# Convex Rationing Solutions
(Incomplete Version, Do not distribute)

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**Abstract**

This paper introduces a notion of convexity on the rationing problem and characterizes the solutions that are convex. We study the composition of convexity with other traditional axioms explored in the literature such as upper and lower composition.

We characterize the class of rules satisfying upper composition and convexity for any number of agents. The dual rules, satisfying lower compositon and convexity are also provided.

A class of solutions meeting upper composition, convexity and consistency is provided. The combination of convexity, scale invariance, consistency and *any one* of the compositions axioms leads to a re-characterization of the priority solutions introduced by Moulin[2000].

## 1 Notation and preliminaries

Let $N$ a finite subset of $\mathbb{N}$, and $\mathbb{R}^N_+$ the positive Euclidean space of the dimension of $N$.

Let $\mathscr{S}^2$ the simplex in $\mathbb{R}^2_+$, that is $\{p \in \mathbb{R}^2_+ \mid p_1 + p_2 = 1\}$. We will denote as $\succ$ the order of the simplex such that for each $x, y \in \mathscr{S}^2$, $x \succ y$ if and only if $x_1 > y_1$.

For $X, Y \subset \mathscr{S}^2$, we say $X \succ Y$ if $x \succ y$ for all $x \in X, y \in Y$. Also, we denote by $(Y, X) \equiv (X, Y) \equiv \{z \mid x \succ z \succ y \text{ for all } x \in X, y \in Y\}$. The definition of the closed and semiopened intervals for subset of $\mathscr{S}^2$ is similarly given.

For $x \in \mathbb{R}^N_+$, $x_N = \sum\limits_{i \in N} x_i$, $\tilde{x} = \frac{x}{x_N}$, and $\parallel x \parallel$ is the euclidean norm of $x$. Notice that $\tilde{x}$ is the multiple of $x$ that belongs to the simplex $\mathscr{S}^2$.

For $x \in \mathbb{R}^N_+$, $A \subset \mathbb{R}^N_+$, the ray with apex $x$ and slopes in $A$ is given by $ray_x(A) = \{x + t \cdot a \mid a \in A, t \geq 0\}$.

Given a function $f : \mathbb{R} \to \mathbb{R}$, the left and right limits at $x$ are denoted by $f_-(x)$ and $f_+(x)$ respectively.

**Definition 1** *(i) A problem is a pair* $[c; b] \in \mathbb{R}_+ \times \mathbb{R}^N_+$ *such that* $\sum\limits_{i \in N} b_i \geq c$.

*We will denote by* $B^N$ *the set of problems thus defined. If* $N = \{1, 2\}$ *we will denote* $B^N$ *only by* $B^2$.

*(ii) A solution is a function* $\varphi : B^N \to \mathbb{R}^N_+$ *such that for* $p_i = \varphi_i[c; b]$,

$$\sum_{i \in N} p_i = c.$$

**Definition 2** *Given a solution $\varphi$ and a solution vector $p$, the level curve of $\varphi$ at $p$ is $C_p^\varphi = \{b \in \mathbb{R}^N \mid \varphi[p_N; b] = p\}$. When there is not danger of confusion, we will denote $C_p^\varphi$ only by $C_p$.*

Our first lemma gives us the conditions that the level curves of a solution must satisfy and conversely, it gives us the conditions that a collection of sets must satisfy to be a solution. In many parts of this paper, I will give a solution by defining its level curves only.

**Lemma 1** *Given a solution $\varphi$, suppose $\{C_p \mid p \in \mathbb{R}_+^N\}$ is the set of level curves of $\varphi$. Then*

*(a) $C_p \neq \emptyset$ for each solution vector $p$.*

*(b) $C_p \cap C_q = \emptyset$ if $p_N = q_N$ and $p \neq q$.*

*(c) For each $c \geq 0$ : $\bigcup_{\{p \mid p_N = c\}} C_p = \{x \in \mathbb{R}_+^N \mid x_N \geq c\}$.*

*(d) For each solution vector $p$ and every $b \in C_p$, $b \geq p$ holds.*

*Moreover, if we have a collection of sets $\{C_p \subset \mathbb{R}_+^N \mid p \in \mathbb{R}_+^N\}$, indexed by each solution vector, that satisfies the four conditions above, then there is a unique solution that has these sets as level curves.*

We will work with solutions that satisfy continuity, individual rationality and the weak-ranking property. The propositions will be given without any explicit mention of these properties, however, we may use some of them in the proofs.

**Axiom 1 (Individual rationality)** *The solution $\varphi$ is individually rational if no one will receive more of what is asking, that is, for any problem $[c; b]$ and $p_i = \varphi_i[c; b]$, $p_i \leq b_i$ for all $i \in N$.*

**Axiom 2 (Weak-ranking)** *For any two vector of claims $b$ and $b'$ such that $b_N = b'_N > c$, if $b'_i > b'_i$ then $\varphi_i[c; b] > \varphi_i[c; b']$.*

Notice that weak-ranking along with continuity imply the level curves are path-connected.

**Axiom 3 (Scale invariance)** *The solution $\varphi$ is scale invariant if for any $[c; b] \in B^N$ and $\lambda > 0$, $\varphi[\lambda c; \lambda b] = \lambda \varphi[c; b]$ holds.*

Notice that if a solution satisfies scale invariance, then given $c$ and $\tilde{c}$ amounts that must be collected, the level curves of a solution on $c$ and $\tilde{c}$ are homothetic. Thus, the level curves of a scale invariant solution are determined only by those in the simplex $\mathscr{S}^2$.

**Axiom 4 (Upper composition)** *The solution $\varphi$ satisfies upper composition (UC) if for any $b \in \mathbb{R}_+^N$ and $c < c' \leq b_N$ :*

$$\varphi[c; b] = \varphi[c; \varphi[c'; b]].$$

The solutions we are going to discuss, are widely related to the classic solutions discussed in the literature: proportional, uniform gains, uniform losses, and priority solutions.

**Definition 3** *$pr$ is the proportional solution if for any problem $[c; b] \in B^N$ :*

$$pr[c; b] = \frac{c}{b_N} b$$

Notice the level curve of $pr$ at $p$ is the ray with slope $p$ that passes trough $p$, that is $ray_p(p)$. Its $t-$distribution is simply the uniform distribution on every $t$.

**Definition 4** *$ul$ is the uniform losses solution if and only if for any $[c;b] \in B^N$:*

$$ul_i[c;b] = (b_i - \mu)_+$$

*where $\mu$ is the solution to $\sum_N (b_i - \mu)_+ = c$ and $(z)_+ = \max(0, z)$.*

In the two agent case, the level curve of $p$ in $ul$ is the ray with slope $(0,1)$ if $p \prec (\frac{1}{2}, \frac{1}{2})$, the ray with slope $(1,0)$ if $p \succ (\frac{1}{2}, \frac{1}{2})$, or the cone with every slope in the simplex if $p = (\frac{1}{2}, \frac{1}{2})$.

The dual solution of the uniform losses solution is the uniform gains solution.

**Definition 5** *$ug$ is the uniform gains solution if and only if for all $[c;b] \in B^N$:*

$$ug_i[c;b] = min(\lambda, b_i)$$

*where $\lambda$ is the solution to $\sum_N \min(\lambda, b_i) = c$.*

In the two agent case, the level curve in $ug$ of every point different than $(0,1)$ or $(1,0)$ are the ray with slope $(\frac{1}{2}, \frac{1}{2})$, furthermore, each border point contains the cone with slopes between this point and $(\frac{1}{2}, \frac{1}{2})$.

**Definition 6** *Let $\sigma$ an order of $N$, then $p = prio^{\sigma}[c;b]$ is the unique solution vector such that:*

$$\text{for all } i,j \in N, \sigma^{-1}(i) < \sigma^{-1}(j) \text{ and } p_j > 0 \text{ imply } p_i = b_i.$$

Finally, a well known concept in the literature is the paths of a solution. For a solution $\varphi$ and a vector of claims $b$, the path to $b$ is the solution of the problem with vector of claims $b$ in every amount to divide $c$ less than $b_N$, that is $path(b) = \{\varphi[c;b] \mid 0 < c \leq b_N\}$. Any solution is uniquely determined by its paths, so this is an alternative way to represent our solutions. These paths have played a key role in the literature, see for example every result in Chambers[2002], Moulin[2000] or Naumova[2002], or many of the constructions given by Thomson[2003]. Contrary to these papers, this concept will not be crucial here. However, in order to compare our new solutions with old solutions, we will alternatively give a representation of the solution by it paths.

# 2 Conic Solutions

This section introduces a new class of solutions not explicitly discussed before, but containing a large class of solutions in the literature. In particular, it contains all solutions discussed by Moulin[2000] and several of the solutions discussed by Thomson[2003]. These solutions are the most pure and intuitive solutions after the definition of level curves, that is, its level curves are lines (rays) or cones. In the next section we will relate this solution with the UC solutions discussed previously.

**Proposition 1** *Let $\varphi$ a solution in $B^2$, then the next four properties are equivalent:*

*i. If $b \in C_p$ then $\varphi[p_N; p + \lambda(b-p)] = p$ for all $1 \geq \lambda \geq 0$ and $p \in \mathbb{R}^N_+$.*

*ii. If $b \in C_p$ then $\varphi[p_N; p + \lambda(b-p)] = p$ for all $\lambda > 1$ and $p \in \mathbb{R}^N_+$.*

*iii. The level curves of $\varphi$ are closed convex sets.*

*iv. The level curves are closed convex cones with apex in their solution vector.*

A solution that satisfies any of the last four properties will be called *conic* solution.

