frac{1}{n})$

L $> (\frac{n+1}{2n} - \frac{1}{n}) + (n-1)(\frac{1}{n} - \frac{1}{2n}) = \frac{n-1}{2n} + \frac{n-1}{2n} = \frac{n-1}{n}$

Then X=$\{u_1, > \frac{n+1}{2n}, u_{i \neq 1} = u\}$ is not an optimal profile. At the optimal $u_1 = \frac{n+1}{2n}$

b) We show that at the optimal $u_{i \neq 1} = \frac{1}{n}$ for every i = 2...n.

Consider $u = \{u_1 = \frac{n+1}{2n}, u_2 \geq, u_3 \geq ... \geq u_k \geq \geq u_{k+1} \geq ... \geq u_n\} for\ k < n.$

$L = \{|u_1{}^* - v_1| + |u_2{}^* - v_2| + ... + |u_n{}^* - v_n|\}$

$L = \frac{n+1}{2n} - \frac{1}{n} + \sum_{i=2}^{k} v_i - \frac{k-1}{n} - \sum_{j=k+1}^{n} v_i + \frac{n-k}{n}$

$L = \frac{3n - 4k + 1}{2n} + 2 \sum_{i=2}^{k} v_i + \frac{n+1}{2n} - 1$

$$L=\frac{2n-4k+2}{2n}+\sum_{i=2}^{k}vi>\frac{2n-4k+2}{2n}+2^{*}\frac{k-1}{n}=\frac{n+1}{2n}$$

This profile does not satisfy proposition 1 then it is not an optimal one. At the optimal, $u_{i\neq 1}\leqslant\frac{1}{n}$

c) However, not all X* $=\{u_1=\frac{n+1}{2n},u_{i\neq 1}\leqslant\frac{1}{n}\}$ are optimal. For example, n = 4 consider profile $\{\frac{5}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8}\}$ that satisfies the optimal condition but is not optimal profile because the loss at v $=\{\frac{1}{3},\frac{1}{3},\frac{1}{3},0\}$

$$L=\{|\frac{5}{8}-\frac{1}{3}|+|\frac{5}{8}-\frac{1}{3}|+|\frac{1}{8}-\frac{1}{3}|+|\frac{5}{8}-0|\}=\frac{5}{6}>\frac{n-1}{n}=\frac{3}{4}$$

Then the condition for optimal profile X* $=\{u_1=\frac{n+1}{2n},u_{i\neq 1}\leqslant\frac{1}{n}\}$ is necessary but not sufficient.

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Proof of Theorem 1.

a) Case 1: $v_1>\frac{n+1}{2n}>v_2>v_3>\ ...\ >v_k>\frac{1}{n}>v_{k+1}>v_{k+2}>\ ...>v_n$

$$L_a=v_1-\frac{n+1}{2n}+\sum_{i=2}^{k}\frac{k-1}{n}+\frac{\frac{n+1}{2}-k}{n}-\sum_{i=k+1}^{\frac{n+1}{2}}v_i+\sum_{i=\frac{n+3}{2}}^{n}v_i$$

$$=-\frac{n+1}{2n}-\frac{k-1}{n}+\frac{\frac{n+1}{2}-k}{n}+v_1+\sum_{i=2}^{k}v_i-\sum_{i=k+1}^{\frac{n+1}{2}}v_i+\sum_{i=\frac{n+3}{2}}^{n}v_i$$

$$=\frac{2-4k}{2n}+2(\sum_{i=1}^{k}v_i+\sum_{i=\frac{n+3}{2}}^{n}v_i)-1=\frac{2-4k-2n}{2n}+2(\sum_{i=1}^{k}v_i+\sum_{i=\frac{n+3}{2}}^{n}v_i)$$

Suppose that $L_a>\frac{n-1}{n}\Rightarrow\frac{2-4k-2n}{2n}+2(\sum_{i=1}^{k}v_i+\sum_{i=\frac{n+3}{2}}^{n}v_i)>\frac{n-1}{n}$

$\Rightarrow 2(\sum_{i=1}^{k}v_i+\sum_{i=\frac{n+3}{2}}^{n}v_i)>\frac{n-1}{n}-\frac{2-4k-2n}{2n}=\frac{4n+4k-2}{2n}$

$\Rightarrow 2(\sum_{i=1}^{k}v_i+\sum_{i=\frac{n+3}{2}}^{n}v_i)>\frac{4n+4k-2}{4n}>1$ because $k>1$

However $\sum_{i=1}^{k}v_i+\sum_{k+1}^{\frac{n+1}{2}}v_i+\sum_{\frac{n+3}{2}}^{n}v_i=1\rightarrow\sum_{i=1}^{k}v_i+\sum_{\frac{n+3}{2}}^{n}v_i=1-\sum_{k+1}^{\frac{n+1}{2}}v_i<1$ Contradiction

Therefore $L_a=|\frac{n+1}{2n}-v_1|+|v_2-\frac{1}{n}|+...+|v_{\frac{n+1}{2}}-\frac{1}{n}|+|v_{\frac{n+3}{2}}-0|+$

$|v_n - 0| \leq \dfrac{n-1}{n}$

Case 2: $\dfrac{n+1}{2n} > v_1 > v_2 > v_3 > \ ... \ > v_k > \ > v_{k+1} > v_{k+2} > \ ... \ > v_n$

$$L_a = \frac{n+1}{2n} - v_1 + \sum_{i=2}^{k} v_i - \frac{k-1}{n} + \frac{\frac{n+1}{2} - k}{n} - \sum_{k+1}^{\frac{n+1}{2}} v_i + \sum_{\frac{n+3}{2}}^{n} v_i$$

$$= \frac{n+1}{2n} - \frac{k-1}{n} + \frac{\frac{n+1}{2} - k}{n} - v_1 + \sum_{i=2}^{k} v_i - \sum_{k+1}^{\frac{n+1}{2}} v_i + \sum_{\frac{n+3}{2}}^{n} v_i$$

$$= \frac{n+1+2-2k+n+1-2k}{2n} - v_1 + \sum_{i=2}^{k} v_i - \sum_{k+1}^{\frac{n+1}{2}} v_i + \sum_{\frac{n+3}{2}}^{n} v_i$$

$$= \frac{2n-4k+4}{2n} - 2v_1 + 2\left(\sum_{i=1}^{k} v_i + \sum_{\frac{n+3}{2}}^{n} v_i\right) - 1 = \frac{4-4k}{2n} + \left(\sum_{i=1}^{k} v_i + \sum_{\frac{n+3}{2}}^{n} v_i\right)$$

Suppose that $L_a > \dfrac{n-1}{n} \Rightarrow \dfrac{4-4k}{2n} + 2(\sum_{i=1}^{k} v_i + \sum_{\frac{n+3}{2}}^{n} v_i) > \dfrac{n-1}{n}$

