Truthful Intermediation with Monetary Punishment

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Abstract

A mechanism chooses an allocation of the resource to intermediaries based on their reported ability to transmit it. We discover and describe the set of incentive compatible mechanisms when a monetary punishment to intermediaries who misreport their ability is possible. This class depends on the punishment function and the probability of punishment. It expands previous characterizations of incentive compatible mechanisms when punishment was no available. Furthermore, when the planner has the ability to select the punishment function, we provide the minimal punishment necessary to achieve incentive compatibility and the corresponding class of first-best mechanism. For any punishment, the optimal mechanism for the planner are provided.

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1 Introduction

Consider a planner interested in transmitting a divisible resource to agents (such as money). Although the planner is not directly linked to the agents, it can do so via a group of intermediaries. Intermediaries differ in their ability to transmit the resource. This ability is represented by the total amount of the resource that an intermediary sends to the agents per unit of resource received, as well as by the proportions in which every agent receives a resource relative to another from a given intermediary. Thus, for instance, two intermediaries might be able to reach different agents, and even when they reach the same group of agents, they may transmit different amounts to the agents.

We study the case where the intermediaries' abilities are private information.[1] Therefore, the planner uses a direct mechanism, where intermediaries report their abilities, which are then used to determine the actual transmission rate to the agents, as well as the distribution of the resource among the different intermediaries. In such cases, intermediaries might be able to game the planner by misrepresenting their ability to transfer the resource to agents. Therefore, incentive compatibility of the mechanism, in our case strategy-proofness, is a desirable requirement.

Strategy-proofness is a very robust property that prevents intermediary to misrepresent their ability regardless of the reports of other intermediaries. Restricting to strategy-proof mechanisms might come with a high cost to the planner.[2] In some settings, the planner might be able to alleviate such a cost by enforcing truthful reporting by other means. Indeed, consider the case of auditing, where the planner has the ability to audit the intermediaries in the game (perhaps with some probability) and assign a punishment (expressed in monetary terms) for the intermediaries who are found misreporting their ability. For a given set of abilities, there is always a large enough punishment such that the intermediaries should not feel compelled to misrepresent their preferences. Indeed, any punishment such that the expected punishment is larger than the expected rewards gained by misrepresenting their preferences satisfies that. Thus a natural generalization of strategy-proofness extends the class of mechanisms that are strategy-proof when such a punishment are available for the planner.

The paper introduces a generalization of strategy-proofness when the planner has the ability to monitor and punish the intermediaries for misrepresenting their preferences. Indeed,

[1]In our companion paper, Han and Juarez[15], we study the case where the abilities of the intermediaries are public information. The planner solicits bids from intermediaries to the use of their links and applies this information to select which intermediaries to contract for the transmission of the resource. The main result of Han and Juarez[15] is the necessary and sufficient conditions for the existence of a free intermediation equilibrium, where there is a perfect transmission of the resource to the agents as if there is no intermediation.

[2]This cost is typically measured in efficiency terms, but can also be measured in equity or other terms.

in the domain of quasilinear preferences where money is available, we consider monetary punishments that will depend on an arbitrary function $h(\alpha_i, \beta_i)$, where $h(\alpha_i, \beta_i)$ is the punishment paid by the intermediary represented by the difference between his true ability α_i and reported ability β_i. A mechanism is h-strategy-proof if there is no incentive for any intermediary to misreport under the punishments h. When the planner does not have the ability to monitor the intermediaries, $h = 0$, our property boils down to the traditional strategy-proofness. On the other hand, when $h > 0$ is large, the intermediaries will be punished a large amount $h(\alpha_i, \beta_i)$ and the amount of misreporting will be substantially reduced. This allow us to capture all mechanisms, when $h \to \infty$.[3]

The main contributions of the paper are three-fold. First, it introduces a notion of strategy-proofness with monitoring and punishment. Second, it introduces a new model of resource transmission and intermediation in networks when the abilities of intermediaries are incomplete information and characterizes the entire class of strategy-proof mechanisms when monitoring and punishment are available to the planner. Furthermore, this paper studies the minimal punishment function for a mechanism to be strategy-proof, and the punishment function to achieve first best efficient allocation. On the other hand, the optimal mechanism for the planner is discovered given an arbitrary punishment function.

1.1 Illustrative Example

To illustrate our mechanisms and main results, consider the example of a planner who is connected to three intermediaries, who themselves are connected to two agents (see Figure 1). The planner is interested in transmitting I units of a resource to the agents, but can only do so via the intermediaries. Intermediaries have different quality of intermediation, represented by the proportion in which they transmit their share to the agents for every unit of resource transmitted. In this case, the abilities of intermediaries are $\alpha_1 = (0.7, 0.4)$, $\alpha_2 = (0.6, 0.6)$ and $\alpha_3 = (0.5, 0.8)$, respectively.

In the absence of information, the planner will ask intermediaries to report their abilities to transmit the resource and determine (a) the amount of resource allocated to every intermediary for transmission to agents and (b) the sharing rate charged for every intermediary to transmit at every link based on the information of report.[4] The intermediary's profit is the difference between his true abilities to transmit the resource and his charged sharing rates multiplied by the amount of resource allocated to him.

For instance, consider the traditional first price auction. When intermediaries report

[3]This is true for punishment functions h such that $h(\alpha_i, \alpha_i) = 0$ for all α_i and $h(\alpha_i, \beta_i) = 0$ for $\beta_i \neq \alpha_i$.

[4]For instance, we can imagine the case where the planner might use a second price auction.

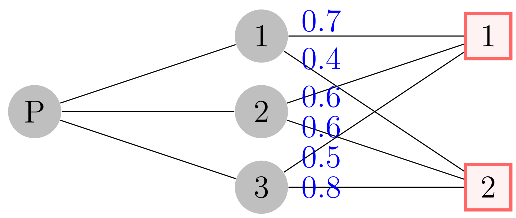

Figure 1: A network with three intermediaries and two agents.

abilities $(\beta_1, \beta_2, \beta_3)$, the planner selects the intermediary with highest reported aggregate ability to transmit all the resource with charged sharing rates $s_i(\beta) = \beta_i$. This mechanism is not strategy-proof. Indeed, when intermediaries report their true abilities $(0.7, 0.4)$, $(0.6, 0.6)$ and $(0.5, 0.8)$, intermediary 3 with aggregate ability 1.3 is selected to transmit all the resource and the charged sharing rates equals his ability $(0.5, 0.8)$. Thus, intermediary 3's profit equals 0. This is not strategy-proof, because he can decrease his report to $(0.5, 0.71)$, where he will get a positive profit equal to $[(0.5, 0.8) - (0.5, 0.71)] * (I, I)^T = 0.09I$.

Now, suppose the planner is able to audit the intermediaries. Suppose that the planner punishes the intermediaries based on the deviation from their true reports with punishment function $h(\alpha_i, \beta_i) = \sum_{m=1}^{M} |\beta_i^m - \alpha_i^m|$. In such a mechanism, intermediaries have no incentive to lie about their reports. Indeed, at the profile α above, when intermediary 3 reports $(0.5, 0.71)$ and planner finds that his true ability is $(0.5, 0.8)$, the expected punishment on intermediary 3 is $(|0.5 - 0.5| + |0.8 - 0.71|) = 0.09$. The expected payoff for intermediary 3 is $(0.09 - 0.09)I = 0$, the same as reporting $\beta_3 = \alpha_3$. Thus, there is no incentive for intermediary 3 to misreport, and first price auction is h-strategy-proof. The set of h-strategy-proof mechanisms expands the set of strategy-proof mechanisms.

1.2 Overview of the Results

We introduce the resource transmission problem of the planner in Section 2 and strategy-proof mechanisms in Section 3. We provide conditions for a mechanism to be h-strategy-proof for arbitrary punishment function h, when the punishment function is differentiable[5] at any truthfully report point (Lemma 1). The class of 0-strategy-proof mechanisms coincide with the class of strategy-proof mechanisms (Theorem 1). Thus, the class of h-strategy-proof mechanisms is largely depending on h, and the comparative static analysis studied in Proposition 1. Furthermore, we study the minimal punishment function for any mechanism in Section 4. Proposition 3 provides the necessary and sufficient condition for minimal

[5]More general punishment functions are discussed in the Appendix.

punishment function, Proposition 2 and Corollary 2 shows the convexity and existence of minimal punishment function. We characterize the first-best efficient allocation in Section 5. Theorem 2 shows that there exists no symmetric, SP, budget balance and first-best efficient mechanism. The minimal punishment function to achieve first-best efficiency is provided. Finally, Theorem 3 discovers the optimal mechanism of any punishment function for the planner.

1.3 Applications

An application of our game theoretical model is the transmission of advertising money in companies. A company looking to promote their product can use different media (the intermediaries) to reach the advertising target of their product; such intermediaries include TV channels, radio stations, Internet websites, and newspapers. The quality of the connections is relevant because, within the media, there are different channels that target to specific demographics of agents and may influence the planner's objective differently. For instance, two local TV stations based in the same city may be connected to all agents in the city, but the audience may be more biased based on demographics or political preferences —e.g. Fox News and CNN reach the same audience, but they target their programming to attract more conservative or liberal viewers, respectively. Nowadays, the printed version of newspapers are read heavily by older people instead of younger people, and the proportions of older to younger readers are typically available to potential purchasers of advertisements. Therefore, it is in the interest of the planner to choose the media channel that best aligns with his preferences.

Alternatively, consider the case of government contracting. For instance, the allocation of government's money to people in need via charities. The government may decide to send the money via charities that will charge an indirect cost for the use of their services. The connections of the charities, as well as their quality, are exogenous information that the planner cannot control, and they are typically taken into account when making a decision on how to allocate the resources. For instance, charities heavily funded by the government include UNICEF or the Red Cross. While both charities overlap in some of the agents that they serve (e.g. children in need), they also have large difference in their recipients.[6] The quality of the connections of the charities is also important when picking a charity. For instance, inefficiencies happen often in charities and universities, where every dollar spent

[6]Thus, for instance, the Federal Emergency Management Agency may be more interested in allocating money to the Red Cross, which distribute a large percentage of their resources to helping domestic citizens affected by disasters, as opposed to UNICEF which helps children around the world.

is often decreased due to indirect cost, which serves to pay for administration.[7] Thus, the planner should care about how their money is distributed to the agents and aligned with its preferences. Our model looks at the case of complete information, which is also the case in this example, as the priorities and activities of the charities are typically reported by them in advance.[8] As such, the planner can make an informed decision on how its money will transmit by the charities chosen.

Finally, the problem has applications to network flow problems. For instance, when there is groundwater that must be distributed to agents via private canals (intermediaries). The planner can decide how to route the water to the canals, but once the water reaches the canal it is distributed to the agents connected to these canals in some fixed proportions that may vary between canals. Conveyance losses are typical in models and may depend on how far the agents are from the source (Jandoc, Juarez, and Roumasset[17] study the optimal allocation of water networks in the presence of these losses). The owners of the canals may charge the planner for the use of their canals, and therefore the planner should consider the trade-offs between allocating goods to cheap canals as opposed to more efficient but expensive canals. The paper studies the case of exogenous quality of the intermediaries.

1.4 Related Literature

The literature on strategy-proofness when money is available has been widely explored. Indeed, the traditional VCG mechanisms in Vickrey[29], Clarke[11], Groves[14] are strategy-proof and efficient. However, one limitation of VCG mechanisms is that they are not budget balance, which does not apply to our model.

The large literature on social choice has been concerned with non-manipulable mechanisms, dating back from Arrow[1] and Gibbard[13], see Barberà[2] for an introduction to strategy-proof social choice functions. Such studies include the case of strategy-proof social choice functions in classical exchange economies (Barberà and Jackson[5]), matching with contracts (Hatfield and Kojima[16]), house allocation with prices (Miyagawa[21]), cost sharing (Moulin and Shenker[25], Moulin[24], Sprumont[27]), preference aggregation (Bossert

[7] This factor in the quality of the charities is so important that all charities in the US are required by law to report the total percentage amount spent in their causes, as opposed to administrative costs. For instance, the current indirect costs for the Red Cross and UNICEF are 9.7% and 4.74%, respectively. Multiple online websites exist that rank charities based on the indirect costs, among other metrics.

[8] The Red Cross publishes at the end of each year 'its activities in the field and at the headquarters during the coming year,' which allow donors to make an informed decision on where the money will go. Earmarking is typically not allowed in such big charities, as 'experience shows that the more restrictive the earmarking policy (whereby donors require that their funds be allocated to a particular region, country, program, project or goods), the more limited the ICRC's operational flexibility, to the detriment of the people that the ICRC is trying to help.'

and Sprumont[8]), social choice (Barberà, Dutta and Sen[4]).

Barberà, Berga and Moreno[3] study group strategy-proof mechanisms, in a general setting that includes the provision of private good and matchings such as house allocation. Moulin[23], Juarez[18] study group strategy-proof in cost sharing problems.