Properties ($i$) and ($ii$) are the classic properties we ask to the production set in elementary microeconomics, namely nonincreasing returns to scale and nondecreasing returns to scale, see chapter 5 of Mas-Colell, Whinston and Green [1992]. The first property only requires the convexity of the level curves on its solution vector, that is, if the income tax of a society is $p$, then any reduction over the income of the agents that is proportional to their net earnings must not change the way to collect the tax. Similarly in property ($ii$), an increase in the income vectors by a proportion to the net incomes of the agents must not change the way to collect. Under scale invariance, properties ($i$) and ($ii$) are dual.

Property ($iii$) is a technical one, but quite interesting, the (convex) merging of two societies where the agents are treated equally (i.e. paying the same), must not change the way to pay. Notice this axioms is weaker than additivity in the amount to distribute and the vector of claims when the vectors are equally treated ($\varphi[c;b]+\varphi[c;\tilde{b}]=\varphi[2c;b+\tilde{b}]$ when $\varphi[c;b]=\varphi[c;\tilde{b}]$), in the context of the rationing problem, if a society of researchers will be equally funded for travel and for research, then when the society request together funds for travel and research must be twice funded as before.

The third property is the combination of ($i$) and ($ii$). The change in the income of a society by a proportion of its net income, must not affect the way to collect income taxes.

The convenience to work with a problem and its dual in the same space of solutions, push us to find an space of conic solutions that is closed under the duality operator. Next proposition gives us this ideal space of conic solutions. The dual of a conic solution will be conic only when the solution satisfies scale invariance.

**Proposition 2** *(a) A solution to $B^N$ is a conic solution that satisfies scale invariance if and only if it's dual is a conic solution that satisfies scale invariance.*

*(b) If a solution to $B^N$ is a conic solution that doesn't satisfy scale invariance then it's dual is not a conic solution.*

**Remark 1**

*The proportional solution, pr, is the unique conic solution in $B^N$ that satisfies self duality.*

# 3 Scale-invariant conic solutions for $n = 2$

For $X, Y \subset \mathscr{S}^2$, we say $X \succeq Y$ if $x \succeq y$ for all $x \in X, y \in Y$. Thus, a correspondence $g : \mathscr{S}^2 \hookleftarrow$ is nondecreasing if $g(x) \succeq g(y)$ for each $x \succeq y$.

A parameterization is an onto and nondecreasing correspondence $f : \mathscr{S}^2 \hookleftarrow$. Since $f$ is nondecreasing and onto then $(0,1) \in f((0,1))$ and $(1,0) \in f((1,0))$. Also, onto guarantees every point in $\mathscr{S}^2$ will belong to the image of $f$, this property along with the monotonicity guarantees that the image of the interval $(p,q) \subset \mathscr{S}^2$ is another interval in $\mathscr{S}^2$. Thus, the graph of $f$ is a continuous and nondecreasing path in $S = \mathscr{S}^2 \times \mathscr{S}^2 \cong [0,1]^2$ that connects $(0,0)$ with $(1,1)$.

On the other hand, the continuity of the graph of $f$ guarantees that the image of a point is a closed interval or a point in $\mathscr{S}^2$. Thus $f$ is uniquely determined by the points whose images are a single value because $f$ has at most a countable number of points whose images are an interval, in particular $f$ is determined by its notched functions $\dot{f}(x) = \min\{z \mid z \in f(x)\}$. Conversely, given a nondecreasing single valued functions $\dot{f} : \mathscr{S}^2 \hookleftarrow$, the correspondence $f(x) = [\dot{f}_-(x), \dot{f}_+(x)]$ is onto and nondecreasing. Therefore the set of parameterizations is isomorphic to the set of nondecreasing solutions in $\mathscr{S}^2$ under the equivalence of solutions that differ in at most a countable number of points.

Similarly to the notched functions, and taking into account that $\mathscr{S}^2$ is isomorphic to $[0,1]$ with the function given by $(x, 1-x) \mapsto x$, a parameterization is simply the correspondence induced by a cumulative distribution in $\mathscr{S}^2$. That is, given a cumulative distribution $\dot{f}$, the induced parameterization is simply $f(x) = [\dot{f}_-(x), \dot{f}_+(x)]$.

Let $\Phi$ the set of parameterizations thus constructed.

Given a parameterization $f \in \Phi$, for each $p \in \mathscr{S}^2$ let $C_p = \{p + tq \mid q \in f(p), t \geq 0\}$, that is, $C_p$ is the cone with apex in $p$ and slope $f(p)$. For any other solution vector $q$, let $C_q = q_N \cdot C_{\tilde{q}}$. We will check the properties of lemma 1, and conclude there is a unique solution that has these level curves. First notice for any solution vector $p$, making $t = 0 : p \in C_p$ holds. Second, since the correspondence is nondecreasing, then $C_p \cap C_q = \emptyset$ for any different solution vector $p$ and $q$ in the simplex $\mathscr{S}^2$. Third, let $b \in \mathbb{R}^N_+$ such that $b_N \geq 1$, and $b^1$ the solution in $\mathscr{S}^2$ whose first coordinate is the closest to the first coordinate of $b$ (i.e. $b^1 = (b_1, 1 - b_1)$ if $b_1 \leq 1$ or $b^1 = (1, 0)$ if $b_1 > 1$), define similarly $b^2$ for the second coordinate, it is easy to check $b^2 \preceq b^1$, and thus the continuous mapping $h : [b^2, b^1] \to \mathscr{S}^2$ given by $h(x) = \widetilde{b - x}$ is decreasing with $h(b^2) \succeq x_2$ and $h(b^1) \preceq x^1$ for some $x_2 \in f(b_2), x_1 \in f(b_1)$ , thus since $f$ is nondecreasing then the graphs of $f$ and $h$ have a common point, say $\widetilde{b - p^*} \in f(p^*)$, hence $b \in C_{p^*}$. Fourth, for any $b \in C_p$, $b \geq p$ clearly holds. Finally, it follows directly from the definition that if $p$ and $q$ are multiples, then the sets $C_p$ and $C_q$ are multiples too, thus this solution is scale-invariant. We will denote by $\varphi^f$ the solution associated to $f \in \Phi$.

Next lemma is a clear consequence of the discussion above.

**Lemma 2** $\varphi^f[c; b] = cp$, *where* $p$ *is the unique solution vector in* $\mathscr{S}^2$ *such that* $\widetilde{b - cp} \in f(p)$.

**Proposition 3** *Any solution* $\varphi^f$ *is a conic solution that satisfies scale invariance. Moreover, any conic solution that satisfies scale invariance can be represented as a* $\varphi^f$ *solution for some* $f \in \Phi$.

**Example 1** *In the proportional solution, the level curve at* $p$ *is the set of multiples of* $p$*. Then* $\varphi^f = pr$ *when* $f(x) = x$.

*In the uniform losses solution, the slope of the level curve of almost every point is* $(\frac{1}{2}, \frac{1}{2})$*. Then,* $f(x) = (\frac{1}{2}, \frac{1}{2}) \forall x \neq (0,1), (1,0)$. $f(0,1) = \{(0,t) \mid 0 \leq t \leq \frac{1}{2}\}$, *and* $f(1,0) = \{(t,0) \mid 0 \leq t \leq \frac{1}{2}\}$.

*In the uniform gains solution, the slope of the level curve of half of the points is* $(1,0)$*, the other half of the points has slope* $(0,1)$*, and* $f(\frac{1}{2}, \frac{1}{2}) = \mathscr{S}^2$*. Notice the representation of the uniform losses solution is the inverse of the representation of the uniform gains solution.*

*In the priority method of* $1 \succ 2$*, the slope of the level curve of every point different than* $(1,0)$ *is* $(0,1)$*, and the level curve of* $(1,0)$ *contains every slope of the simplex* $\mathscr{S}^2$*, thus* $f(x) = (0,1)$ *if* $x \prec (1,0)$ *and* $f(1,0) = \mathscr{S}^2$.

The observation made in the example above, the inverse of the parameterization of $ug$ is precisely the parameterization of its dual $ul$, is a more general property that is satisfied by our parameterizations. The parameterization of the dual is simply the inverse of the parameterization.

**Proposition 4** *If* $f$ *is the parameterization of the scale-invariant conic solution* $\varphi$*, then* $f^{-1}$ *is the parameterization of it dual solution* $\varphi^*$.

Let $\varphi$ a scale-invariant conic solution represented by $f$, and $b$ a vector of claims in $\mathbb{R}^2_+$. As intoduced before, the path to a point $b$ is the solution of the problem with vector of claims $b$ in every amount to divide $c$ less than $b_N$, that is $path(b) = \{\varphi[c; b] \mid 0 < c \leq b_N\}$.

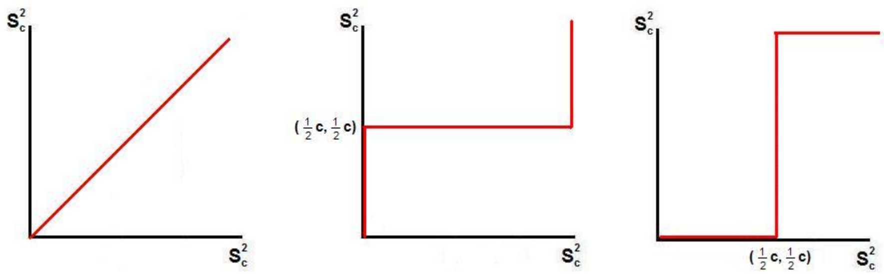


Figure 1: The representation of the proportional, uniform losses and uniform gains solutions.

Since our solutions are individually rational (i.e. $p_i \leq b_i$), then the path to $b$ is in the rectangle bounded by 0 and $b$. Monotonicity in the vector of claims implies this path is a nondecreasing trajectory. In the case of conic solutions, the path to $b$ is above (or below) the segment [0,b] (i.e. our solutions are locally progressive or regressive, see example 3), and the slopes of the path when $c$ increases are increasing (or decreasing) with respect to $b$, and decreasing (or increasing) with respect to 0. Essentially, these properties are sufficient to characterize the conic solutions because given a path that satisfies these properties there is a conic solution that contains this path.