$\Rightarrow 2(\sum_{i=1}^{k} v_i + \sum_{\frac{n+3}{2}}^{n} v_i) > \dfrac{n-1}{n} - \dfrac{4-4k}{2n} = \dfrac{2n+4k-6}{2n} \Rightarrow (\sum_{i=1}^{k} v_i + \sum_{\frac{n+3}{2}}^{n} v_i) > \dfrac{2n+4k-6}{2n}$

On the other hand, we have:

$v_1 + \sum_{i=2}^{k} v_i + \sum_{k+1}^{\frac{n+1}{2}} v_i + \sum_{\frac{n+3}{2}}^{n} v_i = 1 \Rightarrow \sum_{i=2}^{k} v_i + \sum_{\frac{n+3}{2}}^{n} v_i = 1 - \sum_{k+1}^{\frac{n+1}{2}} v_i - v_1$

$\Rightarrow \sum_{i=2}^{k} v_i - \sum_{\frac{n+3}{2}}^{n} v_l = 1 - \sum_{k+1}^{\frac{n+1}{2}} v_i - v_1 < 1 - v_1$ because $\sum_{k+1}^{\frac{n+1}{2}} v_i > 1$

$\Rightarrow \sum_{i=2}^{k} v_i - \sum_{\frac{n+3}{2}}^{n} v_l < 1 - \dfrac{1}{n} = \dfrac{n-1}{n}$ because $v_1 > \dfrac{1}{n}$

Therefore: $\dfrac{2n+4k-6}{2n} < \sum_{i=2}^{k} v_i + \sum_{\frac{n+3}{2}}^{n} v_l < \dfrac{n-1}{n}$.It is impossible because $\dfrac{2n+4k-6}{2n} > \dfrac{n-1}{n} \Leftrightarrow 2n+4k-6 > 2n-2 \Leftrightarrow 4k-4 > 0 \Leftrightarrow k > 1$ true for $k = 2....n$

$\Rightarrow L_a = |\dfrac{n+1}{2n} - v_1| + |v_2 - \dfrac{1}{n}| + ... + |v_{\frac{n+1}{2}} - \dfrac{1}{n}| + |v_{\frac{n+3}{2}} - 0| + |v_n - 0| \leq \dfrac{n-1}{n}$

Or $max\{L_a\} = \dfrac{n-1}{n}$

b)

Case 3: $v_1 > \frac{n+1}{2n} > v_2 > v_3 > ... > v_k > \frac{1}{n} > v_{k+1} > v_{k+2} > ... v_{\frac{n}{2}} > \frac{1}{2n} > ... > v_n$ for $k = 2, 3, ..., n$

$$L_b = v_1 - \frac{n+1}{2n} + \sum_{i=2}^{k} v_i - \frac{\frac{n-2}{2} - k}{n} - \sum_{k+1}^{\frac{n-2}{2}} v_i + v_{\frac{n}{2}} - \sum_{\frac{n+2}{2}}^{n} v_i$$

$$= -\frac{n+1}{2n} - \frac{k-1}{n} + \frac{\frac{n-2}{2} - k}{n} - \frac{1}{2n} + v_1 + \sum_{i=2}^{k} v_i - \sum_{k+1}^{\frac{n-2}{2}} v_i + \sum_{\frac{n}{2}}^{n} v_i$$

$$= \frac{-2-4k}{2n} + 2(\sum_{i=1}^{k} v_i + \sum_{\frac{n}{2}}^{n} v_i) - 1 = \frac{-2-4k-2n}{2n} + 2(\sum_{j=1}^{k} v_i + \sum_{\frac{n}{2}}^{n} v_i)$$

Suppose that $L_b > \frac{n-1}{n} \Rightarrow \frac{-2-4k-2n}{2n} + 2(\sum_{i=1}^{k} v_i + \sum_{\frac{n}{2}}^{n} v_i) > \frac{n-1}{n}$

$\Rightarrow 2(\sum_{i=1}^{k} v_i + \sum_{\frac{n}{2}}^{n} v_i) > \frac{n-1}{n} - \frac{-2-4k-2n}{2n} = \frac{4n+4k}{2n}$

$\Rightarrow (\sum_{i=1}^{k} v_i + \sum_{\frac{n}{2}}^{n} v_i) > \frac{4n+4k}{2n} > 1$ because $k > 1$

However $\sum_{i=1}^{k} v_i + \sum_{k+1}^{\frac{n-2}{2}} v_i + \sum_{\frac{n}{2}}^{n} v_i = 1 \Rightarrow \sum_{i=1}^{k} v_i + \sum_{\frac{n}{2}}^{n} v_i = 1 - \sum_{k+1}^{\frac{n-2}{2}} v_i <$ Contradiction

Therefore:

$L_b = |\frac{n+2}{2n} - v_1| + |v_2 - \frac{1}{n}| + .. + |v_{\frac{n-2}{2}} - \frac{1}{n}| + |v_{\frac{n}{2}} - \frac{1}{2n}| + |v_{\frac{n+2}{2}} - 0| + .. + |v_n - 0| \leq |\frac{n-1}{n}$

Case 4: $v_1 > \frac{n+1}{2n} > v_2 > v_3 > ... > v_k > \frac{1}{n} > v_{k+1} v v_{k+2} > ... > \frac{1}{2n} > v_{\frac{n}{2}} > ... > v_n$ for $k = 2, 3, .., n$

$$L_b = v_1 - \frac{n+1}{2n} + \sum_{i=2}^{k} v_i - \frac{k-1}{n} + \frac{\frac{n-2}{2} - k}{n} - \sum_{k+1}^{\frac{n-2}{2}} v_i - v_{\frac{n}{2}} + \frac{1}{2n} + \sum_{\frac{n+2}{2}}^{n} v_i$$

$$= -\frac{n+1}{2n} - \frac{k-1}{n} + \frac{\frac{n-2}{2} - k}{n} + \frac{1}{2n} + v_1 + \sum_{i=2}^{k} v_i - \sum_{k+1}^{\frac{n}{2}} v_i + \sum_{\frac{n+2}{2}}^{n} v_i$$

$$= \frac{-4k}{2n} + 2(\sum_{j=1}^{k} v_i + \sum_{\frac{n+2}{2}}^{n} v_i) - 1 = \frac{-4k-2n}{2n} + 2(\sum_{j=1}^{k} v_i + \sum_{\frac{n+2}{2}}^{n} v_i)$$

Suppose that $L_b > \dfrac{n-1}{n} \Rightarrow \dfrac{-4k-2n}{2n} + 2(\sum\limits_{i=1}^{k} v_i + \sum_{\frac{n+2}{2}}^{n} v_i) > \dfrac{n-1}{n}$

$\Rightarrow 2(\sum\limits_{i=1}^{k} v_i + \sum_{\frac{n+2}{2}}^{n} v_i) > \dfrac{n-1}{n} - \dfrac{-4k-2n}{2n} = \dfrac{4n+4k-1}{2n}$

$\Rightarrow (\sum\limits_{i=1}^{k} v_i + \sum_{\frac{n+2}{2}}^{n} v_i) > \dfrac{4n+4k-1}{2n}$ because $k>0$

However $\sum\limits_{i=1}^{k} v_i + \sum_{k+1}^{\frac{n}{2}} v_i + \sum_{\frac{n+2}{2}}^{n} v_i = 1 \Rightarrow \sum\limits_{i=1}^{k} v_i + \sum_{\frac{n+2}{2}}^{n} v_i = 1 - \sum_{k+1}^{\frac{n}{2}} v_i < 1$

Contradiction.