There is also strategy-proof mechanisms for restricted domain of preferences, such as the class of single-peaked preferences (Moulin[22]). Our focus in the paper is in the entire domain of quasilinear preferences, where the class of strategy-proof mechanisms that satisfy desirable conditions tends to be small. Hence our work expands the class of strategy-proof mechanisms that the planner can use.

There is also a more recent literature dealing with various relaxations and strengthening of strategy-proof notions. There are approximately strategy-proof mechanisms in voting (Birrell and Pass[7]), matching (Pathak and Sönmez[26]), and more generally, Carrol[9] finds that local strategy-proof with single-crossing ordinal preferences implies full strategy-proof. Obviously strategy-proof mechanisms in Li[19] refine the strategy-proof mechanisms by requiring the strategy to be obviously dominant. Pathak and Sönmez[26] develops a rigorous methodology to compare mechanisms based on their vulnerability to manipulation. Unlike this literature on strategy-proofness, our notion of manipulation depends on the punishment function h. This allows for a weaker notion of manipulation that expands the class of strategy-proof mechanisms that a planner can use, hence providing more flexibility when selecting mechanisms. Indeed, our more general version of strategy-proofness can be easily adapted to these settings.

In contrast with the literature on strategy-proofness, our mechanisms are specifically applied to a novel problem of resource transmission in a network. On this line of work, there is only one closely related, our companion paper, Han and Juarez[15], which study the strategic behavior of intermediaries in a more general resource transmission game. Unlike that paper, our model with incomplete information does not restrict the type of mechanisms to a first-price type of mechanism, instead, it characterizes a large class of mechanisms in a more specific resource transmission game in a network.

Townsend[28] first studies costly verification in a principal-agent model with a risk-averse agent. There is a growing interest in mechanism design problem with state verification, Ben-Porath et al.[6] study the principal allocating an indivisible good among agents with an ability to verify agents' type costly, and they don't allow transfer payments. They study the principal's trade-off between allocating the good more efficiently and incurring the cost of verification, and find the optimal mechanism to be a favored agent mechanism, where a pre-determined agent receives the good, unless another agent reports higher than threshold and agent with highest bid will get the good, if his report is verified to be true. Erlanson and Kleiner[12] study similar problem of costly verification in collective choice problem. Li[20] studies costly verification with limited punishment. On the other hand, Carroll and Egorov[10] studies the mechanism of minimal verification to elicit multidimensional information fully by using a randomized verification strategy and allowing severe punishment. However, we study the mechanism design problem with planner allocating divisible good with report of multidimensional information, allowing exogenous probabilistic verification, and study the minimal punishment as the transfer payments to induce the strategy-proof for a mechanism and also to achieve the first-best allocation.

2 The Model

A planner is endowed with I[9] unit of divisible good. He is interested in transmitting the resource to a group of agents $\mathcal{M} = \{1, \ldots, M\}$, but he can only do so via a set of intermediaries $\mathcal{N} = \{1, \ldots, N\}$. Every intermediary $i \in \mathcal{N}$ has a **quality of intermediation** (or simply refer to as quality) $\alpha_i = (\alpha_i^1, \ldots, \alpha_i^M) \in \mathbb{R}_+^M$ that represents the proportions in which intermediary i can transmit the good to agents. That is, if x_i units are assigned for transmission by intermediary i, then $x_i \alpha_i$ units are received by the agents. Let $\alpha = (\alpha_1, \ldots, \alpha_N)$ be the **quality of intermediaries**. Define $\alpha_i(\mathcal{M}) = \sum_{m \in \mathcal{M}} \alpha_i^m$ to be the aggregate intermediation quality of intermediary i and $\alpha_{-i}(\mathcal{M}) = (\alpha_j(\mathcal{M}))_{j \in \mathcal{N} \setminus i}$ the abilities of the intermediaries in $\mathcal{N} \setminus i$.

We assume that information is asymmetric, the quality of intermediation is private information. Intermediaries know their own quality but do not know others'. Furthermore, the quality of every intermediary is unknown to the planner. Therefore, a mechanism that split the resource to intermediaries for further transmission to the agents is needed. We focus on mechanisms where the planner makes an allocation of the resource to intermediaries based on their reports of quality of intermediation. Assume intermediary i reports his quality $\beta_i \in \mathbb{R}_+^M$, $\beta = (\beta_1, \ldots, \beta_N)$ and $\sum_{m=1}^M \beta_i^m = \beta_i(\mathcal{M})$. The planner also charges the sharing rates of the intermediaries to agents.[10]

Definition 1 (Mechanism)
A mechanism $\phi = (x(\cdot), s(\cdot))$ is a pair of functions $(x(\cdot), s(\cdot))$ such that

i. $x : \mathbb{R}_+^{NM} \mapsto \mathbb{R}_+^N$ allocates the share of a resource to every intermediary based on the

[9] $I = 1$ in most of the discussion.

[10] Thus for instance, the planner might decide to use certain intermediaries to transmit resource to an agent while other intermediaries to transmit resource to a different agent. Since every agent is allowed a charge, this model extends the case where intermediaries are allowed to charge a single price for their links (Han and Juarez[15]) or the case where intermediaries are allowed a proportional constraint.

reported quality of intermediation β. That is, for a quality of intermediation β and $x(\beta) = (x_1(\beta), \ldots, x_N(\beta))$, the amount $x_i(\beta) \in \mathbb{R}_+$ represents the resource allocated to intermediary i.[11]

ii. *$s : \mathbb{R}_+^{NM} \mapsto \mathbb{R}^{NM}$ represents the rates at which the planner transmits resource through the intermediaries. Thus, for a quality of intermediation β and $s(\beta) = (s_1(\beta), \ldots, s_N(\beta))$, the vector $s_i(\beta) \in \mathbb{R}_+^M$ is the sharing rates charged by the planner to transmit via intermediary i.*[12]

For a mechanism $\phi(\cdot) = (x(\cdot), s(\cdot))$ and reported quality of intermediaries β, the final allocation to agents is $\sum_{i=1}^N s_i(\beta)x_i(\beta) \in \mathbb{R}_+^M$.

Intermediaries gain by the surplus of resource transmitted between the charged rates $s_i(\beta)$ and their quality of intermediation α_i. Thus, intermediary i receives a profit $(\alpha_i - s_i(\beta))^T \mathbf{1} x_i(\beta)$ when the reported quality of intermediaries is β. Note that $\mathbf{1} = (1, \ldots, 1)_{M\times 1}$.

The generality of a mechanism allows for a variety of properties not covered in previous literature. For instance, we do not assume that the charged rates of any intermediary i is always below the reported sharing rates, $s_i(\beta) \leq \beta_i$ for any i. Our analysis allows for some intermediaries to be charged to only transmit resource to some agents.

Example 1

The mechanism $\phi = (x(\cdot), s(\cdot))$ is the

i. ***Equally-Sharing (ES)***: $x_i(\beta) = \frac{1}{N}$, $s_i(\beta) = \beta_i$, $\forall i$.

ii. ***Equally-Sharing rates (ESR)***: $x_i(\beta) = \frac{\beta_i(\mathcal{M})}{\sum_{i=1}^N \beta_i(\mathcal{M})}$, $s_i^m(\beta) = \min_{n\in\mathcal{N}} \beta_n^m$, $\forall i$.

iii. ***Second price mechanism (SPM)***: *$x(\beta)$ satisfies: there exists i, s.t. $\beta_i(\mathcal{M}) = \max_{n\in\mathcal{N}} \beta_n(\mathcal{M})$ and $x_i(\beta) = 1$, $\forall j \neq i$, $x_j(\beta) = 0$. $s(\beta)$ satisfies: $s_i(\beta) = \max_{n\neq i} \beta_n(\mathcal{M})$ and $s_j(\beta) = \beta_j$, $\forall j \neq i$.*

iv. ***First price mechanism (FPM)***: *$x(\beta)$ satisfies: there exists i, s.t. $\beta_i(\mathcal{M}) = \max_{n\in\mathcal{N}} \beta_n(\mathcal{M})$ and $x_i(\beta) = 1$, $\forall j \neq i$, $x_j(\beta) = 0$. $s(\beta)$ satisfies: $s_i(\beta) = \beta_i$.*

ES always allocates the resource equally through each intermediary. The sharing rates $s_i(\beta)$ equal to the reported quality of intermediation β_i.

The ESR mechanism always allocates resource with the same sharing rates $s_i(\beta)$ through all intermediaries with the share equal to the ratio of intermediary i's aggregate intermediation quality $\beta_i(\mathcal{M})$ over the aggregate intermediation quality of all intermediaries $\sum_{i=1}^N \beta_i(\mathcal{M})$.

[11] $x(\beta)$ is fully differentiable except some points with measurement 0.

[12] It is equivalent with intermediaries charging cost proportional to the amount allocated to intermediary i, rather than a fixed cost discussed in Han and Juarez[15].

The second price mechanism always allocates the resource through intermediary with highest sum of intermediation quality and chooses the sharing rates $s_i(\beta)$ equal to second highest sum of quality.

A mechanism is budget balance when all the resource is allocated through intermediaries to the agents.

Definition 2 (Budget Balance)
The mechanism $\phi = (x(\beta), s(\beta))$ is budget balance if the resource allocated to the intermediaries sums up to the total resource, which means $\sum_{i=1}^{N} x_i(\beta) = 1$, for any β.

All the mechanisms discussed above are budget balance.

Definition 3 (Symmetric)
The mechanism $\phi = (x(\beta), s(\beta))$ is symmetric if the resource allocated to the intermediaries $x_i(\beta)$ and the sharing-rates $s_i(\beta)$ satisfy: $x_i(\beta) = x_j(\beta')$, $s_i(\beta) = s_j(\beta')$ for any $\beta_i = \beta'_j$, $\beta_j = \beta'_i$ and $\beta_m = \beta'_m$ for any $m \neq i, j$.

All the mechanisms discussed above are symmetric.

3 h-Strategy-Proof Mechanisms

A punishment function $h : \mathbb{R}_+^{2M} \mapsto \mathbb{R}$, $h(a, b)$ can be interpreted as the punishment of an intermediary to report $b \in R_+^M$, if when the true quality of intermediation is $a \in R_+^M$. Assume there is no punishment for truthful report, $h(a, b) = 0$ if $b = a$.[13]

Definition 4 (h-Strategy-Proof)
The mechanism $\phi = (x(\cdot), s(\cdot))$ is h-strategy-proof (h-SP) if for any intermediary i and for any quality of intermediation α_i and β_i, there is

$$(\alpha_i - s_i(\alpha_i, \beta_{-i}))^T \mathbf{1} x_i(\alpha_i, \beta_{-i}) \geq (\alpha_i - s_i(\beta_i, \beta_{-i}))^T \mathbf{1} x_i(\beta_i, \beta_{-i}) - h(\alpha_i, \beta_i) \cdot x_i(\beta_i, \beta_{-i}), \forall \beta_{-i}$$

h-strategy-proof mechanisms can be understood in a way that planner has ability to audit the report β_i and finds out true value α_i. Planner imposes a punishment $h(a, b)$ when report and true value are different. The intermediaries choose to report the intermediation quality to maximize the expected profit.

Lemma 1
A mechanism $\phi = (x(\cdot), s(\cdot))$ is h-SP, then there exists a function $\Phi : \mathbb{R}_+^{NM} \mapsto \mathbb{R}_+^N$ such that:

[13] We do not assume that the punishment is negative, as will be illustrated below.

i. *The aggregate rate that intermediary i charged equals* $\sum_{m=1}^{M} s_i^m(\beta) = \beta_i(\mathcal{M}) - \frac{\Phi_i(\beta)}{x_i(\beta)}$.

ii. *Assume punishment function $h(\alpha, \beta)$ is differentiable at each point $h(\alpha, \alpha)$, where $\alpha = \beta$.*[14] *For each i, m and α_i, $\frac{\partial \Phi_i(\alpha_i, \beta_{-i})}{\partial \alpha_i^m} = (1 + h_{2m}(\alpha_i))x_i(\alpha_i, \beta_{-i})$ with $h_{2m}(\alpha_i) = \lim_{\beta_i^m \to \alpha_i^m} \frac{h(\alpha_i,\alpha_i) - h(\alpha_i,\beta_i)}{\alpha_i^m - \beta_i^m}$, for $\alpha_i^{-m} = \beta_i^{-m}$.*

iii. *If $h_{2m}(\alpha_i) = d$, then $\sum_{m=1}^{M} s_i^m(\beta) = \beta_i(\mathcal{M}) - \frac{\int_0^{\beta_i(\mathcal{M})}(1+d)\hat{x}_i(t,\beta_{-i})dt}{\hat{x}_i(\beta_i(\mathcal{M}),\beta_{-i})}$, with $x_i(\beta) = \hat{x}_i(\beta_i(\mathcal{M}), \beta_{-i})$ and $\frac{\partial \Phi_i(\alpha_i, \beta_{-i})}{\partial \beta_i^m} = (1 + d) \cdot x_i(\alpha_i, \beta_{-i})$.*

From part i, the function

$$\Phi_i(\beta) = \sum_{m=1}^{M} (\beta_i^m - s_i^m(\beta))x_i(\beta)$$

is the profit of intermediary i when he truthfully reports $\beta_i = \alpha_i$ at the profile $\beta = (\beta_i, \beta_{-i})$. From part ii, the profit function Φ_i is monotonic as β_i increases.