**Corollary 1** *Let $\varphi$ a conic solution and $b$ in $\mathbb{R}^2_+$ a vector of claims, then the path to $b$ satisfies the following properties:*

*i. It is strictly above (or below) the segment $[0, b]$.*

*ii. If $x, y \in path(b)$ and $x_N \leq y_N$, then $\tilde{y} \in [\tilde{x}, \tilde{b}]$, that is, the slopes are increasing (or decreasing) with respect to the origin.*

*iii. If $x, y \in path(b)$ and $x_N \leq y_N$, then $\widetilde{b-x} \in [\tilde{b}, \widetilde{b-y}]$, that is, the slopes are decreasing (or increasing) with respect to $b$.*

*Moreover, if there is a path $\gamma$ that satisfies (i),(ii) and (iii), then there is a conic solution where $\gamma$ is one of its paths.*

In particular, notice any convex or concave path satisfies (i), (ii) and (iii), thus these type of paths fit in a conic solution. However, the conversely is not true, there are conic solutions with nonconvex or nonconcave paths, we check this in example 4.

Given the parameterization $f$ of a conic solution, we can construct the path to $b$ in the following way, for simplicity we will assume $b \notin f(b)$ (otherwise its path is just the segment $[0, b]$) and $b \in \mathscr{S}^2$, the path to any point outside $\mathscr{S}^2$ is simple the multiple of the path to someone in $\mathscr{S}^2$. Let $x$ in $\mathscr{S}^2$ the closest element to $b$ such that $b \in f(x)$. Let $y \in [x, b)$, then the point $z \in [0, y]$ belongs to $path(b)$ if and only if $b - z$ is parallel to some slope of $f(y)$. Indeed, suppose $y \in (x, b)$ and notice the slopes of $f(y)$ are between $b$ and $f(b)$, thus any slope in $f(y)$ will intersect $ray_b(b)$. Therefore $b - z$ is parallel to some slope of $f(y)$, if and only if, $b - z$ is parallel to $s - y$ for some $s \in ray_b(b)$, if and only if, $\varphi[1; s] = y$ and $s_N z_N = 1$, if and only if, $\varphi[z_N; b] = z$.

Finally, the construction of the path allows us to easily compute $path(b)$ as a function of the parameterization $f$, we recall, if $b \in f(b)$ then $path(b) = [0, b]$, otherwise:

$$path(b) = \{\frac{1}{1+t}y \;\mid\; y \in [x, b),\; m \in f(y) \text{ and } t(m - b) = b - y\},$$

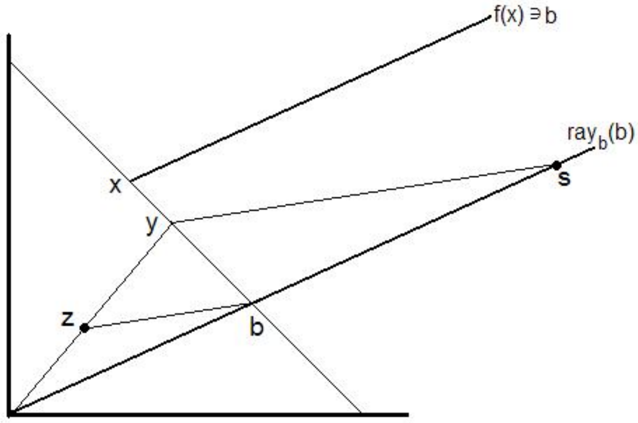


Figure 2: Construction of $path(b)$

where $t$ may be calculated, for instance by $t = \frac{\|b-y\|}{\|m-b\|}$.

### 3.1 Composition axioms and convexity

Let $\kappa$ an ordered cover of $\mathscr{S}^2$, then for any problem $[c;b]$, we associate a rationing method as follows:

$$\varphi[c;b] = \begin{cases} c\tilde{b} & \text{if } \{\tilde{b}\} \in \kappa \\ ca_1 & \text{if } \tilde{b} \in (a_1,a_2) \in \kappa, x = t_1a_1 + t_2a_2 \text{ and } c \le t_1 \\ t_1a_1 + (c-t_1)a_2 & \text{if } \tilde{b} \in (a_1,a_2) \in \kappa, x = t_1a_1 + t_2a_2 \text{ and } t_1 < c \le b_{\{1,2\}} \end{cases} \quad (1)$$

The rationing methods thus constructed is denoted by $\Delta$. These rationing methods were defined by Moulin[2000], and he proved that any scale-invariant solution that satisfies the composition axioms must be one of these methods.

As noted by Moulin, a given element in the oriented open interval $(a_1, a_2)$ has a path which is constructed by a pieces of the segment $[0, a_1]$ and a piece in the parallel to $[0, a_2]$ by the given point. More generally, these methods are a very particular case of the methods discussed previously for $UC$ solutions. In the oriented interval $(a_1, a_2)$, the generator path is simply the parallel line of $[0, a_2]$ by $a_1$, thus the level curves are only parallel lines to $[0, a_2]$. More technically, the method in $\Delta$ with associated partition $\kappa$ is associated to the method in $\Pi$ given by $(\kappa, \phi)$, where $\phi = \{u_{(a_1,a_2)}\}_{(a_1,a_2)\in\kappa}$ and $u_{(a_1,a_2)}(x) = \frac{\|a_1-a_2\|}{\|x-a_2\|}$.

**Theorem 1** *Any solution in $\Delta$ satisfies scale-invariance, upper and lower composition and is conic. Conversely, any continuous and scale-invariant method that meets* ***any two*** *of the properties: upper-composition, lower-composition or conic, must be a rationing method in $\Delta$.*

To construct the conic representations of the methods in $\Delta$, let $f : \mathscr{S}^2 \to \mathscr{S}^2$ a representation of a conic solution. We say that $f$ has a break in the points $a, b \in \mathscr{S}^2$, if and only if $a \in f(a), b \in f(b)$, and either $a \in f(x) \forall a < x < b$ or $b \in f(x) \forall a < x < b$. That is, $f$ has a break in the points $a$ and $b$

if it is constant equal $a$ or $b$ in the interval $(a,b)$, and the graph of $f$ intersects the identity function in $a$ and $b$.

For example, the representation of the $ug$ $(ul)$ has two breaks, one in the points $(0,c)$ and $(\frac{c}{2},\frac{c}{2})$, and the other in the points $(c,0)$ and $(\frac{c}{2},\frac{c}{2})$. On the other hand, the representation of the priority method has only one break in the points $(0,c)$ and $(c,0)$.

**Corollary 2** *A scale-invariant solution that satisfies upper-composition and is conic (lower-composition and is conic, or upper-composition and lower-composition) has a conic representation which is the identity function except for a countable number of breaks.*

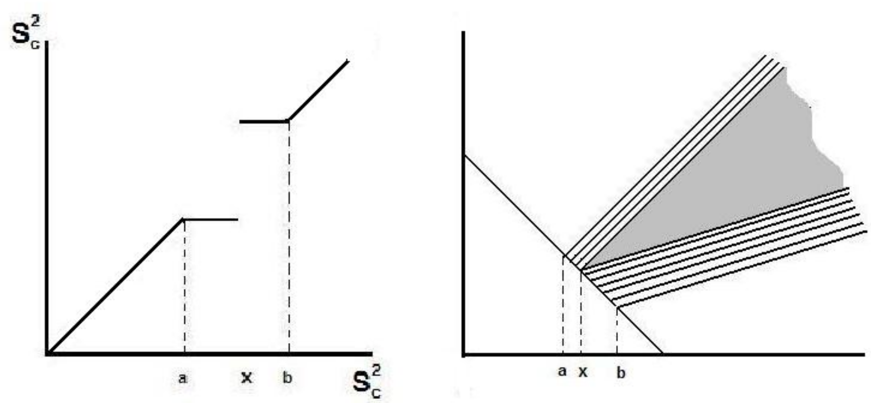


Figure 3: Left figure shows the conic representation of a method in $\Delta$ with two breaks. The first in $a$ and $x$, and the second in $x$ and $b$. Figure at right shows the corresponding solution of left figure.

## 4 Composition and Convexity for $n > 2$

The class os solutions that satisfies upper composition and convexity is very large. The Uniform gains, Uniform losses and Proportional solution discussed above meet these properties. We discuss the class of upper composition and convex solutions.

Recall that for $p, b \in \mathbb{R}^N_+ : ray_p b = \{p + tb | t \geq 0\}$ is the ray with apex $p$ and slope $b$. Let $Rays$ be the set of rays.

**Definition 7** *A **cover by rays** $P$ of $\mathbb{R}^N_+$ is such that:*

*i.* $P \subset Rays$

*ii.* $\cup_{r \in P}\{r\} = \mathbb{R}^N_+$

*iii. If* $ray_x(b), ray_{x'}(b') \in P$ *and* $ray_x(b) \cap ray_{x'}(b') \neq$ *then* $x' \in ray_x(b)$ *or* $x \in ray_{x'}(b')$.

A cover by rays is a set of rays that cover the entire positive orthant and any two rays intersect in at most the apex of one of the rays.