Therefore:

$L_b = |\dfrac{n+1}{2n} - v_1| + |v_2 - \dfrac{1}{n}| + .. + |v_{\frac{n-2}{2}} - \dfrac{1}{n}| + |v_{\frac{n}{2}} - \dfrac{1}{2n}| + |v_{\frac{n+2}{2}} - 0| + .. + |v_n - 0| \le \dfrac{n-1}{n}$

Case 5: $\dfrac{n+1}{2n} > v_1 > v_2 > v_3 > ... > v_k > \dfrac{1}{n} > v_{k+1} > v_{k+2} > ... v_{\frac{n}{2}} > \dfrac{1}{2n} > ... > v_n$ for $k = 2, 3, ..., n$

$$L_b = \frac{n+1}{2n} - v_1 + \sum_{i=2}^{k} v_i - \frac{k-1}{n} + \frac{\frac{n-2}{2} - k}{n} - \sum_{k+1}^{\frac{n-2}{2}} v_i + v_{\frac{n}{2}} - \frac{1}{2n} + \sum_{\frac{n+2}{2}}^{n} v_i$$

$$= \frac{n+1}{2n} - \frac{k-1}{n} + \frac{\frac{n-2}{2} - k}{n} - \frac{1}{2n} - v_1 + \sum_{i=2}^{k} v_i - \sum_{k+1}^{\frac{n-2}{2}} v_i + \sum_{\frac{n}{2}}^{n} v_i$$

$$= \frac{n+1+2-2k+n-2-2k-1}{2n} - v_1 + \sum_{i=2}^{k} v_i - \sum_{k+1}^{\frac{n-2}{2}} v_i + \sum_{\frac{n}{2}}^{n} v_i$$

$$= \frac{2n-4k}{2n} - 2v_1 + 2(\sum_{i=2}^{k} v_i + \sum_{\frac{n}{2}}^{n} v_i) - 1 = \frac{-4k}{2n} + 2(\sum_{i=2}^{k} v_i + \sum_{\frac{n}{2}}^{n} v_i)$$

Suppose that $L_b > \dfrac{n-1}{n} \Rightarrow \dfrac{-4k}{2n} + 2(\sum\limits_{i=2}^{k} v_i + \sum_{\frac{n}{2}}^{n} v_i) > \dfrac{n-1}{n}$

$\Rightarrow 2(\sum\limits_{i=2}^{k} v_i + \sum_{\frac{n}{2}}^{n} v_i) > \dfrac{n-1}{n} + \dfrac{4k}{2n} = \dfrac{2n+4k-2}{2n} \Rightarrow 2(\sum\limits_{i=2}^{k} v_i + \sum_{\frac{n}{2}}^{n} v_i) > \dfrac{2n+4k-2}{4n}$

On the other hand,we have

$v_1 + \sum\limits_{i=2}^{k} v_i + \sum_{k+1}^{\frac{n-2}{2}} v_i + \sum_{\frac{n}{2}}^{n} v_i = 1 \Rightarrow \sum\limits_{i=2}^{k} v_i + \sum_{\frac{n}{2}}^{n} v_i = 1 - \sum_{k+1}^{\frac{n-2}{2}} v_i - v_1$

$\Rightarrow \sum\limits_{i=2}^{k} v_i + \sum_{\frac{n}{2}}^{n} v_i = 1 - \sum_{k+1}^{\frac{n-2}{2}} v_i - v_1 < 1 - v_1$ because $\sum_{k+1}^{\frac{n-2}{2}} v_i \ge 0$

$\Rightarrow \sum\limits_{i=2}^{k} v_i + \sum_{\frac{n}{2}}^{n} v_i = 1 - \sum_{k+1}^{\frac{n-2}{2}} v_i - v_1 < 1 - v_1 < 1 - v_1 < 1 - \dfrac{1}{n} = \dfrac{n-1}{n}$ because

$v_1 > \frac{1}{n}$

Therefore: $\frac{2n+4k-2}{2n} < \sum_{i=2}^{k} v_i + \sum_{\frac{n}{2}}^{n} v_i < \frac{n-1}{n}$.It is impossible because $\frac{2n+4k-2}{2n} > \frac{n-1}{n} \Leftrightarrow 2n+4k-2 > 2n-2 \Leftrightarrow 4k>0 \Leftrightarrow k>0$ true for $k = 2..n$

$L_b = |\frac{n+1}{2n} - v_1| + |v_2 - \frac{1}{n}| + .. + |v_{\frac{n-2}{2}} - \frac{1}{n}| + |v_{\frac{n}{2}} - \frac{1}{2n}| + |v_{\frac{n+2}{2}} - 0| + .. + |v_n - 0| \leq \frac{n-1}{n}$

or $max\ \{L_a\} = \frac{n-1}{n}$

Case 6: $\frac{n+1}{2n} > v_1 > v_2 > v_3 > ... > v_k > \frac{1}{n} > v_{k+1} > v_{k+2} > ... > \frac{1}{2n} > v_{\frac{n}{2}} ... > v_n$ for $k = 2, 3, ..., n$

$$L_b = \frac{n+1}{2n} - v_1 + \sum_{i=2}^{k} v_i - \frac{k-1}{n} + \frac{\frac{n-2}{2} - k}{n} - \sum_{k+1}^{\frac{n-2}{2}} v_i + v_{\frac{n}{2}} + \frac{1}{2n} + \sum_{\frac{n+2}{2}}^{n} v_i$$

$$= \frac{n+1}{2n} - \frac{k-1}{n} + \frac{\frac{n-2}{2} - k}{n} + \frac{1}{2n} - v_1 + \sum_{i=2}^{k} v_i - \sum_{k+1}^{\frac{n}{2}} v_i + \sum_{\frac{n+2}{2}}^{n} v_i$$

$$= \frac{n+1+2-2k+n-2-2k+1}{2n} - v_1 + \sum_{i=2}^{k} v_i - \sum_{k+1}^{\frac{n}{2}} v_i + \sum_{\frac{n+2}{2}}^{n} v_i$$

$$= \frac{2n-4k+2}{2n} - 2v_1 + 2(\sum_{j=1}^{k} v_i + \sum_{\frac{n+2}{2}}^{n} v_i) - 1 = \frac{-4k+2}{2n} + 2(\sum_{i=2}^{k} v_i + \sum_{\frac{n+2}{2}}^{n} v_i)$$