These are local conditions of strategy-proof for deviation of α to β. The conditions are not sufficient for h-SP, and global conditions of strategy-proof are needed.

Theorem 1
The following three conditions are equivalent:

i. *A mechanism is 0-SP.*

ii. *For any any punishment function $h(a, b)$, such that the derivative at the truthful report is zero,*[15] *$h_{2m}(\alpha_i) = 0$ for any α_i.*

iii. *There exists a function $\hat{x}_i : \mathbb{R}_+ \times \mathbb{R}_+^{M(N-1)} \mapsto \mathbb{R}_+$ non-decreasing in the first coordinate such that for any β: $x_i(\beta) = \hat{x}_i(\beta_i(\mathcal{M}), \beta_{-i})$ and $\sum_{m=1}^{M} s_i^m(\beta) = \beta_i(\mathcal{M}) - \frac{\int_0^{\beta_i(\mathcal{M})} \hat{x}_i(t,\beta_{-i})dt}{\hat{x}_i(\beta_i(\mathcal{M}),\beta_{-i})}$.*

There are two important consequences of Theorem 1. On one hand, it provides precise conditions for a mechanisms to be 0-SP, the traditional strategy-proof condition discussed in the literature. First, the allocation of the resource to an intermediary should depend on his aggregate intermediation quality instead of specific quality of transmission to agents. Second, the charged share to an intermediary depend on the average allocation over all the qualities $\int_0^{\beta_i(\mathcal{M})} \hat{x}_i(t, \beta_{-i})dt$.

[14]The more general case, when h is not differentiable, will be discussed in Appendix.

[15]This happens, for instance, at the large class of polynomial punishment functions $h(\alpha_i, \beta_i) = \sum_{m=1}^{M} \gamma_m(\alpha_i^m - \beta_i^m)^{k_m}$ for some $k_m > 1$ for all m.

On the other hand, another consequence of Theorem 1 shows that any punishment function h, whose derivative at the truthful report is zero, is ineffective. That is, such a punishment function h will generate exactly the same class as if there is no punishment, the set of mechanisms in 0-SP. There is a large number of functions that meet this condition, including the class of polynomial punishments. Moreover, if the punishment function approaches infinite, any mechanism is strategy-proof.[16]

Corollary 1

Given the payoff function of intermediaries $(\alpha_i - s_i(\beta_i, \beta_{-i}))^T \mathbf{1} x_i(\beta_i, \beta_{-i})$, the charge rate $s_i^m(\beta)$ is perfectly substituted among the links between intermediaries and agents when $\sum_{m=1}^{M} s_i^m(\beta)$ is fixed.

Proposition 1

For any punishment functions h, h', s.t. $h(a,b) \leq h'(a,b)$, $\forall a, b \in \mathbb{R}_+^M$. Then any h-SP mechanism $\phi = (x(\cdot), s(\cdot))$ is h'-SP.

This proposition shows the result of comparative static analysis of h-SP mechanisms. As the punishment function h increases, the set of h-SP mechanisms expands. The result is consistent with intuition that punishment would decrease the incentives of intermediaries to misreport.

4 Minimal Punishment Function

It is often the case that a mechanism to allocates goods and services is given, whereas the designer of the mechanism has the flexibility to design the punishment function h. In this section, we ask the question: What is the class of punishment functions that makes the mechanism h-strategy-proof.

The following proposition shows the linear combination of punishment functions, which make mechanisms h-SP, also guarantees the linear combination of the mechanisms to be h-SP.

Proposition 2 (Convexity of Punishment Function h)

Suppose the mechanism $\phi_1 = (x_1(\cdot), s_1(\cdot))$ and $\phi_2 = (x_2(\cdot), s_2(\cdot))$ satisfy $x_1(\beta) = x_2(\beta)$ for any β, and ϕ_1 is h-SP for punishment function h_1, ϕ_2 is h-SP for punishment function h_2, then $\phi = (x(\cdot), s(\cdot)) = \lambda\phi_1 + (1-\lambda)\phi_2$, for which $x(\beta) = x_1(\beta) = x_2(\beta)$, and $s(\beta) = \lambda s_1(\beta) + (1-\lambda) s_2(\beta)$. Then ϕ is h-SP for punishment function h, with $h(\alpha_i, \beta_i) = \lambda h_1(\alpha_i, \beta_i) + (1-\lambda) h_2(\alpha_i, \beta_i)$ for any α_i, β_i.

[16] Any mechanism is ∞-SP.

In the rest of this section, we are going to discuss the minimal punishment function h for any mechanism ϕ, such that ϕ is h-SP.

For any mechanism $\phi = (x(\cdot), s(\cdot))$, assume the sum of sharing rates charged by planner is $s_i(\beta, \mathcal{M}) = \sum_{m=1}^{M} s_i^m(\beta)$, and the profit function of intermediary i is $v_i : \mathbb{R}_+^{M(N+1)} \mapsto \mathbb{R}_+$. If the intermediation quality of intermediary i is α_i and reports of all intermediaries are β, the profit of intermediary i is $v_i(\alpha_i, \beta) = (\alpha_i(\mathcal{M}) - s_i(\beta, \mathcal{M}))x_i(\beta)$. The profit of intermediary i for truthfully report is $v_i(\alpha_i, \alpha_i, \beta_{-i}) = (\alpha_i(\mathcal{M}) - s_i(\alpha_i, \beta_{-i}, \mathcal{M}))x_i(\alpha_i, \beta_{-i})$. So intermediary i has incentive to report truthfully if $v_i(\alpha_i, \alpha_i, \beta_{-i}) \geq v_i(\alpha_i, \beta) - h(\alpha_i, \beta_i)x_i(\beta)$, for any α, β.

Definition 5 (Minimal Punishment Function)
For any mechanism $\phi = (x(\cdot), s(\cdot))$, $h_i^{\min} : \mathbb{R}_+^{2M} \mapsto \mathbb{R}_+$ is minimal punishment function for intermediary i, if for any punishment function $h(\alpha_i, \beta_i)$, such that ϕ is h-SP for intermediary i with punishment h, then $h(\alpha_i, \beta_i) \geq h_i^{\min}(\alpha_i, \beta_i)$, $\forall \alpha_i, \beta_i$.

From the definition of minimal punishment function, if mechanism ϕ is symmetric, the minimal punishment function is the same for all intermediaries. We focus on symmetric mechanism in the following. For simplicity, assume the minimal punishment function is denoted as $h^{\min} = h_i^{\min}$.

The following result shows that there exists a minimal punishment at every profile and misreport in order to achieve strategy-proofness.

Proposition 3 (Minimal Punishment Function)
Consider the mechanism $\phi = (x(\cdot), s(\cdot))$, the profit of intermediary i is $v_i(\alpha_i, \beta)$, $v(\alpha, \beta) = (v_1(\alpha_1, \beta), \ldots, v_N(\alpha_N, \beta))$ when the true profile is α and reported profile is β. The punishment function h, which guarantees mechanism ϕ to be h-SP, satisfies: $h \geq h^{\min}(\alpha_i, \beta_i) = \max_{\beta_{-i}}[\frac{v_i(\alpha_i,\beta) - v_i(\alpha_i,\alpha_i,\beta_{-i})}{x_i(\beta)}]$, for any $\beta_i, \alpha_i, \beta_{-i}$, such that $x_i(\beta) > 0$.

We can interpret the function $h^{\min}(\alpha_i, \beta_i) = \max_{\beta_{-i}}[\frac{v_i(\alpha_i,\beta) - v_i(\alpha_i,\alpha_i,\beta_{-i})}{x_i(\beta)}]$ as the minimal punishment that intermediary i needs to incur, when the true profile is α_i but he actually reports β_i.

In particular, we note that the minimal punishment for a strategy proof mechanism satisfies $h(\alpha_i, \beta_i) = 0$, for any α_i, β_i. Thus any strategy proof mechanism is h-SP. On the other hand, if a mechanism is not strategy proof, the minimal punishment function for the mechanism has to be non-zero.

Corollary 2 (Properties of Minimal Punishment Function $h^{\min}$)

i. For any symmetric and monotonic mechanism $\phi = (x(\cdot), s(\cdot))$, there exists minimal punishment function $h^{\min}$.

ii. If the mechanism is strategy proof, then $h^{\min}(\alpha_i, \beta_i) = 0$, for any α_i, β_i.

iii. *If the mechanism is not strategy proof, then the minimal punishment function is nonzero. In other words, there exists α_i, β_i, such that $h^{\min}(\alpha_i, \beta_i) > 0$.*

The following example discusses the minimal punishment function for first price mechanism and second price mechanism.

Example 2

Consider the first price mechanism $\phi_F = (x_F, s_F)$, and second price mechanism $\phi_S = (x_S, s_S)$, $x_S = x_F$ satisfies: for any i, $\beta_i(\mathcal{M}) < \max_{n\in\mathcal{N}} \beta_n(\mathcal{M})$, $x_{Si}(\beta) = 0$. For any i, $\beta_i(\mathcal{M}) = \max_{n\in\mathcal{N}} \beta_n(\mathcal{M})$, $x_{Si}(\beta) = \frac{1}{k(\beta)}$, $k(\beta)$ is the number of intermediaries with largest $\beta_i(\mathcal{M})$.

The sharing rates for first price mechanism is $s_F(\beta) = \beta$, which means the intermediaries are charged at the rates they report. For second price mechanism, the sharing rates $s_{Si}(\beta) = \beta_i$ for i with $\beta_i(\mathcal{M}) \leq \max_{j\neq i} \beta_j(\mathcal{M})$, and $s_{Si}(\beta) = \max_{j\neq i} \beta_j(\mathcal{M})$ for i with $\beta_i(\mathcal{M}) > \max_{j\neq i} \beta_j(\mathcal{M})$. The second price mechanism sharing rates s_S and allocation x_S satisfies $s_{Si}(\beta) = \beta_i(\mathcal{M}) - \frac{\int_0^{\beta_i(\mathcal{M})} \hat{x}_{Si}(t,\beta_{-i})dt}{\hat{x}_{Si}(\beta_i(\mathcal{M}),\beta_{-i})}$, thus, second price mechanism is strategy proof, so the minimal punishment function for the second price mechanism is $h(\alpha_i, \beta_i) = 0$.

The minimal punishment function for first price mechanism is $h(\alpha_i, \beta_i) = (\alpha_i(\mathcal{M}) - \beta_i(\mathcal{M}))_+$ with $(\alpha_i(\mathcal{M}) - \beta_i(\mathcal{M}))_+ = \max\{\alpha_i(\mathcal{M}) - \beta_i(\mathcal{M}), 0\}$.

Consider a mechanism ϕ, which is linear combination of ϕ_F and ϕ_S, satisfies $\phi = \epsilon\phi_F + (1-\epsilon)\phi_S$ with $\epsilon \in [0, 1]$. From Corollary 2, the minimal punishment function for the mechanism ϕ is $h^{\min}(\alpha_i, \beta_i) = \epsilon(\alpha_i(\mathcal{M}) - \beta_i(\mathcal{M}))_+$.

Example 3

Consider the equally sharing rule of resource allocation, $x_i(\beta) = \frac{1}{N}$, which means the planner always allocates $\frac{1}{N}$ to each intermediary. The equally sharing mechanism ϕ_1 in Example 1, satisfying $s_{1i}(\beta) = \beta_i$, is not strategy proof. The intermediary has higher profit reporting lower than the true quality of intermediation. The h-SP condition requires $h(\alpha_i, \beta_i)x_i(\beta) \geq \max_{\beta_{-i}}[v_i(\alpha_i, \beta) - v_i(\alpha_i, \alpha_i, \beta_{-i})]$,substitute allocation x_i and sharing rates s_i into the inequality, $\frac{h(\alpha_i,\beta_i)}{N} \geq \frac{\alpha_i(\mathcal{M})-\beta_i(\mathcal{M})}{N} - 0$, thus, $h(\alpha_i, \beta_i) \geq \alpha_i(\mathcal{M}) - \beta_i(\mathcal{M})$. The minimal punishment function $h^{\min}(\alpha_i, \beta_i) = (\alpha_i(\mathcal{M}) - \beta_i(\mathcal{M}))_+$

The equally sharing strategy proof mechanism ϕ_2 satisfies $x_{2i}(\beta) = \frac{1}{N}$ and $s_{2i}(\beta) = \mathbf{0}$. $\mathbf{0} = (0, \ldots, 0)_{M\times 1}$. The minimal punishment function for this mechanism is $h_2^{\min} = 0$, but no resource will be transmitted to agents.