**Definition 8** *Given a cover by rays $P$, the associated solution $\varphi$ is constructed by constructing the level sets of $\varphi$.*

*Let $c > 0$ and $\Delta_c^N$ be the simplex at $c$.*

*Let $ray_x(b) \in P$ be such that $x_N \geq c$. Let $p = ray_x(b) \cap \Delta_c^N$. Then $ray_x(b) \subset C_p$.*

*Clearly, then set of $\{p' \in \Delta_c^N | ray_x(b) \cap \Delta_c^N \neq \emptyset\}$ is dense in $\Delta_c^N$. The rest of the level sets are approximated by continuity.*

**Theorem 2** *For $n \geq 2$, the solutions constructed above satisfy upper composition and convexity. Moreover, any solution that satisfies these axioms can be represented as above for some cover of rays.*

**Corollary 3** *A solution satisfies scale invariance, scale invariance, upper composition and convexity if and only if $P$ is composed of parallel rays, that is, if $Ray_p(b) \in P$ then $\lambda Ray_p(b) \in P$ for any $\lambda > 0$.*

## 4.1 Composition, Scale Invariance and consistency

Finally, by incorporating consistency and scale invariance to the composition axioms lead to a familiar class of Priority solution studied by Moulin[2000].

**Theorem 3** *A consistent and scale-invariant solution $\varphi$ that satisfies upper-composition and convexity (lower-composition and convexity, or upper-composition and lower-composition) if and only if it equals a priority solution defined by Moulin[2000].*

## Appendix: Proofs

**Lemma 1**

**Proof.**

Let $E_c^N = \{x \in \mathbb{R}_+^N \mid x_N \geq c\}$ and $\varphi$ a solution to $B^N$ with $\{C_p \mid p \in S^N\}$ its level curves.

Since $\varphi$ satisfies individual rationality, for each $p \in S^N$ $p \in C_p$ holds, then $C_p \neq \emptyset$.

To prove $(b)$, suppose that exist $p, q \in S_c^N$ s.t. $C_p \cap C_q \neq \emptyset$, then exists $b \in C_p \cap C_q$, so $p = \varphi[c;b] = q$.

To prove $(c)$, suppose $E_c^N \not\subset \cup_{p \in S_c^N} C_p$. Then, exists a point $b \in E_c^N$ s.t. $\varphi[c;b] \notin S_c^N$, but it cannot occur because $[c;b] \in B^N$, so $[c;b]$ is in the dominium of $\varphi$, and since $\varphi[c;b]$ is a nonnegative vector s.t. $\sum_{i \in N} \varphi_i[c;b] = c$, $\varphi[c;b] \in S_c^N$. The inclusion $\cup_{p \in S_c^N} C_p \subset E_c^N$ is clear.

$(d)$ is clear because $\varphi$ satisfies individual rationality.

Now, suppose the collection of subsets from $E^N$, $\{C_p \subset E^N \mid p \in S_c^N\}$ satisfies the last four conditions. By $(b)$ and $(c)$ the set $\{C_p \mid p_N = c\}$ is a partition of $E_c^N$, then the function $\varphi[c;b] = p$ if and only if $b \in C_p$ and $p_N = c$ is well defined. Moreover, by $(d)$ $\varphi$ satisfies individual rationality. Then $\varphi$ is a solution to $B^N$. The uniqueness is clear from the definition of level curves. ■

**Proposition 1**

**Proof.**

First notice the solutions clearly satisfy scale invariance. On the other hand, it can be easily verified that the solution satisfies lemma 2, hence upper composition.

The proof of the converse statement is constructive, and it is divided in several small steps.

Lets start with a solution $\varphi$ that satisfies the named axioms and suppose the solution has level curves $\{C_p\}_{p \in \mathbb{S}}$.

**Step 1. If** $\lambda p \in C_p$ **and** $\lambda > 1$, **then** $ray_p(p) \subset C_p$.

We will prove it by contradiction. If $ray_p(p) \not\subset C_p$ and $\lambda p \in C_p$. Then we can find $\lambda_1 p, \lambda_2 p \in C_p$, such that $\lambda_1 > \lambda_2 > 1$ (by scale invariance and lemma 1 we can assume $\lambda_2 > 1$), and there is not a point between them that have solution $p$, that is, there is not $\tilde{\lambda}$, $\lambda_1 > \tilde{\lambda} > \lambda_2$ such that $\tilde{\lambda} \in C_p$.

Consider the path $\gamma \in C_p$ that connects $\lambda_1 p$ and $\lambda_2 p$. By the choice of $\lambda_1$ and $\lambda_2$, $\gamma$ does not contain any other point in $ray_p(p)$. Consider the set $B$ inside the solution space $\mathscr{S}^2$ such that every point in $B$ has a multiple in $\gamma$, that is $B = \{q \in \mathscr{S}^2 \mid ray_q(q) \cap \gamma \neq \emptyset\}$.

Consider the extreme point (infimum or supremum) of $B$ different than $p$. The ray by this extreme point cuts the path $\gamma$ once, or a continuous number of times. Therefore, by choosing a point $q$ close to the extreme point, we can guarantee that $ray_q(q)$ cuts $\gamma$ at least twice, say $\tilde{\lambda}_1 q, \tilde{\lambda}_2 q \in \gamma$, $\tilde{\lambda}_1 > \tilde{\lambda}_2$ and $\frac{\tilde{\lambda}_1}{\tilde{\lambda}_2} < \frac{\lambda_1}{\lambda_2}$. Then by scale invariance

$$\frac{\tilde{\lambda}_2}{\tilde{\lambda}_1}(\tilde{\lambda}_1 q) \in \frac{\tilde{\lambda}_2}{\tilde{\lambda}_1} C_p \cap C_p.$$

Thus by lemma 2,

$$\frac{\tilde{\lambda}_2}{\tilde{\lambda}_1} C_p \subset C_p.$$

Therefore, $\frac{\tilde{\lambda}_2}{\tilde{\lambda}_1}(\lambda_1 p) \in C_p$ with $\lambda_1 > \frac{\tilde{\lambda}_2}{\tilde{\lambda}_1}\lambda_1 > \lambda_2$, and this contradicts the election of $\lambda_1$ and $\lambda_2$.

Notice that from step 1 and continuity, if $ray_p(p) \not\subset C_p$ then for a neighborhood of $p$, say $(p - \epsilon, p + \epsilon)$, $ray_q(q) \not\subset C_q$ for each $q \in (p - \epsilon, p + \epsilon)$.

Consider the partition $\kappa$ of $\mathscr{S}^2$ given by: 1) $(a_1, a_2) \in \kappa$ if and only if $ray_p(p) \not\subset C_p$ for all $p \in (a_1, a_2)$, $ray_{a_1}(a_1) \subset C_{a_1}$ and $ray_{a_2}(a_2) \subset C_{a_2}$ or 2) $\{x\} \in \kappa$ if and only if $ray_x(x) \subset C_x$.

Observe that by the remark above, $\kappa$ is well defined and by step 1 it is a partition of $\mathscr{S}^2$.

**Step 2. The function $h : R_+^N \to \mathscr{S}^2$, s.t. $h(x) = \frac{x}{x_N}$ is continuous.**
This statement is clear because it is the multiplication of two continuous functions.

For $p \in \mathscr{S}^2$, let $A(p) = \{x \in [a_1, a_2] \mid C_p \cap ray_x(x) \neq \emptyset\}$.

**Step 3. For any $p \in \mathscr{S}^2$, $A(p)$ is path-connected.**
Let $x \in A(p)$, then there is $\lambda > 1$ s.t. $\lambda x \in C_p$. By connectedness, there is path $\gamma \subset C_p$ s.t. $\gamma(0) = \lambda x$ and $\gamma(1) = p$. Since $\gamma$ is continuous, then by step 2.5, $h \circ \gamma$ is continuous. Clearly, $h \circ \gamma$ is a continuous path in $A(p)$ that connects $x$ and $p$. Therefore $A(p)$ is path-connected.

Notice that since $A(p)$ is path-connected and $p \in \mathscr{S}^2$, then $A(p)$ is an interval (or a point).

For $B \subseteq S^2$, we define $coneB \equiv \cup_{b \in B} ray_b(b)$. By connectedness, for any $p \in (a_1, a_2) \in \kappa$, $C_p \subset cone(a_1, a_2)$. Therefore, we only need to analyze how the level curves are on each $(a_1, a_2) \in \kappa$.

We will divide the rest of the proof in two cases, when some of the level curves $C_{a_1}$ or $C_{a_2}$ has points in $cone(a_1, a_2)$, or when the union of the level curves in $(a_1, a_2)$ covers $cone(a_1, a_2)$.

***Case 1.*** $C_{a_1} \cap cone(a_1, a_2) \neq \emptyset$.

**Step 4. Suppose $\lambda_p p \in C_{a_1}$ for some $\lambda_p > 1$ and $p \in S^2$, then $ray_{\lambda_p p}(p) \subset C_{a_1}$.**
By lemma 2, $C_{\lambda_p p} \subset C_{a_1}$. Let $\lambda_p p + tp \in ray_p(\lambda_p p)$ where $t > 0$. Then by scale invariance:

$$C_{(\lambda_p + t)p} = \frac{(\lambda_p + t)}{\lambda_p} C_{\lambda_p p} \subset \frac{(\lambda_p + t)}{\lambda_p} C_{a_1} = C_{\frac{\lambda_p + t}{\lambda_p} a_1} \tag{2}$$

Since $ray_{a_1}(a_1) \subset C_{a_1}$ then by lemma 2,

$$C_{\frac{\lambda_p + t}{\lambda_p} a_1} \subset C_{a_1}. \tag{3}$$

Therefore, (2) and (3) imply that $(\lambda_p + t)p \in C_{a_1}$ for any $t > 0$.

**Step 5.** $A(a_1) = [a_1, a_2)$.
First, notice if $p \in A(a_1)$ then $A(p) \subset A(a_1)$. Indeed, let $\lambda_p p \in C_{a_1}$, then by lemma 2 $C_{\lambda_p p} \subset C_{a_1}$. By scale invariance $C_p \subset \frac{1}{\lambda_p} C_{a_1}$. Therefore if $y \in A(p)$, then $\tilde{\lambda} y \in C_p$ for some $\tilde{\lambda}$, thus $\tilde{\lambda} \lambda_p y \in C_{a_1}$, hence $y \in A(a_1)$.