Suppose that $L_b > \frac{n-1}{n} \Rightarrow \frac{-4k+2}{2n} + 2(\sum_{i=2}^{k} v_i + \sum_{\frac{n+2}{2}}^{n} v_i) > \frac{n-1}{n}$

$\Rightarrow 2(\sum_{i=2}^{k} v_i + \sum_{\frac{n+2}{2}}^{n} v_i) > \frac{n-1}{n} + \frac{4k-2}{2n} = \frac{2n+4k-4}{2n} \Rightarrow (\sum_{i=2}^{k} v_i + \sum_{\frac{n+2}{2}}^{n} v_i) > \frac{2n+4k-4}{2n}$

$\Rightarrow 2(\sum_{i=2}^{k} v_i + \sum_{\frac{n+2}{2}}^{n} v_i) > \frac{n-1}{n} + \frac{4k-2}{2n} = \frac{2n+4k-4}{2n} \Rightarrow (\sum_{i=2}^{k} v_i + \sum_{\frac{n+2}{2}}^{n} v_i) > \frac{2n+4k-4}{2n}$

On the other hand,we have

$v_1 + \sum_{i=2}^{k} v_i + \sum_{k+1}^{\frac{n}{2}} v_i + \sum_{\frac{n+2}{2}}^{n} v_i = 1 \Rightarrow \sum_{i=2}^{k} v_i + \sum_{\frac{n+2}{2}}^{n} v_i = 1 - \sum_{k+1}^{\frac{n}{2}} v_i - v_1$

$\Rightarrow \sum_{i=2}^{k} v_i + \sum_{\frac{n+2}{2}}^{n} v_i = 1 - \sum_{k+1}^{\frac{n}{2}} v_i - v_1 < 1 - v_1$ because $\sum_{k+1}^{\frac{n}{2}} v_i \geq 0$

$\Rightarrow \sum_{i=2}^{k} v_i + \sum_{\frac{n+2}{2}}^{n} v_i = 1 - \sum_{k+1}^{\frac{n}{2}} v_i - v_1 < 1 - v_1 < 1 - v_1 < 1 - \frac{1}{n} = \frac{n-1}{n}$ because

$v_1 > \frac{1}{n}$

Therefore: $\frac{2n+4k-2}{2n} < \sum_{i=2}^{k} v_i + \sum_{\frac{n+2}{2}}^{n} v_i < \frac{n-1}{n}$.It is impossible because

$\frac{2n+4k-2}{2n} > \frac{n-1}{n} \Leftrightarrow 2n+4k-4 > 2n-2 \Leftrightarrow 4k > 2 \Leftrightarrow k > \frac{1}{2}$ true for $k = 2..n$

$L_b = |\frac{n+1}{2n} - v_1| + |v_2 - \frac{1}{n}| + .. + |v_{\frac{n-2}{2}} - \frac{1}{n}| + |v_{\frac{n}{2}} - \frac{1}{2n}| + |v_{\frac{n+2}{2}} - 0| + .. + |v_n - 0| \leq \frac{n-1}{n}$

or $max\ \{L_a\} = \frac{n-1}{n}$

c)

Case 7: Suppose that we have

$v_i \geq u_i{}^* \forall i \in [1, k]$

$v_i \leq u_i{}^* \forall i \in [k+1, 1]$

$v_i \geq u_i{}^* \forall i \in [l+1, n]$

$$L_c = \sum_{j=1}^{k} v_i - \frac{n+1}{2n} - \frac{1}{2n} - \sum_{3}^{k}(\frac{1}{i-1} - \frac{1}{i}) + \sum_{k+1}^{l}(\frac{1}{i-1} - \frac{1}{i}) + \sum_{l+1}^{n} v_i - \sum_{l+1}^{n}(\frac{1}{i-1} - \frac{1}{i})$$

$$= \sum_{j=1}^{k} v_i - \sum_{k+1}^{l} v_i + \sum_{l+1}^{n} v_i - \frac{n+2}{2n} - (\frac{1}{2} - \frac{1}{k}) + (\frac{1}{k} - \frac{1}{l}) - (\frac{1}{l} - \frac{1}{n})$$

$$= 2(\sum_{j=1}^{k} v_i + \sum_{l+1}^{n} v_i) - 1 - \frac{1}{2} - \frac{n+2}{2n} + \frac{2}{n} - \frac{2}{l} + \frac{l}{n}$$

$$= 2(\sum_{j=1}^{k} v_i + \sum_{l+1}^{n} v_i) - 2 + \frac{2}{k} - \frac{2}{l}$$

If $L_c > \frac{n-1}{n}$ then we have $L_b = 2(\sum_{i=l}^{k} v_i + \sum_{i=l+1}^{n} v_i) - 2 + \frac{2}{k} - \frac{2}{l} \geq \frac{n-1}{n}$

$\Rightarrow 2(\sum_{i=l}^{k} v_i + \sum_{i=l+1}^{n} v_i) > 3 - \frac{2}{k} + \frac{2}{l} - \frac{1}{n}$

$\Rightarrow (\sum_{i=l}^{k} v_i + \sum_{i=l+1}^{n} v_i) > \frac{3}{2} - \frac{1}{k} + \frac{1}{l} - \frac{1}{2n}$

However we can prove that $\frac{3}{2} - \frac{1}{k} + \frac{1}{l} + \frac{1}{2n} > 1$ because

$\frac{3}{2} - \frac{1}{k} + \frac{1}{l} - \frac{1}{2n} > 1 \Leftrightarrow \frac{3}{2} - \frac{1}{k} + \frac{1}{l} - \frac{1}{2n} > 0$

$\Leftrightarrow (knl - 2nl) + (2kn - kl) > 0$

$\Leftrightarrow nl(k-2) + k(2n-l) > 0$

It is true because $n \geq 1 \geq k \geq 3$.