Finally, notice that Example 3 shows the strategy proof mechanism may transmit nothing to agents. The goal of planner is to send resource to agents in need via intermediaries, so the class of strategy proof mechanisms may not be good in some circumstances, punishment based on verification is necessary to achieve larger resource allocated to agents. The following section will discuss the preferences of planner, how will the punishment help improving the resource sent to agents.

5 First-Best Efficiency

We need to emphasize that we are in a setting of transmitting scarce resource to people in need, where the planner does not care about intermediaries, but only the agents. The resource allocation to agents $y \in \mathbb{R}_+^M$ satisfies $y = \sum_{i=1}^N s_i(\beta)x_i(\beta)$. Assume planner's preferences $\succeq$ over the resource allocation to agents y is monotonic, and there exists utility function $u : \mathbb{R}_+^M \mapsto \mathbb{R}$ to represent the preferences.

The following definition states the first best outcome from the perspective of planner.

Definition 6 (First Best Efficient (FBE))
Given the preferences of planner $\succeq$ and the utility function $u : \mathbb{R}_+^M \mapsto \mathbb{R}_+$. A mechanism ϕ is first best efficient (FBE), if for any profile of intermediaries α, the resource allocated to the agents y maximizes the planner's utility as if there is no intermediation, under the condition of individual rationality. The individual rationality means that intermediaries have nonnegative profit. If the preferences is strictly convex, then there exists a unique resource allocation that maximizes planner's utility given any profile $\alpha = (\alpha_1, \ldots, \alpha_N)$.

Given the intermediation quality α, the maximal utility $\bar{u}(\alpha)$ equals $\max_x u(\sum_{i=1}^N x_i\alpha_i)$, such that $\sum_{i=1}^N x_i = 1$. Assume $\bar{x} : R_+^{MN} \mapsto \mathbb{R}_+^N$ is the allocation of resource among intermediaries, which maximizes planner's utility when profile of intermediaries is α, $\bar{x}(\alpha) = (\bar{x}_1(\alpha), \ldots, \bar{x}_N(\alpha)) = \arg\max_x u(\sum_{i=1}^N x_i\alpha_i)$.

Notice that FBE implies the planner allocates resource through intermediaries optimally to achieve maximal utility with the true quality of intermediation.

Given mechanism $\phi = (x(\cdot), s(\cdot))$, and the quality of intermediation is α, the utility of planner is $u^*(\alpha, \phi) = u(\sum_{i=1}^N s_i(\alpha)x_i(\alpha))$, when intermediaries truthfully report their quality of intermediation $\beta = \alpha$.

Theorem 2
Assume the preferences of the planner $\succeq$ is strongly monotonic, continuous, and there exists utility function $u : \mathbb{R}_+^M \mapsto \mathbb{R}$ representing the preferences.

i. There is no symmetric, SP, budget balance and first best efficient mechanism.

ii. For any FBE mechanism ϕ, the minimal punishment function $h^{\min}$ for ϕ to be h-SP satisfies $h^{\min}(\alpha_i, \beta_i) = (\alpha_i(\mathcal{M}) - \beta_i(\mathcal{M}))_+$. Any punishment function h that implements *a FBE mechanism if and only if $h(\alpha_i, \beta_i) \geq h^{\min}(\alpha_i, \beta_i) = (\alpha_i(\mathcal{M}) - \beta_i(\mathcal{M}))_+$ for any $\alpha_i \geq \beta_i$.*

Theorem 2 shows there exists no mechanism satisfying symmetric, budget balance, 0-SP and FBE. It also provides the condition of minimal punishment function for mechanism to achieve the first best efficient. The minimal punishment function is the same with the one in Example 2 to guarantee the first price mechanism to be h-SP, which is not surprising because the first price mechanism is FBE for some special planner's preferences. The following example shows that the first price mechanism is the first best efficient when the resource transmitted to agents is perfect substitute for planner, while the second price mechanism is SP, but not FBE.

Example 4

Consider the preferences of planner is perfect substitute and represented by utility function $u(y) = \sum_{i=1}^{M} y_i$. Then the first price mechanism ϕ_F is FBE but not SP, and the second price mechanism ϕ_S is SP but not FBE.

When the planner only cares about the sum of resource transmitted to all agents, the first best outcome is to allocate the resource through a intermediary with largest aggregate intermediation quality $\alpha_i(\mathcal{M})$ and charge the sharing rates equal to his true quality of intermediation. Assume the aggregate intermediation quality ranking from high to low is $\alpha^1(\mathcal{M}) \geq \cdots \geq \alpha^N(\mathcal{M})$, the maximal utility $\bar{u}(\alpha) = \max_{i \in \mathcal{N}} \alpha_i(\mathcal{M}) = \alpha^1(\mathcal{M})$.

Thus, the first price mechanism, which allocates all resource to intermediary i with largest aggregate intermediation quality $\alpha_i(\mathcal{M})$ and transmit with sharing rates equal to report. The final allocation $y = \alpha^1$ and utility $u^*(\alpha, \phi_F) = \alpha^1(\mathcal{M})$, which is FBE. However, the second price mechanism, which is strategy proof, but it has to pay the intermediary i with largest aggregate intermediation $\alpha_i(\mathcal{M}) - \max_{j \neq i} \alpha_j(\mathcal{M})$ as information rent, The planner's utility $u^*(\alpha, \phi_S) = \alpha^2(\mathcal{M})$. When $\alpha^2(\mathcal{M}) < \alpha^1(\mathcal{M})$, $u^*(\alpha, \phi_S) < \bar{u}(\alpha)$, the second price mechanism can not achieve FBE.

This section shows that punishment is necessary to achieve the first best efficient for the planner, and the minimal punishment function for first best efficient mechanism to be h-SP coincides with the minimal punishment function for first price mechanism to be h-SP. Verification and punishment could be used to expand the class of strategy proof mechanisms and achieve higher efficiency for planner.

6 Optimal Mechanism

Definition 7

*Consider an arbitrary monotonic preferences $\succeq$ of planner, and let h be an arbitrary punishment function. We say the h-SP mechanism ϕ^* is h-**optimum** if for any h-SP mechanism $\bar{\phi}$, we have that $y^*(\beta) \succeq \bar{y}(\beta)$ for any β.*

Given $t \in [0,1]$, consider the punishment function $h^t(a,b) = t|\sum_{i=1}^{M}(a_i - b_i)|$ and mechanism $\phi^t = t\phi_F + (1-t)\phi_S$, which is the convex combination of the first price mechanism ϕ_F and second price mechanism ϕ_S. For example, first price mechanism ϕ_F is h-optimum with $h(a,b) = |\sum_{i=1}^{M}(a_i - b_i)|$.

Rank the aggregate quality of intermediation from high to low, assume $\alpha^1(\mathcal{M}) \geq \alpha^2(\mathcal{M}) \geq \cdots \geq \alpha^n(\mathcal{M})$.

Definition 8

Consider an arbitrary punishment function h, let function $t^h : R_+^{MN} \mapsto [0,1]$ satisfy $t^h(\alpha) = \min_{\alpha^2(\mathcal{M}) \leq \gamma(\mathcal{M}) \leq \alpha^1(\mathcal{M})} \frac{h(\alpha^1,\gamma)}{|\alpha^1(\mathcal{M}) - \gamma(\mathcal{M})|}$. Define the mechanism $\phi^h(\beta) = t^h(\alpha)\phi_F(\alpha) + (1-t^h(\alpha))\phi_S(\alpha)$.

This mechanism selects the worst possible ability for the planner to monitor, in relation to punishment function h with profile α. For profile α, the mechanism $\phi^h(\alpha) = t^h(\alpha)\phi_F(\alpha) + (1-t^h(\alpha))\phi_S(\alpha)$ allocates all resource to intermediary with highest aggregate intermediation quality $\alpha^1(\mathcal{M})$, and transmit at the rate of $t^h(\alpha)\alpha^1 + (1-t^h(\alpha))\alpha^2$. It is worth noting $t^h(\alpha)$ only depends on α^1 and α^2.

Theorem 3

For an arbitrary punishment function h, the mechanism ϕ^h is h-optimal among mechanisms that allocate to the intermediary with highest aggregate intermediation quality.

Resource allocating to the intermediary with highest aggregate intermediation quality is necessary for the result. See the following example.

Example 5

Suppose there is no punishment, that is $h(a,b) = 0$. Consider the strategy-proof mechanisms. From Theorem 1, the strategy-proof mechanism satisfies $s_i(\beta, \mathcal{M}) = \beta_i(\mathcal{M}) - \frac{\int_0^{\beta_i(\mathcal{M})} \hat{x}_i(t,\beta_{-i})dt}{\hat{x}_i(\beta_i(\mathcal{M}),\beta_{-i})}$.

Given the quality of intermediation α, assume $\alpha_1(\mathcal{M}) \geq \cdots \geq \alpha_N(\mathcal{M})$. The second price mechanism ϕ_S will transmit aggregate amount $\alpha_2(\mathcal{M})$ of resource to the agents. Consider another strategy-proof mechanism $\phi_0 = (x_0(\cdot), s_0(\cdot))$, the allocation rule satisfies $x_0(\beta_1, \alpha_{-1}) = (\frac{1}{N}, \ldots, \frac{1}{N})$ when $\alpha_2(\mathcal{M}) \leq \beta_1(\mathcal{M}) < \alpha_1(\mathcal{M})$, and otherwise $x_0(\beta) = x_S(\beta)$. Then the aggregate rates of transmission $s_{01}(\alpha, \mathcal{M}) = \alpha_1(\mathcal{M}) - \frac{\int_0^{\alpha_i(\mathcal{M})} \hat{x}_{01}(t,\alpha_{-i})dt}{x_{01}(\alpha)} = \frac{(N-1)\alpha_1(\mathcal{M}) + \alpha_2(\mathcal{M})}{N} > \alpha_2(\mathcal{M}) = s_S(\alpha, \mathcal{M})$.

However, for α' with $\alpha'_{-1} = \alpha_{-1}$ and $\alpha_2(\mathcal{M}) < \alpha'_1(\mathcal{M}) < \alpha_1(\mathcal{M})$, the aggregate resource sent in ϕ_S is $\alpha_2(\mathcal{M})$, and $\frac{2\alpha_2(\mathcal{M})+\sum_{i=3}^{N}\alpha_i(\mathcal{M})}{N}$ in ϕ_0. Thus, neither of the mechanisms are optimal for any SP mechanism.

Corollary 3

i. The mechanism ϕ^t is the unique h^t-optimal mechanism among the mechanisms that allocate resource to the intermediary with highest aggregate intermediation quality.

ii. h^t is the minimal punishment function of the mechanism ϕ^t.

Corollary 3 discusses the special case of punishment function $h(a, b) = t|a(\mathcal{M}) - b(\mathcal{M})|$, and $\phi^t = t\phi_F + (1-t)\phi_S$ is h-optimal mechanism.

Corollary 4

The second price mechanism ϕ_S is h-optimal for a punishment function h that satisfies either of the following conditions:

i. The derivative of h equals to zero, $\frac{\partial h(a,b)}{\partial b}|_{b=a} = 0$, for any a. All polynomial functions satisfy this.

ii. Any punishment function such that for each a, there exists $b \in U_\epsilon(a)$ in neighborhood of a with any $\epsilon > 0$, such that $h(a, b) = 0$. This includes, punishment functions that do not punish small deviations.

The first result of Corollary 4 is a natural extension of the results in Theorem 1. The set of strategy-proof mechanisms does not expand if the punishment function h has derivative 0, thus, there is no h-SP mechanism, which allocates resource to intermediary with highest aggregate intermediation quality, can transmit more resource than second price mechanism ϕ_S.

The second result shows if there exists small deviation without any punishment, then the punishment function h does not improve the h-optimal mechanism.

Corollary 5

The first price mechanism ϕ_F is optimal for any punishment function h that satisfies: $h(a, b) \geq \max\{a(\mathcal{M}) - b(\mathcal{M}), 0\}$ for any a, b.

This section discovers the optimal mechanism for the planner given an arbitrary punishment function, which is a convex combination of first price mechanism and second price mechanism, and the weight of the combination at each profile depends on the punishment function.

7 Conclusion

This paper investigates the strategy-proof mechanisms for the problem of resource transmission with intermediation on network. The mechanism requires the intermediaries to report their quality of intermediation, transmits the resource according to the sharing rates based on the report, and imposes punishment for misreporting.