To prove step 5, we know $A(a_1)$ is a connected subset that contains $a_1$, thus $A(a_1) = [a_1, p]$ or $A(a_1) = [a_1, p)$ for some $p \in (a_1, a_2]$.

If $A(a_1) = [a_1, p]$ then $\lambda_p p \in C_{a_1}$ for some $\lambda_p > 1$. Since there is not a multiple of $p$ that has solution $p$ then any point in $ray_p(p)$ has solution in the interval $[a_1, p]$ (otherwise, some multiple of $C_{a_1}$ contains a multiple of a point outside $[a_1, p]$). This implies along with the unboundness of the level curves (which follows by continuity, see step 5 of proposition 2), the level curve of $p$ lies in the $cone([p, a_2))$, thus $A(p) \cap (p, a_2) \neq \emptyset$. Hence by the initial observation of this step, $A(a_1) \cap (p, a_2) \neq \emptyset$, and this contradicts the initial election of $A(a_1)$.

If $A(a_1) = [a_1, p)$ and $p \neq a_2$, then $C_x \cap ray_p(p) = \emptyset$ for each $x \in A(p)$. Therefore $C_p$ is exclusively contained in $cone((p, a_2)) \cup \{p\}$. This observation along with the unboundness of $C_p$ imply none of the level curves in of $(p, a_2]$ intersect $ray_p(p)$, which is a contradiction.

Finally, we will construct the weight solution $u_{(a_1, a_2)}$ which we will denote simply by $u$. Let $u : [a_1, a_2) \to [1, \infty)$ given by

$$u(a) = \min\{\lambda_a \mid \lambda_a a \in C_{a_1}\} \tag{4}$$

By step 5, for any point $a \in [a_1, a_2)$ the set $\{\lambda_a \mid \lambda_a a \in C_{a_1}\}$ is not empty, moreover, the level curve of $C_{a_1}$ is closed, so the minimum of each set exists, therefore $u$ is well defined. Moreover, if $a \neq a_1$ then $u(a) > 1$ because $a \in C_a$ for any $a$.

**Step 7. For any** $a \in [a_1, a_2)$ $C_a \subset \frac{1}{u(a)}\{u(x)x \mid u(x) \geq u(a)\}$ **holds.**

**Step 8.** $u$ **is onto, that is** $A(a_1) = [1, \infty)$.

Suppose there is $c > 1$ s.t. $u(x) \neq c$ for all $x \in (a_1, a_2)$. By continuity, for each $x \in (a_1, a_2)$, $C_x$ is not bounded above. Then, by step 7, there is $p \in (a_1, a_2)$ s.t. $u(p) > c$. Then $\frac{u(p)}{c}p \in cone(a_1, a_2)\backslash C_{a_1}$. Then, there is $z \in (a_1, a_2)$ s.t. $\frac{u(x)}{c}x \in \frac{1}{u(z)}C_{u(z)z}$. Then by step 7, $\frac{u(x)}{c}x = \frac{1}{u(z)}(u(x)x)$. Therefore $u(z) = c$, which is a contradiction with the assumption.

**Step 9. For each** $x \in (a_1, a_2)$, $u$ **is bounded in** $[a_1, x)$

Since $x \in A(a_1)$, then there exist $\lambda > 1$ s.t. $\lambda x \in C_{a_1}$. Let $\gamma$ a path in $C_{a_1}$ that connects $a_1$ and $\lambda x$. Clearly, for every point $y \in [a_1, x)$, $u(y) \leq \max_k(\gamma(k)_N)$ holds. Since $\gamma$ is defined in a closes interval, then $\max_k(\gamma(k)_N) < \infty$.

**Step 10.** $u$ **is injective.**

Suppose there exist $p, q \in (a_1, a_2)$ s.t. $u(p) = u(q)$. Then $u(p)C_p \cap u(q)C_q = \emptyset$.

Clearly, $A(p) \cap A(q) = \emptyset$, otherwise by step 7, for some $z$: $\frac{1}{u(p)}u(z)z \in C_p \cap C_q$, which cannot occur.

On the other hand, since $C_p$ and $C_q$ are non-bounded, then $u$ is not bounded in the disjointed intervals $A(p)$ and $A(q)$. However, by step 9, one of this intervals must be bounded, which is a contradiction.

Finally, steps 8 and 10 along with the connectedness of the level curves, guarantees the solution is strictly monotone and continuous. Step 9 guarantees $lim_{x \to a_2} u(x) = \infty$.

***Case 2.*** $C_{a_1} \cap cone(a_1, a_2) = \emptyset$ ***and*** $C_{a_2} \cap cone(a_1, a_2) = \emptyset$

**Step 11.** $A(x) = \cup_{y \in C_x} A(\tilde{y}) = \cup_{\tilde{y} \in A(x)} A(\tilde{y})$

Let's prove the first equality, if $\tilde{y} \in A(x)$ then $\lambda_y \tilde{y} \in C_x$ for some $\lambda_y \geq 1$, since $\tilde{y} \in A(\tilde{y})$, then $A(x) \subset \cup_{y \in C_x} A(\tilde{y})$. On the other side, let $y \in C_x$, then by upper composition $C_y \subset C_x$, then $A(\tilde{y}) \subset A(x)$, therefore $A(x) \supset \cup_{y \in C_x} A(\tilde{y})$. The second equality is straightforward from the definition of $A(\cdot)$.

**Step 12. For each** $x \in (a_1, a_2)$ **either** $A(x) \subset [a_1, x]$ **or** $A(x) \subset [x, a_2]$ **holds.**

Suppose there is $x, y, z \in (a_1, a_2)$ such that $y, z \in A(x)$, $y \in (a_1, x)$ and $z \in (x, a_2)$. By connectedness $(y, z) \subset A(x)$. Let $\gamma$ the path $[2x, x]$, then $\varphi[1, \gamma]$ is a path in $\mathscr{S}^2$. By continuity, $\varphi[1; \lambda x] \in (y, z) \subset A(x)$ for some $2 \geq \lambda > 1$. Therefore $\lambda x \in C_p$ for some $\tilde{\lambda} p \in C_x$. Finally by scale invariance and upper composition $\lambda \tilde{\lambda} x \in C_x$, which is a contradiction with step 2.

**Step 13. If** $b \in C_p$ **then** $A(\tilde{b}) \subset A(p)$

Since $b \in C_p$ then $C_b \subset C_p$, which implies $A(\tilde{b}) \subset A(p)$.

To prove the next step, suppose there are $x < y^* \leq a_2$ such that $y^* \in A(x)$. We will prove that $A(x^*) = [x^*, a_2)$ for all $x^* \in (a_1, a_2)$. This will be done in several steps:

**Step 14.** $A(x) = [x, a_2)$

By step 12, $[x, a_2) \supset A(x)$ holds. We will prove the other side by contradiction. Since $A(x)$ is connected, then it is an interval, let $y = inf\{p \in (x, a_2) \mid p \notin A(x)\}$. Since $y^* \in A(x)$, then $a_2 \succeq y \succeq y^* \succ x$. Therefore either $A(x) = [x, y)$ or $A(x) = [x, y]$ holds.

Case 1. Suppose $[x, y) = A(x)$, $x \prec y \prec a_2$.

First observe that for every $\lambda \geq 1$, $\varphi[1; [\lambda y, y]]$ is a connected subset in $\mathscr{S}^2$ that contains $y$. Also, $\varphi[1; [\lambda y, y]] \cap [a, y) = \emptyset$ because $y \notin [x, y) \subset A(x)$. Therefore $\varphi[1; [\lambda y, y]] \subset [y, a_2)$. Moreover, by step 14, and connectedness $\varphi[1; [\lambda y, y)] = (y, p]$ for some $p \in (y, a_2)$. Then by step 13, $A_y \subset A(h)$ for each $h \in (y, p]$. This fact, along with step 12 and connectedness, implies $A(y) \subset (a_1, y]$. Finally, $A(x) \subset A(y)$, which means that $k \cdot C_x \subset C_y$ for some $k > 1$, however, by continuity $C_x$ is not bounded in $A(x) = [x, y)$ which is a contradiction because $C_y$ is clearly bounded in $[x, y]$.

Case 2. Suppose $A(x) = [x, y]$.

Let $\lambda_y$ such that $\lambda_y y \in C_x$, then by connectedness there is a path in $C_x$ that connects $\lambda_y y$ and $x$, this means $C_x$ is bounded, which is a contradiction with the unboundedness of the level curves.

**Step 15.** $A(y) = [y, a_2)$ **for all** $y \in (a_1, a_2)$.

In step 14, we proved that if $y^* \in A(x)$, $x \prec y^* \prec a_2$ then $A(x) = [x, a_2)$. Let $y \in (x, a_2)$, then $A(y) \subset A(x) = [x, a_2)$, so $C_y$ is a multiple of $C_x$. Since $C_x$ is bounded in every closed interval, then $C_y$ can only be unbounded if $A(y) = [y, a_2)$. Therefore if $y \in (x, a_2)$, then $A(y) = (y, a_2]$. From this, it is clear that $\{x \mid A(x) = [x, a_2)\}$ is a connected set. Without loss of generality let's assume that $[x^*, a_2) = \{x \mid A(x) = [x, a_2)\}$.

Finally, for each $y \in (a_1, x^*)$, $A(y) = (a_1, y]$ holds, and similarly, for each $z \in (x^*, a_2)$ $A(z) = [z, a_2)$ holds. Therefore, $\varphi[a, 2x^*] \notin (a_1, x^*)$ and $\varphi[a, 2x^*] \notin (a_1, x^*)$ holds, which means that $\varphi[1; 2x^*] = x^*$, and it is a contradiction with step 2.