Then $(\sum_{i=l}^{k} v_i + \sum_{i=l+1}^{n} v_i) \geq \frac{3}{2} - \frac{1}{k} + \frac{1}{l} - \frac{1}{2n} > 1$

On the otherh and: $\sum_{i=1}^{k} v_i + \sum_{k+1}^{l} v_i + \sum_{i=l+1}^{n} v_i = 1 \Rightarrow \sum_{i=1}^{k} v_i + \sum_{i=l+1}^{n} v_i = 1 - \sum_{k+1}^{l} v_i < 1$

because $\sum_{k+1}^{l} v_i > 0$ a Contradiction

Then $L_c = |u_1{}^* - v_1| + |u_2{}^* - v_2| + |u_3{}^* - v_3| + ... + |u_n{}^* - v_n| \leq \frac{n-1}{n}$ Case 8:

Suppose that we have

$\frac{n+1}{2n} > v_1 > v_2 > \frac{1}{2n}$

$v_i \geq u_i{}^* \forall i \in [3, k]$

$v_i \leq u_i{}^* \forall i \in [k+1, l]$

$v_i \geq u_i{}^* \forall i \in [l+1, n]$

$$L_c = \frac{n+1}{2n} - v_1 + v_2 - \frac{1}{2n} + \sum_{3}^{k} v_i - \sum_{3}^{k} (\frac{1}{i-1} - \frac{1}{i}) + \sum_{k+1}^{l} (\frac{1}{i-1} - \frac{1}{i}) - \sum_{k+1}^{l} + \sum_{i=l+1}^{n} v_i - \sum_{i=l+1}^{n} (\frac{1}{i-1} - \frac{1}{i})$$

$L_c = -v_1 + v_2 + \sum_{3}^{k} v_i - \sum_{k+1}^{l} v_i + \sum_{i=l+1}^{n} v_i + \frac{n+1}{2n} - \frac{1}{2n} - (\frac{1}{2} - \frac{1}{k}) + (\frac{1}{k} - \frac{1}{l}) - (\frac{1}{l} - \frac{1}{n})$

$L_c = 2(\sum_{i=2}^{k} v_i + \sum_{i=l+1}^{n} v_i) - 1 + \frac{2}{k} - \frac{2}{l} + \frac{1}{n}$

If $L_c > \frac{n-1}{n}$ then we have $L_c = 2(\sum_{i=2}^{k} v_i + \sum_{i=l+1}^{n} v_i) - 1 + \frac{2}{k} - \frac{2}{l} + \frac{1}{n} \geq \frac{n-1}{n}$

$\Rightarrow 2(\sum_{i=2}^{k} v_i + \sum_{i=l+1}^{n} v_i) > 1 - \frac{2}{k} + \frac{2}{l} + \frac{1}{n} + \frac{n-1}{n}$

$\Rightarrow 2(\sum_{i=2}^{k} v_i + \sum_{i=l+1}^{n} v_i) > 2 - \frac{2}{k} + \frac{2}{l} - \frac{2}{n}$

$(\sum_{i=2}^{k} v_i + \sum_{i=l+1}^{n} v_i) > 1 - \frac{1}{k} + \frac{1}{l} - \frac{1}{n}$

In addition $v_1 + \sum_{i=2}^{k} v_i + \sum_{k+1}^{l} v_i + \sum_{i=l+1}^{n} v_i = 1$

$\Rightarrow \sum_{i=2}^{k} v_i + \sum_{i=l+1}^{n} v_i = 1 - v_1 - \sum_{k+1}^{l} v_i \Rightarrow \sum_{i=2}^{k} v_i + \sum_{i=l+1}^{n} v_i \leq 1 - v_1$ because $\sum_{k+1}^{l} v_i > 0$

However we have :$v_1 > v_2 > v_3 = \frac{1}{6} \Rightarrow \sum_{i=2}^{k} v_i + \sum_{i=l+1}^{n} v_i \leq 1 - v_1 < 1 - \frac{1}{6} = \frac{5}{6}$

Then $$1 - \frac{1}{n} - \frac{1}{k} + \frac{1}{l} < (\sum_{i=2}^{k} v_i + \sum_{i=l+1}^{n} v_i) < \frac{5}{6} (^{**})$$

For $k \geq 6 (^{**})$ can not happen because $1 - \frac{1}{n} - \frac{1}{k} + \frac{1}{l} \geq \frac{5}{6}$

We also prove that

$L_b = |u_1{}^* - v_1| + |u_2{}^* - v_2| + |u_3{}^* - v_3| + ... + |u_n{}^* - v_n| \leq \frac{n-1}{n}$ for $k < 6$

■

Proposition 4

Proof. Assume that X* and Y* are two profile that satisfy the following condition with all true profile $\{v_1, v_2, ..., v_n\}$

$L_u = |u_1{}^* - v_1| + |\ u_2{}^* - v_2| + ... + |\ u_n{}^* - v_n| \leq \frac{n-1}{n}$

$L_Y = |Y_1{}^* - v_1| + |Y_2{}^* - v_2| + ... + |Y_n{}^* - v_n| \leq \frac{n-1}{n}$

Consider a combination profile $\alpha u^* + (1-\alpha)Y^*$, for any $0 < \alpha < 1$, then we have

$L_c = |\alpha u_1{}^* + (1-\alpha)Y^* - v_1| + |\alpha u_2{}^* + (1-\alpha)Y^* - v_2| + ... + |\alpha u_n{}^* + (1-\alpha)Y^* - v_n|$

$\Rightarrow L_c = |\alpha u_1{}^* + (1-\alpha)Y^* - \alpha v_1 - (1-\alpha)v_1| + |\alpha u_2{}^* + (1-\alpha)Y^* - \alpha v_2 - (1-\alpha)v_2| + ... + |\alpha u_n{}^* + (1-\alpha)Y^* - \alpha v_n - (1-\alpha)v_n|$

$\Rightarrow L_c \leq [|\alpha(u_1{}^* - v_1)| + |\alpha(u_2{}^* - v_2)| + ... + |\alpha(u_n{}^* - v_n)|] + |(1-\alpha)(Y_1{}^* - v_1)| + |(1-\alpha)(Y_2{}^* - v_2)| + ... + |(1-\alpha)(Y_n{}^* - v_n)|$

$\Rightarrow L_c \leq [|\alpha(u_1 * - v_1)| + |\alpha(u_2{}^* - v_2)| + ... + |\alpha(u_n{}^* - v_n)|] + [|(1-\alpha)(Y_1{}^* - v_1)| + |(1-\alpha)(Y_2{}^* - v_2)| + ... + |(1-\alpha)(Y_n{}^* - v_n)|]$

$\Rightarrow L_c \leq \alpha[u_1{}^* - v_1| + u_t 2^* - v_2| + ... + |u_n{}^* - v_n|] + (1-\alpha)[Y_1 * - v_1| + Y_2{}^* - v_2| + ... + |Y_n{}^* - v_n|]$

$\Rightarrow L_c \leq \alpha L_u + (1-\alpha)L_Y \leq \alpha \frac{n-1}{n} + (1-\alpha)\frac{n-1}{n}) = \frac{n-1}{n}$

■

Proposition 5

Proof. In order to show that, assume there exist an optimal profile Z* = $\{Z_1, Z_2, Z_3\}$. It means that for every$\{v_1, v_2, v_3\}$,the loss must be satisfied:

$L = |Z_1 - v_1| + |Z_2 - v_2| + |Z_3 - v_3|\ (^{***})$

Then $(^{***})$must be true with $v = \{1, 0, 0\}$, we have:

$L = 1 - Z_1 + Z_2 + Z_3 \leq \frac{2}{3}$

$\Leftrightarrow 2(1 - Z_1) \leq \frac{2}{3} \Leftrightarrow Z_1 \geq \frac{2}{3}$

Suppose that $Z_1 > \frac{2}{3} \Rightarrow Z_2 + Z_3$ therefore both Z_2 *and* $Z_3 < \frac{1}{3}$. Because $(^{***})$ also true with $v = \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$

$L = Z_1 - \frac{1}{3} + \frac{1}{3} - Z_2 + \frac{1}{3} - Z_3 = \frac{1}{3} + 1 - 2(Z_2 + Z_3) = \frac{4}{3} - 2(Z_2 + Z_3) \leq \frac{2}{3}$

$\Rightarrow (Z_2 + Z_3) \geq \dfrac{1}{3}$ a Contradiction

Therefore $Z_1 = \dfrac{2}{3}$ and $Z_2 + Z_3 = \dfrac{1}{3}$

We also have two original profiles as: $u^* = \{\dfrac{2}{3}, \dfrac{1}{3}, 0\}$ and $Y^* = \{\dfrac{2}{3}, \dfrac{1}{6}, \dfrac{1}{6}\}$then

$\Phi = \{\alpha u^* + (1-\alpha)Y^*, 0<\alpha<1\}$.

$\Phi = \{\dfrac{2}{3}, (\alpha\dfrac{1}{3} + (1-\alpha)\dfrac{1}{6}), (\alpha_0 + (1-\alpha)\dfrac{1}{6}))$

■

Proposition 6

Proof. *Step 1:* Prove that $U^* = \{\dfrac{5}{8}, \dfrac{25}{120}, \dfrac{1}{12}, \dfrac{1}{12}\}$, $u^* = \{\dfrac{5}{8}, \dfrac{1}{4}, \dfrac{1}{16}, \dfrac{1}{16}\}$, $Y^* = \{\dfrac{5}{8}, \dfrac{3}{16}, \dfrac{3}{16}, 0\}$and

$Z^* = \{\dfrac{5}{8}, \dfrac{7}{48}, \dfrac{7}{48}, \dfrac{1}{12}\}$ are optimal

a) We prove that $U^* = \{\dfrac{5}{8}, \dfrac{25}{120}, \dfrac{1}{12}, \dfrac{1}{12}\}$ is an optimal profile Case 1: $v_1 > \dfrac{5}{8} > v_2 > \dfrac{25}{120} > v_3 > \dfrac{1}{12} > v_4$

$L = v_1 - \dfrac{5}{8} + v_2 - \dfrac{25}{120} + v_3 - \dfrac{1}{12} + \dfrac{1}{12} - v_4$

$L = 1 - 2v_4 - \dfrac{5}{8} - \dfrac{25}{120} = \dfrac{20}{120} - 2v_4 < \dfrac{20}{120} < \dfrac{3}{4}$

because $v_4 \geq 0$ Case 2: $v_1 > \dfrac{5}{8} > v_2 > \dfrac{25}{120} > v_3 > \dfrac{1}{12} > v_4$

$L = v_1 - \dfrac{5}{8} - v_2 + \dfrac{25}{120} + v_3 - \dfrac{1}{12} + \dfrac{1}{12} - v_4$

$L = 1 - 2(v_2 + v_4) - \dfrac{5}{8} + \dfrac{25}{120} = \dfrac{50}{120} - 2(v_2 + v_4) < \dfrac{50}{120} < \dfrac{3}{4}$

because $v_2 \geq v_4 \geq 0$ Case 3: $v_1 > \dfrac{5}{8} > \dfrac{25}{120} > v_2 > \dfrac{1}{12} v_3 > v_4$

$L = v_1 - \dfrac{5}{8} - v_2 + \dfrac{25}{120} - v_3 + \dfrac{1}{12} + \dfrac{1}{12} - v_4$

$L = 1 - 2(v_2 + v_3 + v_4) - \dfrac{5}{8} + \dfrac{25}{120} + \dfrac{1}{6} = \dfrac{90}{120} - 2(v_2 + v_3 + v_4) \leq \dfrac{90}{120} < \dfrac{3}{4}$

because $v_2 + v_3 + v_4 \geq 0$ Case 4: $\dfrac{5}{8} > v_1 > v_2 > \dfrac{25}{120} > v_3 > \dfrac{1}{12} > v_4$

$L = -v_1 + \dfrac{5}{8} + v_2 - \dfrac{25}{120} + v_3 - \dfrac{1}{12} + \dfrac{1}{12} - v_4$

$L = -1 + 2(v_2 + v_3) + \dfrac{5}{8} - \dfrac{25}{120} = 2(v_2 + v_3) - \dfrac{70}{120} \leq 2(\dfrac{1}{4} + \dfrac{25}{120}) - \dfrac{70}{120} = \dfrac{40}{120} < \dfrac{3}{4}$

because $v_2 \leq \dfrac{1}{4}$ (proposition 2) and $v_3 < \dfrac{25}{120}$(assumption). Case 5: $\dfrac{5}{8} > v_1 > \dfrac{25}{120} > v_2 > v_3 > \dfrac{1}{12} > v_4$

$L = -v_1 + \dfrac{5}{8} - v_2 + \dfrac{25}{120} + v_3 - \dfrac{1}{12} + \dfrac{1}{12} - v_4$

$L = -1 + 2v_3 + \dfrac{5}{8} + \dfrac{25}{120} = 2v_3 - \dfrac{20}{120} \leq \dfrac{50}{120} - \dfrac{20}{120} = \dfrac{30}{120} < \dfrac{3}{4}$

because $v_3 < \dfrac{25}{120}$(assumption). Case 6: $\dfrac{5}{8} > v_1 > v_2 > \dfrac{25}{120} > \dfrac{1}{12} > v_3 > v_4$

$L = -v_1 + \dfrac{5}{8} + v_2 - \dfrac{25}{120} - v_3 + \dfrac{1}{12} + \dfrac{1}{12} - v_4$

$L = -1 + 2v_2 + \frac{5}{8} - \frac{25}{120} + \frac{1}{6} = 2v_2 - \frac{50}{120} \leq \frac{2}{4} - \frac{50}{120} = \frac{10}{120} < \frac{3}{4}$
because $v_2 \leq \frac{1}{4}$ (proposition 2)

We can see that in the proof above, maximum loss occurs at case 3 and case 4. Therefore, for remaining profiles we consider case 3 and case 4 only.
b) $u^* = \{\frac{5}{8}, \frac{1}{4}, \frac{1}{16}, \frac{1}{16}\}$ Case 3b: $v_1 > \frac{5}{8} > \frac{1}{4} > v_2 > \frac{1}{16} > v_3 > v_4$