This paper is a start to study the strategy-proof mechanisms with punishments, we discover and describe the sets of strategy-proof mechanisms with various punishment functions. The conditions of strategy-proof mechanism, that require share of resource transmission and sharing rates of intermediaries to satisfy, are provided. We also demonstrate the strategy-proof, symmetric, budget balance mechanisms with three cases of punishment function. With linear punishment function, the strategy-proof, symmetric, budget balance mechanisms require the sharing rule to depend only on the sum of quality of links. The minimal punishment function for a mechanism to be strategy-proof are discussed. On the other hand, given the punishment function, the optimal mechanism for the planner is discovered.

The mechanism design approach in this paper is a complement to game theoretical approach. It studies how to make a plan providing incentives for intermediaries to report the true quality of intermediation, rather than competing in the price charging for using their links.

Appendix: Proofs

In order to prove Theorems and Corollaries, first introduce the follow Lemmas.

Lemma 2

For function $f(x_1, \ldots, x_M)$ satisfying $\frac{\partial f(x)}{\partial x_i} = \frac{\partial f(x)}{\partial x_j}, \forall i, j$. Then there exists function $g : \mathbb{R} \mapsto \mathbb{R}$, s.t. $f(x) = g(\sum_{i=1}^{M} x_i)$.

Proof.

Assume $u_j = \sum_{i=1}^{j} x_i, \ \forall j$. $x_1 = u_1$, $x_j = u_j - u_{j-1}$ for $j \geq 2$. Define function $\hat{f}(u_1, \ldots, u_M)$ s.t. $\hat{f}(u_1, \ldots, u_M) = f(x_1, \ldots, x_M)$.

Then $\frac{\partial f(x)}{\partial x_j} = \sum_{i=1}^{M} \frac{\partial \hat{f}(u)}{\partial u_i} \frac{\partial u_i}{\partial x_j} = \sum_{i=j}^{M} \frac{\partial \hat{f}(u)}{\partial u_i}$.

$\frac{\partial f(x)}{\partial x_i} = \frac{\partial f(x)}{\partial x_j}, \forall i, j$.

$\sum_{i=j}^{M} \frac{\partial \hat{f}(u)}{\partial u_i} = \sum_{i=j+1}^{M} \frac{\partial \hat{f}(u)}{\partial u_i}$, then $\frac{\partial \hat{f}(u)}{\partial u_j} = 0, \forall j = 1, \ldots, M-1$.

So $\hat{f}(u)$ only depends on $u_M = \sum_{i=1}^{M} x_i$.

Thus, there exists function $g : \mathbb{R} \mapsto \mathbb{R}$, s.t. $f(x) = \hat{f}(u) = g(\sum_{i=1}^{M} x_i)$. ■

Lemma 3

For function $f(x_1,\ldots,x_N)$ satisfying $\int_0^{x_1} f(t,0)dt+\int_0^{x_2} f(t,x_1,0)dt+\cdots+\int_0^{x_N} f(t,x_{-N})dt = \int_0^{\sum_{i=1}^N x_i} f(t,0)dt$, $\forall x$. Then there exists function $g:\mathbb{R}\mapsto\mathbb{R}$, s.t. $f(x)=g(\sum_{i=1}^N x_i)$.

Proof.

For any $x=(x_1,\ldots,x_N)$, $\int_0^{\sum_{i=1}^N x_i} f(t,0)dt = \int_0^{x_1} f(t,0)dt + \int_0^{x_2} f(t,x_1,0)dt + \cdots + \int_0^{x_N} f(t,x_{-N})dt$.

Consider $\Delta>0$, $\int_0^{\sum_{i=1}^N x_i+\Delta} f(t,0)dt = \int_0^{x_1} f(t,0)dt+\int_0^{x_2} f(t,x_1,0)dt+\cdots+\int_0^{x_N+\Delta} f(t,x_{-N})dt$.

Then $\int_{\sum_{i=1}^N x_i}^{\sum_{i=1}^N x_i+\Delta} f(t,0)dt = \int_{x_N}^{x_N+\Delta} f(t,x_{-N})dt$. Let $\Delta\to 0$, $f(\sum_{i=1}^N x_i,0)=f(x)$.

Thus, $f(x)$ could be written as a function of $\sum_{i=1}^N x_i$. There exists function $g:\mathbb{R}\mapsto\mathbb{R}$ and $f(x)=g(\sum_{i=1}^N x_i)$. ■

Lemma 4

For the mechanism of planner $\phi=(x(\beta),s(\beta))$, functions $x_i(\beta),s_i(\beta)$ satisfying the First Order Condition, and $x_i(\beta)=\hat{x}_i(\beta_i(\mathcal{M}),\beta_{-i})$, $s_i(\beta)=\hat{s}_i(\beta_i(\mathcal{M}),\beta_{-i})$. If the mechanism is strategy-proof for each dimension of quality of intermediation β_i^m, then it is strategy-proof for any $\beta_i=(\beta_i^1,\ldots,\beta_i^M)$.

Proof.

Since $x_i(\beta)$, $s_i(\beta)$ depend only on $\beta_i(\mathcal{M})$ of the report β_i of intermediary i, deviation of intermediary i to β_i is equivalent as deviating to $\bar{\beta}_i$, with $\bar{\beta}_i(\mathcal{M})=\sum_{m=1}^M \bar{\beta}_i^m$. $\bar{\beta}_i^m=\alpha_i^m$ for $m\neq j$. Since the mechanism ϕ satisfies F.O.C., it is strategy-proof for deviation in each dimension of sharing-rates β_i^m. Thus, it is strategy-proof for deviation to any β_i. ■

Lemma 5

When $x_i(\beta)$ is symmetric, then there exists $f:\mathbb{R}_+^{NM}\mapsto\mathbb{R}_+$, s.t. $x_i(\beta)=\frac{f(\beta_i,\beta_{-i})}{\sum_{i=1}^N f(\beta_i,\beta_{-i})}$. If $x_i(\beta)=\hat{x}_i(\beta_i(\mathcal{M}),\beta_{-i})$, and the mechanism is symmetric, then there exists function $f:\mathbb{R}_+\times\mathbb{R}_+^{M(N-1)}\mapsto\mathbb{R}_+$, s.t. $x_i(\beta)=\frac{f(\beta_i(\mathcal{M}),\beta_{-i})}{\sum_{j=1}^N f(\beta_j(\mathcal{M}),\beta_{-j})}$.

Proof. $x_i(\beta)=x_i(\beta_i,\beta_{-i})$, there exists f_i, s.t. $x_i(\beta_i,\beta_{-i})\times k=f_i(\beta_i,\beta_{-i})$, k is a constant.

For β and $\hat{\beta}$ with $\beta_i=\hat{\beta}_j$, $\beta_j=\hat{\beta}_i$ and $\beta_m=\hat{\beta}_m$, $\forall m\neq i,j$.

Since the mechanism is symmetric, $x_i(\beta)=x_j(\hat{\beta})$, by definition of f_i, $f_i(\beta_i,\beta_{-i}) = f_j(\hat{\beta}_j,\hat{\beta_{-j}})$ and $\sum_{n=1}^N f_n(\beta)=\sum_{n=1}^N f_n(\hat{\beta})$.

Thus, $f_i(\beta_i,\beta_{-i})=f_j(\hat{\beta}_j,\hat{\beta_{-j}})=f(\beta_i,\beta_{-i})$.

$x_i(\beta_i,\beta_{-i})=\frac{f_i(\beta_i,\beta_{-i})}{k}$, budget balance requires $\sum_{i=1}^N x_i(\beta_i,\beta_{-i})=1$. $\sum_{i=1}^N\frac{f_i(\beta_i,\beta_{-i})}{k}=1$, and $k=\sum_{i=1}^N f_i(\beta_i,\beta_{-i})$. So $x_i(\beta_i,\beta_{-i})=\frac{f_i(\beta_i,\beta_{-i})}{\sum_{i=1}^N f_i(\beta_i,\beta_{-i})}$.

The proof for the case of $x_i(\beta)=\hat{x}_i(\beta_i(\mathcal{M}),\beta_{-i})$ is the same.

$x_i(\beta) = \hat{x}_i(\beta_i(\mathcal{M}), \beta_{-i})$, there exists f_i, s.t. $\hat{x}_i(\beta_i(\mathcal{M}), \beta_{-i}) \times k = f_i(\beta_i(\mathcal{M}), \beta_{-i})$, k is a constant.

If there exists β, $\hat{\beta}$ with $\beta_i = \hat{\beta}_j$, $\beta_j = \hat{\beta}_i$ and $\beta_m = \hat{\beta}_m$, $\forall m \neq i, j$.

s.t. $f_i(\beta_i(\mathcal{M}), \beta_{-i}) \neq f_j(\hat{\beta}_i(\mathcal{M}), \hat{\beta}_{-j})$.

Since $\sum_{n=1}^{N} f_n(\beta_n(\mathcal{M}), \beta_{-n}) = \sum_{n=1}^{N} f_n(\hat{\beta}_n(\mathcal{M}), \hat{\beta}_{-n})$, then $x_i(\beta) = \frac{f_i(\beta_i(\mathcal{M}),\beta_{-i})}{\sum_{n=1}^{N} f_n(\hat{\beta}_n(\mathcal{M}),\beta_{-n})} \neq \frac{f_j(\hat{\beta}_j(\mathcal{M}),\hat{\beta}_{-j})}{\sum_{n=1}^{N} f_n(\hat{\beta}_n(\mathcal{M}),\hat{\beta}_{-n})} = x_j(\hat{\beta})$, this is contradiction with symmetric mechanism.

Thus, $f_i(\beta_i(\mathcal{M}), \beta_{-i}) = f_j(\hat{\beta}_j(\mathcal{M}), \hat{\beta}_{-j}) = f(\beta_i(\mathcal{M}), \beta_{-i})$. ■

Proof of Lemma 1

Proof.

Part i. Given the mechanism $\phi = (x(\cdot), s(\cdot))$, assume $\Phi_i(\beta) = \sum_{m=1}^{M} (\beta_i^m - s_i^m(\beta)) x_i(\beta)$, which represents the profit of intermediary i.

Part ii. h-SP mechanism satisfies $(\alpha_i - s_i(\alpha_i, \beta_{-i}))^T \mathbf{1} x_i(\alpha_i, \beta_{-i}) \geq (\alpha_i - s_i(\beta_i, \beta_{-i}))^T \mathbf{1} x_i(\beta_i, \beta_{-i}) - h(\alpha_i, \beta_i) \cdot x_i(\beta_i, \beta_{-i})$, $\forall \beta_{-i}$.

For β, with $\beta_i^j > \alpha_i^j$, and $\beta_i^{-j} = \alpha_i^{-j}$. Then h-SP requires $\frac{\Phi_i(\alpha_i,\beta_{-i}) - \Phi_i(\beta_i,\beta_{-i})}{\alpha_i^j - \beta_i^j} \leq x_i(\beta_i, \beta_{-i}) + \frac{(h(\alpha_i,\alpha_i) - h(\alpha_i,\beta_i)) \cdot x_i(\beta_i,\beta_{-i})}{\alpha_i^j - \beta_i^j}$.

For β, with $\beta_i^j < \alpha_i^j$, and $\beta_i^{-j} = \alpha_i^{-j}$. Then h-SP requires $\frac{\Phi_i(\alpha_i,\beta_{-i}) - \Phi_i(\beta_i,\beta_{-i})}{\alpha_i^j - \beta_i^j} \geq x_i(\beta_i, \beta_{-i}) + \frac{(h(\alpha_i,\alpha_i) - h(\alpha_i,\beta_i)) \cdot x_i(\beta_i,\beta_{-i})}{\alpha_i^j - \beta_i^j}$.

From these two inequalities: $\frac{\Phi_i(\alpha_i,\beta_{-i}) - \Phi_i(\beta_i,\beta_{-i})}{\alpha_i^j - \beta_i^j} = x_i(\beta_i, \beta_{-i}) + \frac{(h(\alpha_i,\alpha_i) - h(\alpha_i,\beta_i)) \cdot x_i(\beta_i,\beta_{-i})}{\alpha_i^j - \beta_i^j}$.

Let $\beta_i \to \alpha_i$, with $\beta_i^{-j} = \alpha_i^{-j}$. The equation is equivalent with $\frac{\partial \Phi_i(\alpha_i,\beta_{-i})}{\partial \beta_i^m} = x_i(\beta_i, \beta_{-i}) + h_{2m}(\alpha_i) \cdot x_i(\beta_i, \beta_{-i}) = (1 + h_{2m}(\alpha_i)) x_i(\alpha_i, \beta_{-i})$, $\forall m$.

Part iii. If $h_{2m}(\alpha_i) = h_{2k}(\alpha_i)$, $\frac{\partial \Phi_i(\alpha_i,\beta_{-i})}{\partial \beta_i^m} = \frac{\partial \Phi_i(\alpha_i,\beta_{-i})}{\partial \beta_i^k}$, $\forall m, k$.

From Lemma 2, there exists $\hat{\Phi}_i(\beta_i(\mathcal{M}), \beta_{-i}) = \Phi_i(\beta)$.