The remaining of the proof is easy, let $t_k = a_1 + \frac{1}{k+1}(a_2 - a_1)$ for $k = 1, 2, ...$ and define $u_k : [t_k, a_2) \to [0, \infty)$ as:

$$u_1(x) = \lambda \text{ where } \lambda \cdot x \in C_{t_1}$$

Similarly, for $k \geq 1$, let $\lambda_{k+1}(a_k)$ the number such that $\lambda_{k+1}(a_k) \cdot t_k \in C_{a_{k+1}}$ (by step 14, we know there is only one value that satisfies this condition). Let $m_{k+1} = \frac{u_k(a_k)}{\lambda_{k+1}(a_k)}$.

$$u_{k+1}(x) = \begin{cases} u_k(x) & \text{if } x \in [a_k, a_2) \\ m_k \lambda & \text{where } \lambda \cdot x \in C_{t_{k+1}} \end{cases} \tag{5}$$

By the monotonicity of $A(a_k)$, and hence of the level curves, these solutions will be monotonic. $u = \lim_{k \to \infty} u_k$ meets the requirements of the theorem in this case.

**Remark 2** *The level curve of the point* $a \in (a_1, a_2)$ *is given by:*

$$C_a = \{\frac{u(x)}{u(a)} x \mid x \in (a, a_2)\} \tag{6}$$

■

**Proposition 2**

**Proof.**

**Step 1. The level curves of** $\varphi$ **are closed sets.**

Let $s \in S$ then $C_s = \varphi^{-1}[\{s\}]$, since $\varphi$ is continuous and $\{s\}$ is closed then the inverse image of closed set is a closed set. Then $C_s$ is closed.

***Implication*** $(i) \Rightarrow (iii)$.

First let's prove for it for $t = 1$, the other case is a clear consequence of the fact that any bounded and decreasing sequence is convergent.

Let $a, b \in C_p$ and let's define $\triangle(p; a, b) \equiv \{t_1 a + t_2 b + (1 - t_1 - t_2)p \mid 0 < t_1 + t_2 \leq 1\}$. $\triangle(p; a, b)$ is the triangle of $p, a, b$ excluding the segment $[a, b]$.

Suppose there is $d \in \triangle(p; a, b)$ s.t. $q \equiv \varphi[c; d] \neq p$. Then $d$ is an interior point of the triangle and the line segment $[d, q]$ doesn't cut the segments $[b, p]$ and $[a, p]$.

For each $w \in [p, q]$, let $\gamma_w = [d, w]$. Since $\gamma_q$ doesn't intersect the segments $[b, p]$ and $[a, p]$, and $\gamma_p$ intersects both segments, then by continuity there is $\hat{q} \in (p, q)$ such that $\gamma_{\hat{q}}$ cuts some of the segments $[b, p]$ or $[a, p]$, let's say it cuts $[b, p]$. Then the segment $[b, p]$ divides in two parts the triangle with points $d, p$ and $\hat{q}$.

On the other hand, let $\gamma$ the path $[d, p]$, then $\varphi[c; \gamma] = [q, p]$. By continuity, there is $\hat{w} \in \gamma$ s.t. $\varphi[c; w] = \hat{q}$. Then the segment $[\hat{w}, q]$ is inside the triangle with points $d, p$ and $\hat{q}$. However, since $[b, p]$ divides in two parts the triangle with points $d, p$ and $\hat{q}$, then the segment, $[\hat{w}, \hat{q}]$ is divided by $[b, p]$. Therefore, since $[\hat{w}, \hat{q}] \subset C_{\hat{q}}$ and $[b, q] \subset C_p$ then $C_{\hat{q}} \cap C_p \neq \emptyset$, which is a contradiction with lemma 1. Therefore $\triangle(p; a, b) \subset C_p$.

Since the level curves are closed sets, then the segment $[a, b] \subset C_p$.

***Implication*** $(iii) \Rightarrow (iv)$**.**

This part will be given in steps two to five. It is an aplication of the theorem of separability of convex sets.

*Step 2. Let* $[a, b]$ *and* $[c, d]$ *be segments. Then for any* $p \in [a, b]$,

$$dist(p, [c, d]) \leq max(|a - c|, |b - d|).$$

Since $dist(p, [c, d])$ is a differentiable function which is definied for all $p \in \mathbb{R}^N$. The critic points of the function are in the segment $[c, d]$, then the function restricted to $[a, b]$ has it maximum in the border of $[a, b]$. Then the inequality holds.

**Step 3. Suppose** $C_s$ **is a level curve of** $\varphi$**. Let** $p \in bdC_s$ **then** $[s, p] \subset bdC_s$.

$[s, p] \subset C_s$ because $C_s$ is convex. Suppose $[s, p] \not\subset bdC_s$ then exists $t \in (s, p)$ and $\gamma > 0$ s.t. $B_\gamma(t) \subset intC_S$. Since $p \in bdC_s$ and $\varphi$ is continuous, there exists $\delta < \gamma$ s.t. if $|p - x| < \delta$ then $|s - \varphi[c; x]| < \gamma$ . Take $x \in B_\delta(p) \backslash C_s$ (exists an element because $p \in bdC_s$). Take the level curves $C_s$ and $C_{\varphi[c;x]}$, these curves are convex and haven't common points. Then exists an hyperplane that separates the sets. This hyperplane must cut the segments $[p, x]$ and $[s, \varphi[c; x]]$ on $f$ and $g$ respectively. Then $|f - p| < \gamma$ and $|g - s| < \gamma$ so, the distance from $t$ to $line(f, g)$ is less then $\gamma$. Then by step 0, the hyperplane must cut the $B_\gamma(t)$, which is a contradiction because the hyperplane does not cut the interior of $C_s$.

**Step 4. If** $intC_s \neq \emptyset$ **then** $C_s$ **is cone.**

(a) Let $x \in intC_s$ then $ray_s(x)$ does not cut $bdC_s$ (otherwise, using **step 2**, part of $(s, x]$ belongs to the border and it can't be true because $(s, x]$ is in the interior of $C_s$). Then $ray_s(x) \subset intC_s$.

(b) Let $x \in bdC_s$, then exists $\{x_k\}_{k \in N} \subset intC_s$ s.t. $x_k \to x$. Then $ray_s(x_k) \to ray_s(x)$. Since $\{ray_s(x_k)\}_{k \in N} \subset intC_s$ and $C_S$ is closed then $ray_s(x) \subset C_s$.

Then $C_s$ is cone.

**Step 5. Suppose** $\varphi \in B^2$**, then the level curves are not bounded.**

Let $n \in N$, then $\varphi[c; (n, 0)] = (c, 0)$ and $\varphi[c; (0, n)] = (0, c)$. Then using the mean value theorem $\varphi[c; [(n, 0), (0, n)]] = S_c$. Then the level curves are not bounded.

Since $\varphi \in B^2$, then $intC_s = \emptyset$ and $C_s$ convex imply $C_s$ is a segment. Using **step 4**, if $C_s$ is a segment then it is a ray, then $C_s$ is cone.

The equivalence $(iv) \Rightarrow (ii)$ is obvious.

***Implication*** $(ii) \Rightarrow (i)$

Let $p \in \mathscr{S}^2$, and $b \in C_p$. Assume $ray_p(b) \not\subset C_p$. Then by continuity and by $(ii)$, there is a monotone sequence in $ray_p(b)$ whose image under the solution is not $p$, and it converges to $p$. Let's say $\{x_k\}_{k\geq 1}$, such that $| x_k - p | > | x_{k+1} - p | \underset{k\to\infty}{\to} 0$ and $\varphi[c; x_k] \neq p$.

By $(ii)$, $A_{x_k} \equiv \{\varphi[c; x_k] + t \cdot (x_k - \varphi[c; x_k]) \mid t \geq 1\} \subset C_{\varphi[c;x_k]}$. Since $x_k$ goes to $p$, then the limit of the sets $A_{x_k}$ has solution $p$. However, the limit of the sets $A_{x_k}$ is a ray with apex in $p$. Therefore, the level curve $C_p$ contains a ray with apex in $p$.

Therefore every level curve contains a ray, thus by weak ranking —this is the unique place in the proof I will use, however, a messier proof without this axiom can be given— the level curves must be cones because every ray divides the space of problems in two parts so every level curves is bounded by rays only. This is a contradiction with the assumption. Thus the ray $ray_p(b - p) \subset C_p$.

The remaining implication clearly holds because $(i) \Leftrightarrow (iv) \Rightarrow (ii)$. ∎

**Proposition 3**

**Proof. (a)** Suppose $\varphi$ is a *conic solution* that satisfies scale invariance. It is easy to prove that $\varphi^*$ satisfies scale invariance too. Let $p \in S_c^N$ and $b \in C_p$ of $\varphi^*$, then exist unique $q \in S_c^N$ and $t > 0$ such that $b = p + tq$. Then $\varphi^*[c; p + tq] = p$ and by Proposition 2$a$, $\varphi[c; \frac{1}{t}p + q] = q$. Since $q + \frac{1}{t}p \in C_q$ of $\varphi$, and $\varphi$ is a *conic function* then $q + lp \in C_q$ of $\varphi$ for all $l \in (0, \infty)$. By Proposition 2$b$, $p + lq \in C_p$ of $\varphi^*$ for all $l \in (0, \infty)$. Hence $C_p$ in $\varphi^*$ is a cone and furthermore $\varphi^*$ is a *conic function.* The other way of the implication is analogous.

**(b)** Suppose $\varphi$ is a *conic solution* that does not satisfy scale invariance.

**Step 1.** Since $\varphi$ does not satisfy scale invariance then exists $\lambda, c > 0$ and $b \in E_c^N$ such that $\varphi[\lambda c; \lambda b] \neq \lambda\varphi[c; b]$. Let $\varphi[c; b] = p$, by lemma 1 exist $q \in S_c^N, t > 0$ s.t. $b = p + tq$. Let $\tilde{t} \neq t$ then $\varphi[\lambda c; \lambda p + \tilde{t}\lambda q)] \neq \lambda p$, otherwise $ray_{\lambda p}(\lambda q) \subset C_{\lambda p}$ and $\lambda b \in ray_{\lambda p}(\lambda q)$ imply $\lambda b \in C_{\lambda p}$!. In particular, if $\tilde{t} = \frac{1}{\lambda}$ we have $\varphi[\lambda c; \lambda p + q] \neq \lambda p$.