$L = v_1 - \frac{5}{8} - v_2 + \frac{1}{4} - v_3 + \frac{1}{16} + \frac{1}{16} - v_4$

$L = 1 - 2(v_2 + v_3 + v_4) - \frac{5}{8} + \frac{1}{4} + \frac{1}{8} = \frac{6}{8} - 2(v_2 + v_3 + v_4) \leq \frac{6}{8} = \frac{3}{4}$
because $v_2 + v_3 + v_4 \geq 0$ Case 4b: $\frac{5}{8} > v_1 > v_2 > \frac{1}{4} > v_3 > \frac{1}{16} > v_4$

$L = -v_1 + \frac{5}{8} + v_2 - \frac{1}{4} + v_3 - \frac{1}{16} + \frac{1}{16} - v_4$

$L = -1 + 2(v_2 + v_3) + \frac{5}{8} - \frac{1}{4} = 2(v_2 + v_3) - \frac{5}{8} < 2(\frac{1}{4} + \frac{1}{4}) - \frac{5}{8} = \frac{3}{8} < \frac{3}{4}$
because $v_2 \leq \frac{1}{4}$ (proposition 2) and $v_3 < \frac{1}{4}$ (assumption). $Y^* = \{\frac{5}{8}, \frac{3}{16}, \frac{3}{16}, 0\}$
Case 3c: $v_1 > \frac{5}{8} > \frac{3}{16} > v_2 > v_3 >$

$L = v_1 - \frac{5}{8} - v_2 + \frac{3}{16} - v_3 + \frac{3}{16} + v_4$

$L = 1 - 2(v_2 + v_3) - \frac{5}{8} + \frac{3}{8} = \frac{6}{8} - 2(v_2 + v_3) \leq \frac{6}{8} = \frac{3}{4}$
because $v_2 + v_3 \geq 0$ Case 4c: $\frac{5}{8} > v_1 > v_2 > \frac{3}{16} > v_3 > v_4$

$L = -v_1 + \frac{5}{8} + v_2 - \frac{3}{16} - v_3 + \frac{3}{16} + v_4$

$L = -1 + 2(v_2 + v_3) + \frac{5}{8} = 2(v_2 + v_3) - \frac{3}{8} < 2(\frac{1}{4} + \frac{3}{16}) - \frac{3}{8} = \frac{11}{16} < \frac{3}{4}$
because $v_2 \leq \frac{1}{4}$ (proposition 2) and $v_3 < \frac{3}{16}$(assumption).
d)$Z^* = \{\frac{5}{8}, \frac{7}{48}, \frac{7}{48}, \frac{1}{12}\}$

Case 3d: $v_1 > \frac{5}{8} > \frac{7}{48} > v_2 > v_3 > \frac{1}{12} > v_4$

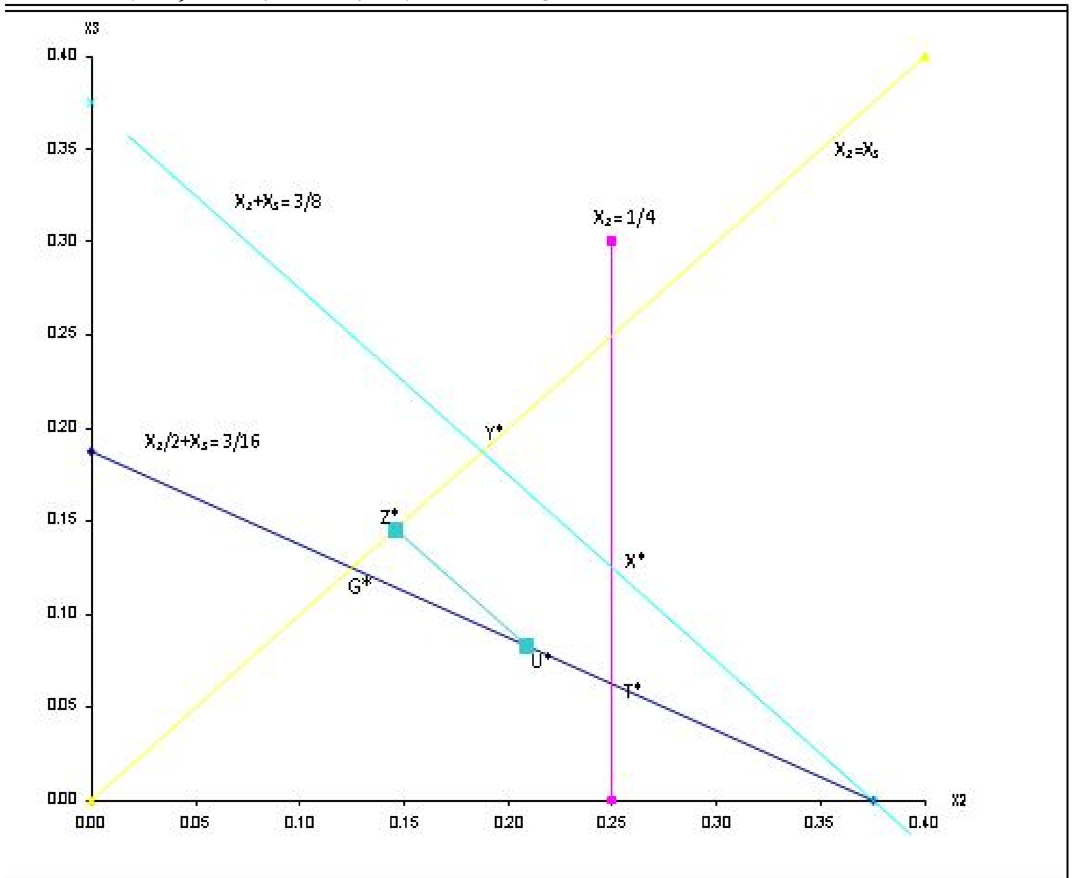
$L = v_1 - \frac{5}{8} - v_2 + \frac{7}{48} - v_3 + \frac{7}{48} + \frac{1}{12} - v_4$

$L = 1 - 2(v_2 + v_3 + v_4) - \frac{5}{8} + \frac{14}{48} + \frac{1}{12} = \frac{36}{48} - 2(v_2 + v_3 + v_4) \leq \frac{36}{48} = \frac{3}{4}$
because $v_2 + v_3 + v_4 \geq 0$ Case 4d: $\frac{5}{8} > v_1 > v_2 > \frac{7}{48} > v_3 > \frac{1}{12} > v_4$

$L = -v_1 + \frac{5}{8} + v_2 - \frac{7}{48} - v_3 + \frac{7}{48} + \frac{1}{12} - v_4$

$L = -1 + 2(v_2 + v_4) + \frac{5}{8} + \frac{1}{12} = 2(v_2 + v_4) - \frac{14}{48} \leq 2(\frac{1}{4} + \frac{7}{48}) - \frac{14}{48} = \frac{24}{48} < \frac{3}{4}$
because $v_2 \leq \frac{1}{4}$(proposition 2) and $v_4 < \frac{7}{48}$(assumption). *Step 2:* Prove that Φ

$= \{\alpha\, T^{*}+\beta\, U^{*}+\chi X^{*}+\delta\, Y^{*}+\gamma\ Z^{*}\}$ for any α, β, χ, δ,γ are non negative and $\alpha+\beta$



$+\chi +\delta +\gamma\ =1.$

a)According to proposition 4, we have $\Phi \supset \{\alpha\, T^{*}+\beta\, U^{*}+\chi X^{*}+\delta\, Y^{*}+\gamma\ Z^{*}\}$ then on the flat limited by u_2 and $u_3$all points belong to the area (X*Y*Z*U*T*) are optimal. b)We show that all point out side that area are not optimal.