$\frac{\partial \hat{\Phi}_i(\beta_i(\mathcal{M}),\beta_{-i})}{\partial \beta_i^m} = x_i(\beta_i, \beta_{-i}) + h_{2m}(\alpha_i) \cdot x_i(\beta_i, \beta_{-i}) = (1 + h_{2m}(\alpha_i)) x_i(\beta)$.

If $h_{2m}(\alpha_i) = d$, $x_i(\beta) = \frac{1}{1+d} \cdot \frac{\partial \hat{\Phi}_i(\beta_i(\mathcal{M}),\beta_{-i})}{\partial \beta_i^m}$, there exists $\hat{x}_i(\beta_i(\mathcal{M}), \beta_{-i}) = x_i(\beta)$.

$\frac{\partial \Phi_i(\beta_i,\beta_{-i})}{\partial \beta_i^m} = (1 + d) \hat{x}_i(\beta_i(\mathcal{M}), \beta_{-i})$, from Lemma 3, $\Phi_i(\beta) = \int_0^{\beta_i(\mathcal{M})} \hat{x}_i(t, \beta_{-i}) dt$.

From definition of $\Phi_i(\beta)$, $\Phi_i(\beta) = \sum_{m=1}^{M} (\beta_i^m - s_i^m(\beta)) x_i(\beta)$.

So $\sum_{m=1}^{M} s_i^m(\beta) = \frac{\beta_i(\mathcal{M}) x_i(\beta) - \Phi_i(\beta)}{x_i(\beta)} = \frac{\beta_i(\mathcal{M}) \hat{x}_i(\beta_i(\mathcal{M}),\beta_{-i}) - \hat{\Phi}_i(\beta_i(\mathcal{M}),\beta_{-i})}{\hat{x}_i(\beta_i(\mathcal{M}),\beta_{-i})}$ depend only on $\beta_i(\mathcal{M})$ instead of β_i.

There exist functions $\hat{s}_i^m : \mathbb{R}_+ \times \mathbb{R}_+^{M(N-1)} \mapsto \mathbb{R}_+$, $\sum_{m=1}^{M} s_i^m(\beta) = \sum_{m=1}^{M} \hat{s}_i^m(\beta_i(\mathcal{M}), \beta_{-i})$.

Thus, $\sum_{m=1}^{M} s_i^m(\beta) = \beta_i(\mathcal{M}) - \frac{(1+d) \int_0^{\beta_i(\mathcal{M})} \hat{x}_i(t,\beta_{-i}) dt}{\hat{x}_i(\beta_i(\mathcal{M}),\beta_{-i})}$.

From Lemma 4, the mechanisms $\phi = (x(\cdot), s(\cdot))$ with functions $x_i(\beta)$ and $s_i(\beta)$ satisfying the conditions above are strategy-proof for intermediary i. ■

Proof of Theorem 1

Proof.

When $c = 0$, consider report β with $\beta_i^j > \alpha_i^j$ and $\beta_i^{-j} = \alpha_i^{-j}$. $\frac{\Phi_i(\alpha_i,\beta_{-i})-\Phi_i(\beta_i,\beta_{-i})}{\alpha_i^j-\beta_i^j} \leq x_i(\beta_i, \beta_{-i})$. Take the limit, $\frac{\partial \Phi_i(\alpha_i,\beta_{-i})}{\partial \beta_i^m} \leq x_i(\beta)$.

Consider report β with $\beta_i^j < \alpha_i^j$, $\beta_i^{-j} = \alpha_i^{-j}$, $\frac{\Phi_i(\alpha_i,\beta_{-i})-\Phi_i(\beta_i,\beta_{-i})}{\alpha_i^j-\beta_i^j} \geq x_i(\beta_i, \beta_{-i})$. Take the limit, $\frac{\partial \Phi_i(\alpha_i,\beta_{-i})}{\partial \beta_i^m} \geq x_i(\beta)$.

From the inequalities above, $\frac{\partial \Phi_i(\alpha_i,\beta_{-i})}{\partial \beta_i^m} = x_i(\alpha_i, \beta_{-i})$, $\forall m$ with $\beta_i \to \alpha_i$.

From Lemma 2, there exists $\hat{x}_i(\beta_i(\mathcal{M}), \beta_{-i}) = x_i(\beta)$.

$\frac{\partial \Phi_i(\alpha_i,\beta_{-i})}{\partial \beta_i^m} = \hat{x}_i(\beta_i(\mathcal{M}), \beta_{-i})$, from Lemma 3, $\Phi_i(\beta) = \int_0^{\beta_i(\mathcal{M})} \hat{x}_i(t, \beta_{-i})dt$.

Since $\Phi_i(\beta) = \sum_{m=1}^M (\beta_i^m - s_i^m(\beta))x_i(\beta)$, then $\sum_{m=1}^M s_i^m(\beta) = \beta_i(\mathcal{M}) - \frac{\int_0^{\beta_i(\mathcal{M})} \hat{x}_i(t,\beta_{-i})dt}{\hat{x}_i(\beta_i(\mathcal{M}),\beta_{-i})}$.

From Lemma 4, the mechanisms $\phi = (x(\cdot), s(\cdot))$ with functions $x_i(\beta)$ and $s_i(\beta)$ satisfying the conditions above are strategy-proof for intermediary i. A mechanism is 0-SP is equivalent with the condition $\sum_{m=1}^M s_i^m(\beta) = \beta_i(\mathcal{M}) - \frac{\int_0^{\beta_i(\mathcal{M})} \hat{x}_i(t,\beta_{-i})dt}{\hat{x}_i(\beta_i(\mathcal{M}),\beta_{-i})}$.

From Lemma 1, h-SP mechanism satisfies $\frac{\partial \Phi_i(\beta_i,\beta_{-i})}{\partial \beta_i^m} = (1 + h_{2m}(\alpha_i))x_i(\beta)$.

When $h(\alpha_i, \beta_i) = \sum_{m=1}^M \gamma_m(\alpha_i^m - \beta_i^m)^{k_m}$, there is $h_{2m}(\alpha_i) = 0$. For any $h_{2m}(\alpha_i) = 0$, the condition is equivalent with $\frac{\partial \Phi_i(\beta_i,\beta_{-i})}{\partial \beta_i^m} = x_i(\alpha_i, \beta_{-i})$. ■

Results for General Punishment Function h

- For $\beta_i(\mathcal{M}) > \alpha_i(\mathcal{M})$:

  $\sum_{m=1}^M \frac{(\alpha_i^m-\beta_i^m)}{\alpha_i(\mathcal{M})-\beta_i(\mathcal{M})} \frac{\partial \Phi_i(\alpha_i,\beta_{-i})}{\partial \beta_i^m} \leq (1 + \lim_{t\to 0} \frac{(h(\alpha_i,\alpha_i)-h(\alpha_i,\beta_i(t)))}{t(\alpha_i(\mathcal{M})-\beta_i(\mathcal{M}))})x_i(\alpha_i, \beta_{-i})$.

- For $\beta_i(\mathcal{M}) < \alpha_i(\mathcal{M})$:

  $\sum_{m=1}^M \frac{(\alpha_i^m-\beta_i^m)}{\alpha_i(\mathcal{M})-\beta_i(\mathcal{M})} \frac{\partial \Phi_i(\alpha_i,\beta_{-i})}{\partial \beta_i^m} \geq (1 + \lim_{t\to 0} \frac{(h(\alpha_i,\alpha_i)-h(\alpha_i,\beta_i(t)))}{t(\alpha_i(\mathcal{M})-\beta_i(\mathcal{M}))})x_i(\alpha_i, \beta_{-i})$. [17]

Proof.

$(\alpha_i - s_i(\alpha_i, \beta_{-i}))^T \mathbf{1} x_i(\alpha_i, \beta_{-i}) \geq (\alpha_i - s_i(\beta_i, \beta_{-i}))^T \mathbf{1} x_i(\beta_i, \beta_{-i}) - h(\alpha_i, \beta_i) \cdot x_i(\beta_i, \beta_{-i}), \forall \beta_{-i}$

$\Phi_i(\alpha_i, \beta_{-i}) = (\alpha_i - s_i(\alpha_i, \beta_{-i}))^T \mathbf{1} x_i(\alpha_i, \beta_{-i})$, $\Phi_i(\beta_i, \beta_{-i}) = (\beta_i - s_i(\beta_i, \beta_{-i}))^T \mathbf{1} x_i(\beta_i, \beta_{-i})$.

$(\alpha_i - s_i(\beta_i, \beta_{-i}))^T \mathbf{1} x_i(\beta_i, \beta_{-i}) = (\alpha_i - \beta_i)^T \mathbf{1} x_i(\beta_i, \beta_{-i}) + \Phi_i(\beta_i, \beta_{-i})$.

Thus, h-SP is equivalent with $\Phi_i(\alpha_i, \beta_{-i}) - \Phi_i(\beta_i, \beta_{-i}) \geq (\alpha_i - \beta_i)^T \mathbf{1} x_i(\beta_i, \beta_{-i}) - h(\alpha_i, \beta_i) \cdot x_i(\beta_i, \beta_{-i})$.

$\Phi_i(\alpha_i, \beta_{-i}) - \Phi_i(\beta_i, \beta_{-i}) \geq (\alpha_i - \beta_i)^T \mathbf{1} x_i(\beta_i, \beta_{-i}) + (h(\alpha_i, \alpha_i) - h(\alpha_i, \beta_i)) \cdot x_i(\beta_i, \beta_{-i})$.

If $\beta_i(\mathcal{M}) > \alpha_i(\mathcal{M})$, $\frac{\Phi_i(\alpha_i,\beta_{-i})-\Phi_i(\beta_i,\beta_{-i})}{\alpha_i(\mathcal{M})-\beta_i(\mathcal{M})} \leq x_i(\beta_i, \beta_{-i}) + \frac{(h(\alpha_i,\alpha_i)-h(\alpha_i,\beta_i))\cdot x_i(\beta_i,\beta_{-i})}{\alpha_i(\mathcal{M})-\beta_i(\mathcal{M})}$.

If $\beta_i(\mathcal{M}) < \alpha_i(\mathcal{M})$, $\frac{\Phi_i(\alpha_i,\beta_{-i})-\Phi_i(\beta_i,\beta_{-i})}{\alpha_i(\mathcal{M})-\beta_i(\mathcal{M})} \geq x_i(\beta_i, \beta_{-i}) + \frac{(h(\alpha_i,\alpha_i)-h(\alpha_i,\beta_i))\cdot x_i(\beta_i,\beta_{-i})}{\alpha_i(\mathcal{M})-\beta_i(\mathcal{M})}$.

Let $\beta_i(t) = t\beta_i + (1 - t)\alpha_i$, with $0 \leq t \leq 1$. As $t \to 0$, $\beta_i(t) \to \alpha_i$.

If $\beta_i(\mathcal{M}) > \alpha_i(\mathcal{M})$, $\frac{\Phi_i(\alpha_i,\beta_{-i})-\Phi_i(\beta_i(t),\beta_{-i})}{\alpha_i(\mathcal{M})-\beta_i(\mathcal{M})(t)} \leq x_i(\beta_i(t), \beta_{-i}) + \frac{(h(\alpha_i,\alpha_i)-h(\alpha_i,\beta_i(t)))\cdot x_i(\beta_i(t),\beta_{-i})}{\alpha_i(\mathcal{M})-\beta_i(\mathcal{M})(t)}$.

Equivalent with $\frac{\Phi_i(\alpha_i,\beta_{-i})-\Phi_i(\beta_i(t),\beta_{-i})}{t(\alpha_i(\mathcal{M})-\beta_i(\mathcal{M}))} \leq x_i(\beta_i(t), \beta_{-i}) + \frac{(h(\alpha_i,\alpha_i)-h(\alpha_i,\beta_i(t)))\cdot x_i(\beta_i(t),\beta_{-i})}{t(\alpha_i(\mathcal{M})-\beta_i(\mathcal{M}))}$.

[17] $\beta_i(t) = t\beta_i + (1 - t)\alpha_i$.

Take the limit $t \to 0$, $\sum_{m=1}^{M} \frac{(\alpha_i^m - \beta_i^m)}{\alpha_i(\mathcal{M}) - \beta_i(\mathcal{M})} \frac{\partial \Phi_i(\alpha_i, \beta_{-i})}{\partial \beta_i^m} \leq (1 + \lim_{t \to 0} \frac{(h(\alpha_i,\alpha_i) - h(\alpha_i, \beta_i(t)))}{t(\alpha_i(\mathcal{M}) - \beta_i(\mathcal{M}))}) x_i(\alpha_i, \beta_{-i})$.

If $\beta_i(\mathcal{M}) < \alpha_i(\mathcal{M})$, $\frac{\Phi_i(\alpha_i, \beta_{-i}) - \Phi_i(\beta_i(t), \beta_{-i})}{\alpha_i(\mathcal{M}) - \beta_i(\mathcal{M})(t)} \geq x_i(\beta_i(t), \beta_{-i}) + \frac{(h(\alpha_i,\alpha_i) - h(\alpha_i, \beta_i(t))) \cdot x_i(\beta_i(t), \beta_{-i})}{\alpha_i(\mathcal{M}) - \beta_i(\mathcal{M})(t)}$.