**Step 2.** Since $\varphi$ is a *conic solution* $\varphi^*[c; p + q] = p + q - \varphi[c; p + q] = p + q - p = q$.

Suppose $\varphi^*$ is a *conic solution* then by step 2 $\varphi^*[c; xp + q] = q$ for all $x > 0$. Then $q = \varphi^*[c; xp + q] = xp + q - \varphi[xc; xp + q]$ for all $x > 0$, then $\varphi[xc; xp + q] = xp$ for all $x > 0$!, it is a contradiction with step 1. ∎

**Remark 1**

**Proof.** It is clear that $pr$ is a *conic function* that satisfies self duality.

Let us prove the uniqueness. Suppose $\varphi$ is a *conic function* that satisfies self duality. Let $p \in S_c^N$, since $\varphi$ is a conic function exist an slope $q \in S_c^N$ such that $\varphi[c; p + tq] = p$ for all $t \in [0, \infty)$. Then $\varphi[c; p + q] = p$. On the other hand, $\varphi^*[c; p + q] = p + q - \varphi[p_N + q_N - c; p + q] = p + q - \varphi[c; p + q] = q$. Then $q = \varphi^*[c; p + q] = \varphi[c; p + q] = p$. So, exists a unique slope in $C_p$ and it is p, then $C_p$ is the set of multiples of $p$. Therefore $\varphi = pr$. ∎

**Proposition 4**

**Proof.**

Let $f(\cdot)$ the representation of the conic solution $\varphi$. Let $p \in \mathscr{S}^2$ and $q \in f(p)$. Then

$$\varphi[1; p + tq] = p \text{ for all } t > 0.$$

By scale invariance and duality

$$\varphi^*[1; \frac{1}{t}p + q] = q \text{ for all } t > 0.$$

Therefore the level curve of $q$ in $\varphi^*$ contains the slope $p$. Since $f$ is onto, then the level curve of any solution vector $q$ is defined in this way. Hence $f^* = f^{-1}$. ■

**Corollary 1**

**Proof.**

*i.* Let $0 < l \leq 1$, then by scale invariance $\varphi[l; b] = l\varphi[1; \frac{1}{l}b]$. Therefore, to find the path of $b$, we only need to find the the solution of the points in $ray_b(b)$.

The level curve $C_x$ intersects $ray_b(b)$ if and only if $x+t(y) = b(1+t)$ for some $t \geq 0$ and $y \in f(x)$.

If $y = b$, then $x = b$, this means $b \in C_b$.

If $y \neq b$, then, $t = \frac{\|b-x\|}{\|y-b\|}$ this means that $\frac{1}{1+t}x$ is in the path of $b$, that is $\frac{\|y-b\|}{\|b-x\|+\|y-b\|}x$ is in the path of $b$.

■

**Proposition 6.**

**Proof.**

From the construction, it is easy to prove any solution in $\Delta$ satisfies the named axioms. We will prove now the converse.

Let $\varphi$ a continuous, scale-invariant, $UC$ and conic solution. By proposition 1, there is a solution $\varphi^{(\kappa,\phi)} \in \Pi$ s.t. $\varphi = \varphi^{(\kappa,\phi)}$.

**Step 1. Uniqueness if the solution in the oriented interval $(a_1, a_2)$.**

Let $(a_1, a_2) \in \kappa$, and denote by $u = u_{(a_1,a_2)} \in \phi$.

By equation (6), the level curve $C_a$ of the point $a \in (a_1, a_2)$ is of dimension 1, since $C_a$ is a cone, then it is a line.

Also, notice that if $x \succ y, x, y \in (a_1, a_2)$, then $C_y \subset \frac{u(x)}{u(y)}C_x$. Therefore, the level curves of the points in $(a_1, a_2)$ are parallel lines with apex on it solution vector.

Let $z \in [a_1, a_2]$ such that $C_a = \{a + tz \mid t \geq 0\}$. If $z \in (a_1, a_2)$, then $C_z = ray_z(z)$ which clearly does not coincide with equation (6): $C_z = \frac{1}{u(z)}\{u(y)y \mid y \geq z\}$. If $z = a_2$, we obtain the solution described in (1). Similarly, if $z = a_1$, we obtain the solution described in (1) except for the orientation of $(a_1, a_2)$. Therefore $\varphi \in \Delta$

**Step 2. Equivalence with $LC$**

Since $LC$ is the dual property of $UC$, we only need to check the dual of $\varphi$ is a solution in $\Delta$.

Let $(a_1, a_2) \in \kappa$ an oriented interval. By step 1, $C_a = \{a + ta_2 \mid t \geq 0\}$ for each $a \in (a_1, a_2)$. By continuity, $\{a_1 + ta_2 \mid t \geq 0\} \subset C_{a_1}$, since $ray_{a_1}(a_1) \subset C_{a_1}$ and the level curves are convex closed sets, then $cone_{a_1}([a_1, a_2]) \subseteq C_{a_1}$. Therefore, the representation $h$ of $\varphi$ in the oriented interval $(a_1, a_2)$ is given by: $h(x) = a_2$ if $x \in (a_1, a_2)$, $a_2 \in h(a_2)$, and $[a_1, a_2] \in h(a_1)$.

According to proposition 4, the dual solution of $\varphi^*$, has a representation $h^{-1}$. Then $h^{-1}$ in the interval $(a_1, a_2)$ is given by: $h^{-1}(x) = a_1$ if $x \in (a_1, a_2)$, $a_1 \in h^{-1}(a_1)$, and $[a_1, a_2] \in h^{-1}(a_2)$, which is the representation for the oriented interval $(a_2, a_1)$. Therefore $\varphi^*$ is a solution in $\Delta$.

The remaining part was given by Moulin[2000], lemma 4.

■

# Appendix 2: UC and SI

The most representative upper composition solutions are the equal sacrifice solutions (ESS), these solutions in the two agent case are characterized by upper composition, strict-monotonicity (in the amount to distribute), strict-ranking (over the claim of the agents), symmetry and a consistent extension to more agents. Along with scale invariance, the level curves of the solutions can be analyzed by two representative cases. The first one, where the level curves of these solutions are generated by the path of a particular power function, that is the level curves of certain point is only the multiple of the path that crosses that point. In the second case, the level curves of the points (0,1) and (1,0) are the sets between a very specific type of path and its corresponding Cartesian axis, the level curves of the other points are simply the path whose multiple belongs to the border of one of these sets, see example 1 below. Naumova[2002] offers the non-symmetric extension to the ESS, but their representative level curves do not differ too much (except by the symmetry) from the ESS solution. As we will see later, the solutions we will construct are closely related to these solutions.

The key property to analyze the $UC$ solutions is given on its level curves, any two level curves are either mutually disjoint or one contains the other. This property allow us to give an ordering to $\mathbb{R}^N$, that is $x$ succeeds $y$ if the level curve of $x$ contains the level curve of $y$. On the other hand, this lemma tell us the solution are *level curve independent* not *path independent*, that is our solution will be generated by a set of independent (disjoint) level curves. This property breaks off the classic way to represent our solutions by a set of independent paths, which is sometimes misused because there are UC solutions whose paths are not necessarily independent, for example the second case of the ESS solutions, where two paths are not necessarily disjoint and neither one contained in the other.

**Lemma 3** *The solution $\varphi$ satisfies UC if and only if for any two level curves $C_p$ and $C_q$ :*

$$C_p \cap C_q \neq \emptyset \text{ implies } C_p \subset C_q \text{ and } p_N \geq q_N, \text{ or } C_p \supset C_q \text{ and } p_N \leq q_N. \tag{7}$$

**Proof.** $\Rightarrow$)

Suppose $\varphi$ satisfies $UC$, let $C_p$, $C_q$ and $x \in C_p \cap C_q$. We can assume without loss of generality that $q_N > p_N$ because any two level curves with the same amount to distribute are disjoint (Lemma 1(b)). Then by UC

$$p = \varphi[p_N; x] = \varphi[p_N; \varphi[q_N; x]] = \varphi[p_N; q],$$

thus $q \in C_p$. On the other hand, let $y \in C_q$, then by $UC$

$$p = \varphi[p_N; q] = \varphi[p_N; \varphi[q_N; y]] = \varphi[p_N; y],$$

therefore $y \in C_p$.

$\Leftarrow$)

Let $\varphi$ the solution induced by the level curves satisfying property (7). Take $c' > c$, $b \in \mathbb{R}^N_+$ such that $b_N \geq c'$, and $p' = \varphi[c'; b]$ and $p = \varphi[c; b]$, then $C_{p'} \subset C_p$ because $b$ is a common element of the level curves. By individual rationality, $p' \in C_{p'}$ so $p' \in C_p$ holds.

■

Another nice property of the upper composition solutions is that they are invariant with respect to a change of coordinates, that is, given a $UC$ solution $\varphi$ and $a_1, a_2$ vectors in the simplex $\mathscr{S}^2$, the composition of $\varphi$ with the linear transformation that maps the canonical vectors $e_1 = (0, 1)$ and $e_2 = (1, 0)$ to $a_1$ and $a_2$ is also a $UC$ solution (restricted to the closed convex cone $ray_0([a_1, a_2])$), this result is straightforward from lemma 2. In observance of this, given a partition of the simplex $\mathscr{S}^2$ by oriented open intervals and singletons, we can attach several $UC$ solutions to this partition by attaching a $UC$ solution to every interval $(a_1, a_2)$ (in the sense discussed above), and to each vector in $ray_a(a)$ we assign the solution $a$. This partition of the simplex was borrowed from Moulin[2000], and will be called an ordered covering of $\mathscr{S}^2$.