Because

$u_1\ = 5/8, u_2 + u_3 + u_4\ = 3/8 \Rightarrow u_4\ = 3/8\ \ - (u_2 + u_3){\geq}0 \Rightarrow 3/8{\geq}\ (u_2 + u_3)$ (1)

According to proposition 2, $\Rightarrow \dfrac{1}{4}{\geq}u_2$ (2)

Based on our assumption then $u_2{\geq}u_3$ (3)

Based on our assumption then

$u_3{\geq}u_4 \Rightarrow u_3{\geq}3/8 - (u_2 + u_3){\geq}0 \Rightarrow u_2/2 + u_3{\geq}3/16$ (4)

Combine (1), (2), (3), and (4) we have $\Phi \supset$ *(X*Y*G*T*)*

However every point belongs to (G*U*Z*) is not optimal. Because, we can take any point that has $u_2 = u_{2(Z*)}$ and$u_3 = u_{3(Z*)} + \varepsilon$.

Then we have a new profile (5/8, 7/48,7/48+ε, 1/12 -ε), the total loss of this profile

$L = |5/8 - v_1| + |7/48 - v_2| + |7/48 + \varepsilon - v_3| + |1/12 - \varepsilon - v_4|$
$L \geq |5/8 - v_1| + |7/48 - v_2| + |7/48 - v_3| + |1/12 - v_4| + 2|\varepsilon| > L_{(Z*)}$
Because Z* is an optimal file, $\Rightarrow$ G is not optimal profile.
$\{\alpha T^* + \beta U^* + \chi X^* + \delta Y^* + \gamma\ Z^*\} \supset \Phi$
Combine (a) and (b) we have $\Phi = \{\alpha T^* + \beta U^* + \chi X^* + \delta Y^* + \gamma\ Z^*\}$ ■

Proof. Theorem 2 a) For n=3, the scoring profile is (a_1, a_2, a_3)such that $a_1 \geq a_2 \geq a_3 \geq 0$. Assume that every member knows the order $1 \succ 2 \succ 3$. Using the same rule constructed above, we have the scoring matrix as follow:

	1	2	3
1		$a1$	$a2$
2	$a1$		$a2$
3	$a1$	$a2$	

Then, the vector of money distribution is $(\frac{2a_1}{3(a_1+a_2)}, \frac{a_1+a_2}{3(a_1+a_2)}, \frac{2a_2}{3(a_1+a_2)})$or $(\frac{2a_1}{3(a_1+a_2)}, \frac{1}{3)}, \frac{2a_2}{3(a_1+a_2)})$. Because $u_2 = \frac{1}{3}$ then there is only the optimal X*=$(\frac{1}{3}, \frac{1}{3}, 0)^3$ feasible with scoring method. Then we have:

$$\begin{cases} \dfrac{2a2}{3(a1+a2)} = 0 \\ \\ \dfrac{2a2}{3(a1+a2)} = \dfrac{2}{3} \end{cases} \Rightarrow a1 = 1, a2 = 0 = a3$$

Therefore, for group of 3 agents we should use scoring profile (1, 0, 0) for 1^{st}, 2^{nd}, 3^{rd} position respectively.

b) For n=4, the scoring profile is (a_1, a_2, a_3, a_4)such that $a_1 \geq a_2 \geq a_3 \geq a_4 0$ Assume that every member knows the order $1 \succ 2 \succ 3 \succ 4$.

Using the same rule constructed above, we have the scoring matrix as follow

	1	2	3	4
1		$a1$	$a2$	$a3$
2	$a1$		$a2$	$a3$
3	$a1$	$a2$		$a3$
4	$a1$	$a2$	$a3$	

Then, the vector of money distribution is
$(\frac{3a_1}{4(a_1+a_2+a_3)}, \frac{a_1+2a_2}{4(a_1+a_2+a_3)}, \frac{a_2+2a_3}{4(a_1+a_2+a_3)}, \frac{3a_3}{4(a_1+a_2+a_3)})$.

[3] According to profile X*=$(\frac{1}{3}, \frac{1}{3}, 0)$ is optimal

We should write $u_i = f(a_i)$to find which optimal profile is feasible with scoring method. Without loss of general, we can assume that $a_1+a_2+a3+a_4 = 1$

$$\begin{cases} u_1 = \dfrac{3a_1}{4} \\ u_2 = \dfrac{a_1 + 2a2}{4} \\ u_3 = \dfrac{a_2 + 2a3}{4} \\ u_4 = \dfrac{3a_3}{4} \end{cases}$$

According to proposition 2,

$u_1 = \frac{5}{8} \Rightarrow a_1 = \frac{5}{6}$and

$u_2 \leq \frac{1}{4} \Rightarrow a_2 \leq \frac{1}{12}$.

Suppose that $u_2 < \frac{1}{4}$ then $a_2 < \frac{1}{12}$. We also have $a_2 \geq a_3$ then $a_3 < \frac{1}{12}$, that leads to

$u_4 \leq u_3 <$

and

1 a contradiction. Then $u_2 = \frac{1}{4} \Rightarrow a_2 = .\frac{1}{12}$

Since $u_4 + u_3 = \frac{1}{8}$ and $u_4 \leq u_3$ then $u_4 \leq \frac{1}{16}$.

If $u_4 \leq \frac{1}{16}$ then $a_3 < \frac{1}{12}$ that leads to$u_3 < \frac{1}{16}$ and $u_4 + u_3 < \frac{1}{8}$ a contradiction.

Then $u_3 = u_4 = \frac{1}{16}$, $a_3 = \frac{1}{12}$ and $a_4 = 0$

Therefore, only the optimal profile X*=$(\frac{5}{8}, \frac{1}{4}, \frac{1}{16}, \frac{1}{16})$[4] is feasible with scoring method at scoring profile: $(\frac{5}{6}, \frac{1}{12}, \frac{1}{12}, 0)$ ■

[4]Step 1 of the of proposition 6 shows that X* is optimal.

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