Take the limit $t \to 0$, $\sum_{m=1}^{M} \frac{(\alpha_i^m - \beta_i^m)}{\alpha_i(\mathcal{M}) - \beta_i(\mathcal{M})} \frac{\partial \Phi_i(\alpha_i, \beta_{-i})}{\partial \beta_i^m} \geq (1 + \lim_{t \to 0} \frac{(h(\alpha_i,\alpha_i) - h(\alpha_i, \beta_i(t)))}{t(\alpha_i(\mathcal{M}) - \beta_i(\mathcal{M}))}) x_i(\alpha_i, \beta_{-i})$.

■

Proof of Note, $h = \infty$

Proof.

$h = \infty$, when $\beta_i \leq \alpha_i$, same with Lemma 1. $x_i(\beta_i, \beta_{-i})(\alpha_i(\mathcal{M}) - \beta_i(\mathcal{M})) \leq \Phi_i(\alpha_i, \beta_{-i}) - \Phi_i(\beta_i, \beta_{-i})$.

When $\beta_i^m > \alpha_i^m$, h-SP requires $\Phi_i(\beta_i, \beta_{-i}) + x_i(\beta_i, \beta_{-i})(\alpha_i(\mathcal{M}) - \beta_i(\mathcal{M})) + (\alpha_i(\mathcal{M}) - \beta_i(\mathcal{M}))x_i(\beta_i, \beta_{-i}) \leq \Phi_i(\alpha_i, \beta_{-i})$. If $x_i(\beta_i, \beta_{-i}) = 0$, $\Phi_i(\beta_i, \beta_{-i}) = 0$, the inequality of h-SP satisfies. If $x_i(\beta_i, \beta_{-i}) > 0$, $(\alpha_i(\mathcal{M}) - \beta_i(\mathcal{M}))x_i(\beta_i, \beta_{-i}) \to -\infty$, the inequality of h-SP satisfies.

Thus, ∞-SP is equivalent with $\Phi_i(\beta_i, \beta_{-i}) - \Phi_i(\hat{\beta}_i, \beta_{-i}) \geq x_i(\hat{\beta}_i, \beta_{-i})(\beta_i(\mathcal{M}) - \hat{\beta}_i(\mathcal{M}))$, $\forall \hat{\beta}_i \leq \beta_i$.

From Lemma 4, the functions $x_i(\beta)$ and $s_i(\beta)$ satisfying this inequality are strategy-proof for intermediary i.

For any quality of intermediation α_i and β_i, $\alpha_i \neq \beta_i$.

$(\alpha_i - s_i(\alpha_i, \beta_{-i}))^T \mathbf{1} x_i(\alpha_i, \beta_{-i}) \geq (\alpha_i - s_i(\beta_i, \beta_{-i}))^T \mathbf{1}$, $\forall \beta_{-i}$.

When $h = +\infty$, for $\alpha \neq \beta$, then $h(\alpha_i, \beta_i) \cdot x_i(\beta_i, \beta_{-i}) \to \infty$ if $x_i(\beta_i, \beta_{-i}) > 0$, and the h-SP holds. The inequality holds when $x_i(\beta_i, \beta_{-i}) = 0$.

$h = 0$, consider report β with $\beta_i^j > \alpha_i^j$ and $\beta_i^{-j} = \alpha_i^{-j}$. From Lemma 1, $\frac{\Phi_i(\alpha_i, \beta_{-i}) - \Phi_i(\beta_i, \beta_{-i})}{\alpha_i^j - \beta_i^j} \leq x_i(\beta_i, \beta_{-i})$. Take the limit, $\frac{\partial \Phi_i(\hat{\beta}_i, \beta_{-i})}{\partial \beta_i^m} \leq x_i(\beta)$.

Consider report β with $\beta_i^j < \alpha_i^j$, $\beta_i^{-j} = \alpha_i^{-j}$, $\frac{\Phi_i(\alpha_i, \beta_{-i}) - \Phi_i(\beta_i, \beta_{-i})}{\alpha_i^j - \beta_i^j} \leq x_i(\beta_i, \beta_{-i})$. Take the limit, $\frac{\partial \Phi_i(\hat{\beta}_i, \beta_{-i})}{\partial \beta_i^m} \geq x_i(\beta)$.

From above, $x_i(\beta) = \frac{\partial \Phi_i(\hat{\beta}_i, \beta_{-i})}{\partial \beta_i^m}$, $\forall m$.

From Lemma 2, there exists $\hat{x}_i(\beta_i(\mathcal{M}), \beta_{-i}) = x_i(\beta)$.

$\frac{\partial \Phi_i(\hat{\beta}_i, \beta_{-i})}{\partial \beta_i^m} = \hat{x}_i(\beta_i(\mathcal{M}), \beta_{-i})$, from Lemma 3, $\Phi_i(\beta) = \int_0^{\beta_i(\mathcal{M})} \hat{x}_i(t, \beta_{-i}) dt$.

From definition of $\Phi_i(\beta)$, $\Phi_i(\beta) = \sum_{m=1}^{M} (\beta_i^m - s_i^m(\beta)) x_i(\beta)$.

Thus, $\sum_{m=1}^{M} s_i^m(\beta) = \beta_i(\mathcal{M}) - \frac{\int_0^{\beta_i(\mathcal{M})} \hat{x}_i(t, \beta_{-i}) dt}{\hat{x}_i(\beta_i(\mathcal{M}), \beta_{-i})}$.

From Lemma 4, the functions $x_i(\beta)$ and $s_i(\beta)$ satisfying F.O.C. are strategy-proof for intermediary i. ■

Proof of Proposition 1

Proof.

For any punishment function h and h'. $h(a,b) \leq h'(a,b)$, for any a and b.

Any mechanism ϕ is h-SP, satisfies: $(\alpha_i - s_i(\alpha_i, \beta_{-i}))^T \mathbf{1} x_i(\alpha_i, \beta_{-i}) \geq (\alpha_i - s_i(\beta_i, \beta_{-i}))^T \mathbf{1} x_i(\beta_i, \beta_{-i}) - h(\alpha_i, \beta_i) \cdot x_i(\beta_i, \beta_{-i})$, $\forall \beta_{-i}$.

Since $h(\alpha_i, \beta_i) \leq h'(\alpha_i, \beta_i)$, $(\alpha_i - s_i(\alpha_i, \beta_{-i}))^T \mathbf{1} x_i(\alpha_i, \beta_{-i}) \geq (\alpha_i - s_i(\beta_i, \beta_{-i}))^T \mathbf{1} x_i(\beta_i, \beta_{-i}) - h(\alpha_i, \beta_i) \cdot x_i(\beta_i, \beta_{-i}) \geq (\alpha_i - s_i(\beta_i, \beta_{-i}))^T \mathbf{1} x_i(\beta_i, \beta_{-i}) - h'(\alpha_i, \beta_i) \cdot x_i(\beta_i, \beta_{-i})$, $\forall \beta_{-i}$.

So the mechanism $\phi = (x(\cdot), s(\cdot))$ is h'-SP. ■

Proof of Proposition 2

Proof.

ϕ_1 is h-SP for punishment function h_1, so $(\alpha_i(\mathcal{M}) - \sum_{m=1}^{M} s_{1i}^m(\alpha_i, \beta_{-i}))x_{1i}(\alpha_i, \beta_{-i}) \geq (\alpha_i(\mathcal{M}) - \sum_{m=1}^{M} s_{1i}^m(\beta))x_{1i}(\beta) - h_1(\alpha_i, \beta_i)x_i(\beta)$.

ϕ_2 is h-SP for punishment function h_2, so $(\alpha_i(\mathcal{M}) - \sum_{m=1}^{M} s_{2i}^m(\alpha_i, \beta_{-i}))x_{2i}(\alpha_i, \beta_{-i}) \geq (\alpha_i(\mathcal{M}) - \sum_{m=1}^{M} s_{2i}^m(\beta))x_{2i}(\beta) - h_2(\alpha_i, \beta_i)x_i(\beta)$.

Then the linear combination of the two inequalities with weighted of λ and $(1-\lambda)$ results $\lambda[(\alpha_i(\mathcal{M}) - \sum_{m=1}^{M} s_{1i}^m(\alpha_i, \beta_{-i}))x_{1i}(\alpha_i, \beta_{-i})] + (1-\lambda)[(\alpha_i(\mathcal{M}) - \sum_{m=1}^{M} s_{2i}^m(\alpha_i, \beta_{-i}))x_{2i}(\alpha_i, \beta_{-i})] \geq \lambda[(\alpha_i(\mathcal{M}) - \sum_{m=1}^{M} s_{1i}^m(\beta))x_{1i}(\beta) - h_1(\alpha_i, \beta_i)x_i(\beta)] + (1-\lambda)[(\alpha_i(\mathcal{M}) - \sum_{m=1}^{M} s_{2i}^m(\beta))x_{2i}(\beta) - h_2(\alpha_i, \beta_i)x_i(\beta)]$.

$x_1(\beta) = x_2(\beta)$ and $x_1(\alpha_i, \beta_{-i}) = x_2(\alpha_i, \beta_{-i})$, so rearrange the inequality above, there is $(\alpha_i(\mathcal{M}) - \lambda \sum_{m=1}^{M} s_{1i}^m(\alpha_i, \beta_{-i}) - (1-\lambda) \sum_{m=1}^{M} s_{2i}^m(\alpha_i, \beta_{-i}))x_i(\alpha_i, \beta_{-i}) \geq (\alpha_i(\mathcal{M}) - \lambda \sum_{m=1}^{M} s_{1i}^m(\beta) - (1-\lambda) \sum_{m=1}^{M} s_{2i}^m(\beta))x_i(\beta) - [\lambda h_1(\alpha_i, \beta_i) + (1-\lambda)h_2(\alpha_i, \beta_i)]x_i(\beta)$.

It is equivalent with $(\alpha_i(\mathcal{M}) - \sum_{m=1}^{M} s_i^m(\alpha_i, \beta_{-i}))x_i(\alpha_i, \beta_{-i}) \geq (\alpha_i(\mathcal{M}) - \sum_{m=1}^{M} s_i^m(\beta))x_i(\beta) - [\lambda h_1(\alpha_i, \beta_i) + (1-\lambda)h_2(\alpha_i, \beta_i)]x_i(\beta)$. Thus, ϕ is h-SP for punishment function h, with $h(\alpha_i, \beta_i) = \lambda h_1(\alpha_i, \beta_i) + (1-\lambda)h_2(\alpha_i, \beta_i)$. ■

Proof of Proposition 3

Proof.

A mechanism is h-SP if and only if $v_i(\alpha_i, \alpha_i, \beta_{-i}) \geq v_i(\alpha_i, \beta) - h(\alpha_i, \beta_i)x_i(\beta)$, for any β, α_i. It is equivalent with $ch(\alpha_i, \beta_i)x_i(\beta) \geq v_i(\alpha_i, \beta) - v_i(\alpha_i, \alpha_i, \beta_{-i})$, for any β, α_i.

So if $h(\alpha_i, \beta_i)x_i(\beta) \geq \max_{\beta_{-i}}[v_i(\alpha_i, \beta) - v_i(\alpha_i, \alpha_i, \beta_{-i})]$ for any β, α_i, mechanism ϕ is h-SP.

On the other side, if $h(\alpha_i, \beta_i)x_i(\beta) < \max_{\beta_{-i}}[v_i(\alpha_i, \beta) - v_i(\alpha_i, \alpha_i, \beta_{-i})]$ for some β, α_i. Then there exists β_{-i}, when intermediary i has ability of transmission α_i, he will deviate to report β_i and achieve higher profit $v_i(\alpha_i, \beta) - h(\alpha_i, \beta_i)x_i(\beta)$, such that mechanism is not h-SP. ■

Proof of Corollary 2

Proof.

i. By definition, the minimal punishment function $h^{\min}(\alpha_i, \beta_i) = \max_{\beta_{-i}}[\frac{v_i(\alpha_i,\beta)-v_i(\alpha_i,\alpha_i,\beta_{-i})}{x_i(\beta)}]$, for any $\beta_i, \alpha_i, \beta_{-i}$, such that $x_i(\beta) > 0$.

$\frac{v_i(\alpha_i,\beta)-v_i(\alpha_i,\alpha_i,\beta_{-i})}{x_i(\beta)} = \alpha_i(\mathcal{M}) - s_i(\beta, \mathcal{M}) - \frac{(\alpha_i(\mathcal{M})-s_i(\alpha_i,\beta_{-i},\mathcal{M}))x_i(\alpha_i,\beta_{-i})}{x_i(\beta)}$.