Let $(\kappa, \phi)$ a pair where $\kappa$ is an ordered covering of $\mathscr{S}^2$ and $\phi = \{u_{(a_1,a_2)}\}_{(a_1,a_2)\in\kappa}$ is a collection of functions such that each $u_{(a_1,a_2)} : [a_1, a_2] \to [0, \infty]$ is continuous, strictly-increasing, and $u_{(a_1,a_2)}(a_2) = \infty$. We denote by $\Xi$ the set of pairs $(\kappa, \phi)$ thus constructed.

Each $(\kappa, \phi) \in \Xi$ will help us to define a solution by means of its level curves, as we said before, every oriented open interval of $\kappa$ can be associated to a local $UC$ solution, and every function $u_{(a_1,a_2)}$ gives the weight in the path that will generate the local $UC$ solution (it will be called generator path). That is, for each vector $p \in (a_1, a_2)$, $u_{(a_1,a_2)}(p) \cdot p$ will be an element in the generator path of the local $UC$ solution in $(a_1, a_2)$. This generator path, which is given by the image of the function $v_1(p) = u_{(a_1,a_2)}(p) \cdot p$, is simply the reference to homothetically construct the level curves. Formally, the images of the functions in $V = \{v_k \mid v_k(p) = ku_{(a_1,a_2)}(p) \cdot p, k \in (0, \infty), p \in [a_1, a_2]\}$ form a partition of $ray_0(a_1, a_2)$, thus the level curve of $p \in (a_1, a_2)$ is defined as the restricted image of the unique function $v_k$ that crossed $p$, where restricted image means we are considering only those points $q$ such that $v_k(q)_N \geq p_N$. The vectors of claims in $ray_0(a_1, a_2)$ which do not belong to any level curve thus defined (if there is one), will belong to the level curve of $a_1$.

In the sense described above, there are only two representative classes of solutions (again, thinking only in the cone with apex in 0 and slopes in $(a_1, a_2)$), and these are closely related with the cases analyzed in the ESS solutions. The first one, where each level curve is generated by a continuous path (i.e. the restricted image of $v_k$ for some $k$) that goes through the origin, thus $u_{(a_1,a_2)}(a_1) = 0$ and the level curves of neither $a_1$ nor $a_2$ have points in $ray_0(a_1, a_2)$. The second one, where the level curves in $ray_0(a_1, a_2)$ are generated by the border of the level curve of $a_1$ (i.e. by the image of the unique $v_k$ that passes through $a_1$), thus $u_{(a_1,a_2)}(a_1) > 0$ in this case.

A similar way to define these solutions is by means of its paths, the path to a point $p$ is the union of the segment $[0, v_k(a_1)]$ and the image of $v_k$ restricted to the triangle $[0, a_1, a_2]$, where $v_k$ is, as defined before, the unique function in $V$ that goes through $p$. If $u_{(a_1,a_2)}(a_1) = 0$ then $v_k(a_1) = 0$, so the path to $p$ is only the complement of its level curve in the image of $v_k$. If $u_{(a_1,a_2)}(a_1) > 0$ then $v_k(a_1) \neq 0$, so its path is the union of the segment $[0, v_k(a_1)]$ and the difference of the image of $v_k$ and its level curve, see figure 1.

Formally, the associated solution $\varphi^{(\kappa,\phi)}$ to $(\kappa, \phi) \in \Xi$ is defined as:

$$\varphi^{(\kappa,\phi)}[c; b] = \begin{cases} c\tilde{b} & \text{if } \{\tilde{b}\} \in \kappa \\ cp & \text{if } \tilde{b} \in (a_1, a_2) \in \kappa \text{ and } p \in (a_1, a_2) \text{ uniquelly solves } u_{(a_1,a_2)}(p) = \frac{c \cdot u_{(a_1,a_2)}(\tilde{b})}{b_N}, \\ ca_1 & \text{if } \tilde{b} \in (a_1, a_2) \in \kappa \text{ and there is not } p \in (a_1, a_2) \text{ that solves} \\ & u_{(a_1,a_2)}(p) = \frac{c \cdot u_{(a_1,a_2)}(\tilde{b})}{b_N} \end{cases} \tag{8}$$

We denote by $\Pi$ the set of solutions thus constructed.

It is important to notice we are considering solutions that may not satisfy individual rationality, as in Naumova[2002]. In order to get rational solutions, we need to check the generator path (i.e. the image of some $v_k$) does not grow too slow. Formally, we must check $u_{(a_1,a_2)}(x) \cdot x_1 \leq u_{(a_1,a_2)}(y) \cdot y_1$ and $u_{(a_1,a_2)}(x) \cdot (1 - x_1) \leq u_{(a_1,a_2)}(y) \cdot (1 - y_1)$ for every $x \prec_{(a_1,a_2)} y$ (i.e. $x$ precedes $y$ with the order given by the oriented interval $(a_1, a_2)$). Also, notice one of these equations will automatically hold because the dominium of $u$ is decreasing or increasing, so we only need to worry by one of these inequalities.

**Proposition 5** *Any solution $\varphi^{(\kappa,\phi)} \in \Pi$ satisfies upper-composition and scale-invariance.*

*Conversely, any solution for two agents that satisfies upper-composition and scale-invariance is a solution in $\Pi$.*

*Moreover, if $\varphi^{(\kappa_1,\phi_1)} = \varphi^{(\kappa_2,\phi_2)}$ then $\kappa_1 = \kappa_2$, and the functions in $\phi_1$ and $\phi_2$ are affine, that is, there exists positive constants $\{k^{(a_1,a_2)}\}_{(a_1,a_2)\in\kappa_1}$ such that $u_{(a_1,a_2)} = k^{(a_1,a_2)}\tilde{u}_{(a_1,a_2)}$, for each $u_{(a_1,a_2)} \in \phi_1, \tilde{u}_{(a_1,a_2)} \in \phi_2$.*

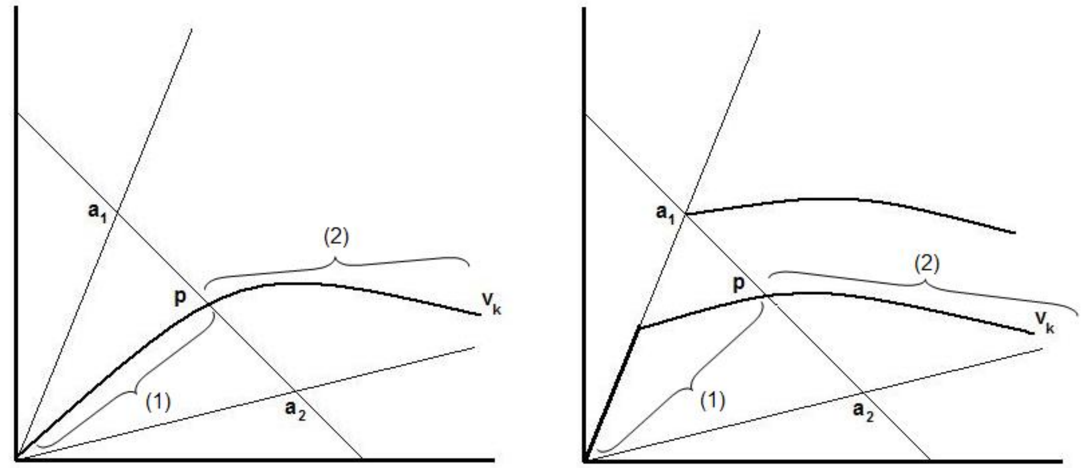


Figure 4: Pictures show the cases $u_{(a_1,a_2)}(a_1) = 0$ and $u_{(a_1,a_2)}(a_1) > 0$ respectively. In both pictures, (1) represents the path to $p$, and (2) its level curve.

**Example 2 (The equal sacrifice solutions)** *The closest result to the last proposition was given by Young[1988]. Young proved that any symmetric solution that satisfies upper-composition, scale-invariance, strictly resource monotonic, strictly-ranking, and a consistent extension to more agents is computed by solving* $\varphi[c;b] = p$, $w(b_1) - w(p_1) = w(b_2) - w(p_2)$ *where* $w(x) = x^k$ *or* $w(x) = -x^{-k}$ *for* $k > 0$.

*First we will assume* $w(x) = x^k$, $k > 0$.

*We can easily check any two points in the set* $\{(x_1, (x_1^k + \lambda^k)^{\frac{1}{k}}) \mid x_1 \geq 0\}$ *belongs to the same level curve. This set is the so called generator path of this solution.*

*Also, this path belongs to the first half of the positive orthant,* $\{b \in \mathbb{R}^2_+ \mid b_2 \geq b_1\}$. *Moreover, for every* $p$ *in* $(0, \frac{1}{2})$, *every line* $ray_0(p)$ *intersects only once the path. Therefore the corresponding ordered cover of the simplex* $\mathscr{S}^2$ *is given by* $\{\{0\}, (0, \frac{1}{2}), \{\frac{1}{2}\}, (\frac{1}{2}, 1), \{1\}\}$.

*Finally, the corresponding weight function is given by* $u_{(0,\frac{1}{2})}(z) = \frac{1}{((1-z)^k - z^k)^{\frac{1}{k}}}$ *for every* $z \in (0, \frac{1}{2})$, *and the symmetric function for* $z \in (\frac{1}{2}, 1)$.

*For the case* $w(x) = -x^{-k}$, $k > 0$. *We can similarly check that the same partition of the simplex works in this case. The corresponding weigth function is given by* $u_{(\frac{1}{2},1)}(z) = ((1-z)^{-k} - z^{-k})^{-\frac{1}{k}}$.

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