Individual rationality for intermediary to participate, $(\alpha_i(\mathcal{M}) - s_i(\alpha_i, \beta_{-i}, \mathcal{M}) \geq 0$, so $\frac{(\alpha_i(\mathcal{M})-s_i(\alpha_i,\beta_{-i},\mathcal{M}))x_i(\alpha_i,\beta_{-i})}{x_i(\beta)} \geq 0$. Then $\frac{v_i(\alpha_i,\beta)-v_i(\alpha_i,\alpha_i,\beta_{-i})}{x_i(\beta)} \leq \alpha_i(\mathcal{M})$.

Thus, $h^{\min}(\alpha_i, \beta_i) = \max_{\beta_{-i}}[\frac{v_i(\alpha_i,\beta)-v_i(\alpha_i,\alpha_i,\beta_{-i})}{x_i(\beta)}] \leq \alpha_i(\mathcal{M})$. For any α_i, β_i, the upper bound exists. There exists minimal punishment function $h^{\min}(\alpha_i, \beta_i)$.

ii. Consider mechanism $\phi = (x(\cdot), s(\cdot))$ is strategy proof. Then $v_i(\alpha_i, \beta) \leq v_i(\alpha_i, \alpha_i, \beta_{-i})$, for any β, α_i. So $0 \geq \max_{\beta_{-i}}[v_i(\alpha_i, \beta) - v_i(\alpha_i, \alpha_i, \beta_{-i})]$ for any β, α_i. Then the minimal punishment function $h^{\min}(\alpha_i, \beta_i) = 0$, for any β_i, α_i.

iii. Consider mechanism $\phi = (x(\cdot), s(\cdot))$ is not strategy proof, there exists α_i, β, such that $v_i(\alpha_i, \alpha_i, \beta_{-i}) < v_i(\alpha_i, \beta)$. Suppose $h^{\min}(\alpha_i, \beta_i) = 0$, for any β_i, α_i. Then $h^{\min}(\alpha_i, \beta_i)x_i(\beta) = 0 < v_i(\alpha_i, \beta) - v_i(\alpha_i, \alpha_i, \beta_{-i})$, contradicts with the h-SP. So $h^{\min}$ is nonzero. ■

Proof of Theorem 2

Proof.

i. To prove there does not exist SP mechanism, such that it is FBE.

Assume mechanism $\phi = (x(\cdot), s(\cdot))$ is SP. Given the allocation $x(\beta)$, the sharing rates $s(\beta)$ of SP mechanism satisfies $s_i(\beta, \mathcal{M}) = \beta_i(\mathcal{M}) - \frac{\int_0^{\beta_i(\mathcal{M})} \hat{x}_i(t,\beta_{-i})dt}{\hat{x}_i(\beta_i(\mathcal{M}),\beta_{-i})}$ from Theorem 1.

Then aggregate resource transmitted to agents is $y(\mathcal{M}) = \sum_{j=1}^{M} y_j$. ϕ is strategy proof, intermediaries will report truthfully, $\beta = \alpha$. The aggregate resource allocated to agents equals $y(\mathcal{M}) = \sum_{i=1}^{N} s_i(\alpha, \mathcal{M})\hat{x}_i(\alpha_i(\mathcal{M}), \alpha_{-i}) = \sum_{i=1}^{N} \alpha_i(\mathcal{M})\hat{x}_i(\alpha_i(\mathcal{M}), \alpha_{-i}) - \sum_{i=1}^{N} \int_0^{\alpha_i(\mathcal{M})} \hat{x}_i(t, \alpha_{-i})dt$.

Given mechanism ϕ, resource allocation is $y = \sum_{i=1}^{N} s_i(\alpha)x_i(\alpha)$. By definition of FBE, $\bar{u} = \max_x u(\sum_{i=1}^{N} x_i\alpha_i)$.

To prove, there exists α, such that $\sum_{i=1}^{N} \int_0^{\alpha_i(\mathcal{M})} \hat{x}_i(t, \alpha_{-i})dt > 0$. Mechanism ϕ is budget balance, $\sum_{i=1}^{N} x_i(\alpha) = 1$. Assume $x_i(\tilde{\alpha}) > 0$, then let $\alpha_{-i} = \tilde{\alpha}_{-i}$ and $\alpha_i > \tilde{\alpha}_i$. $\hat{x}_i(t, \alpha_{-i})$ is monotonic in t, so for any $t > \tilde{\alpha}_i(\mathcal{M})$, $\hat{x}_i(t, \alpha_{-i}) > 0$, thus, $\sum_{i=1}^{N} \int_0^{\alpha_i(\mathcal{M})} \hat{x}_i(t, \alpha_{-i})dt > 0$. The aggregate resource allocated to agents under mechanism ϕ is $y(\mathcal{M}) < \sum_{i=1}^{N} \alpha_i(\mathcal{M})\hat{x}_i(\alpha_i(\mathcal{M}), \alpha_{-i})$, when quality of intermediation is α.

By definition of maximal utility $\bar{u}(\alpha)$, $\bar{u}(\alpha) \geq u(\sum_{i=1}^{N} \alpha_i\hat{x}_i(\alpha_i(\mathcal{M}), \alpha_{-i})) > u(y)$. The first inequality is from definition of maximal utility $\bar{u}$, and the second inequality comes from strongly monotone of preferences.

So there is no symmetric, SP, budget balance, and FBE mechanism.

ii. Consider ϕ is FBE mechanism. If the quality of intermediation is α, $\beta = \alpha$, the intermediaries report truthfully about their quality of intermediation. The first best allocation of resource is $\bar{x}(\alpha)$. To achieve the fist best efficient, the sharing rates s_i should equal to the true quality of intermediation α_i if intermediary i is used by the planner, which means $s_i(\beta) = \beta_i$, if $x_i(\beta) > 0$.

Assume the final allocation of resource to agents that maximizes planner's utility is $\bar{y}(\alpha) = \sum_{i=1}^{N} \bar{x}_i(\alpha)\alpha_i$ maximizes planner's utility. Suppose there exists i, sharing rates $s_i(\beta) \leq \beta_i = \alpha_i$, $s_i \neq \beta_i$ and $x_i(\beta) > 0$, then $y = \sum_{i=1}^{N} \bar{x}_i(\alpha)s_i(\beta) \leq \bar{y}(\alpha)$ and $y \neq \bar{y}(\alpha)$. If the preferences of planner is strongly monotone, then $u(y) < u(\bar{y}(\alpha))$, the fist best efficient will not be achieved. For FBE mechanism ϕ, the sharing rates $s_i(\beta) = \beta_i$, if $x_i(\alpha) > 0$, $\beta = \alpha$.

If $x_i(\alpha_i, \beta_{-i}) = 0$, $(\alpha_i(\mathcal{M}) - s_i(\alpha_i, \beta_{-i}, \mathcal{M}))x_i(\alpha_i, \beta_{-i}) = 0$. If $x_i(\alpha_i, \beta_{-i}) > 0$, there is $s_i(\alpha_i, \beta_{-i}, \mathcal{M}) = \alpha_i(\mathcal{M})$. So $v_i(\alpha_i, \alpha_i, \beta_{-i}) = (\alpha_i(\mathcal{M}) - s_i(\alpha_i, \beta_{-i}, \mathcal{M}))x_i(\alpha_i, \beta_{-i}) = 0$. And $v_i(\alpha_i, \beta) = (\alpha_i(\mathcal{M}) - s_i(\beta, \mathcal{M}))x_i(\beta) = (\alpha_i(\mathcal{M}) - \beta_i(\mathcal{M}))x_i(\beta)$.

Then $v_i(\alpha_i, \beta) - v_i(\alpha_i, \alpha_i, \beta_{-i}) = (\alpha_i(\mathcal{M}) - \beta_i(\mathcal{M}))x_i(\beta)$.

The minimal punishment function $h^{\min} = \max_{\beta_{-i}}[\frac{v_i(\alpha_i,\beta) - v_i(\alpha_i,\alpha_i,\beta_{-i})}{x_i(\beta)}] = \max_{\beta_{-i}}[\alpha_i(\mathcal{M}) - \beta_i(\mathcal{M})] = \alpha_i(\mathcal{M}) - \beta_i(\mathcal{M})$.

From Proposition 3, any punishment function h that implements a FBE mechanism if and only if $h(\alpha_i, \beta_i) \geq h^{\min}(\alpha_i, \beta_i) = (\alpha_i(\mathcal{M}) - \beta_i(\mathcal{M}))_+$ for any $\alpha_i \geq \beta_i$. ■

Proof of Theorem 3

Proof. Consider an arbitrary mechanism ϕ that is h-optimal, and always allocates the full resource to the intermediary with the highest aggregate intermediation quality, so $x = x_F = x_S$. Without loss of generality, assume that $\alpha_1(\mathcal{M}) \geq \alpha_2(\mathcal{M}) \geq \cdots \geq \alpha_n(\mathcal{M})$. $\phi = (x, s)$, $x_1(\alpha) = 1$, $s_1(\alpha, \mathcal{M}) \leq \alpha_1(\mathcal{M})$.

The second price mechanism ϕ_S is strategy-proof, and the resource will be transmitted at the rate $\alpha_2(\mathcal{M})$. So the h-optimal mechanism will transmit no less than $\alpha_2(\mathcal{M})$, $s_1(\alpha, \mathcal{M}) \geq \alpha_2(\mathcal{M})$.

Since the optimal aggregate transmission rate $\alpha_2(\mathcal{M}) \leq s_1(\alpha, \mathcal{M}) \leq \alpha_1(\mathcal{M})$, there exists $t(\alpha) \in [0, 1]$, s.t. $s_1(\alpha, \mathcal{M}) = t(\alpha)\alpha_1(\mathcal{M}) + (1 - t(\alpha))\alpha_2(\mathcal{M})$. The mechanism ϕ can be written as $\phi(\alpha) = t(\alpha)\phi_F(\alpha) + (1 - t(\alpha))\phi_S(\alpha)$, for each α.

By h-SP,

$$\alpha_1(\mathcal{M}) - (t(\alpha)\alpha_1(\mathcal{M}) + (1-t(\alpha))\alpha_2(\mathcal{M})) \geq \alpha_1(\mathcal{M}) - (t(\alpha)\beta_1(\mathcal{M}) + (1-t(\alpha))\alpha_2(\mathcal{M})) - h(\alpha_1, \beta_1).$$

Thus, $t(\alpha) \leq \frac{h(\alpha_1,\beta_1)}{\alpha_1(\mathcal{M}) - \beta_1(\mathcal{M})}$ for any $\alpha_1 \geq \beta_1 \geq \alpha_2$.

Hence, $t(\alpha) \leq t^h(\alpha) = \min_{\alpha_1(\mathcal{M}) \geq \gamma(\mathcal{M}) \geq \alpha_2(\mathcal{M})} \frac{h(\alpha_1,\gamma)}{\alpha_1(\mathcal{M}) - \gamma(\mathcal{M})}$. The mechanism ϕ^h transmit no less than any h-SP mechanism allocating to intermediary with highest aggregate intermediation quality.

Moreover, ϕ^h is h-SP and allocates to intermediary with highest aggregate intermediation quality. Then ϕ^h is h-optimal among mechanisms allocate to intermediary with highest aggregate intermediation quality. ■

Proof of Corollary 3

Proof.

i. Without loss of generality, assume that $\alpha_1(\mathcal{M}) \geq \alpha_2(\mathcal{M}) \geq \cdots \geq \alpha_n(\mathcal{M})$. Assume $\phi = (x, s)$ is h^t-optimal among mechanisms with $x_1(\alpha) = 1$.

The second price mechanism ϕ_S is h^t-SP, and the resource will be transmitted at the rate $\alpha_2(\mathcal{M})$. So the h^t-optimal mechanism will transmit no less than $\alpha_2(\mathcal{M})$, $s_1(\alpha, \mathcal{M}) \geq \alpha_2(\mathcal{M})$. At the same time, $s_1(\alpha, \mathcal{M}) \leq \alpha_1(\mathcal{M})$. There exists $t(\alpha) \in [0, 1]$, s.t. $s_1(\alpha, \mathcal{M}) = t(\alpha)\alpha_1(\mathcal{M}) + (1 - t(\alpha))\alpha_2(\mathcal{M})$. The mechanism ϕ can be written as $\phi(\alpha) = t(\alpha)\phi_F(\alpha) + (1 - t(\alpha))\phi_S(\alpha)$, for each α.

By h-SP,

$$\alpha_1(\mathcal{M})-(t(\alpha)\alpha_1(\mathcal{M})+(1-t(\alpha))\alpha_2(\mathcal{M})) \geq \alpha_1(\mathcal{M})-(t(\alpha)\beta_1(\mathcal{M})+(1-t(\alpha))\alpha_2(\mathcal{M}))-h^t(\alpha_1, \beta_1).$$

Thus, $t(\alpha) \leq \frac{h(\alpha_1,\beta_1)}{\alpha_1(\mathcal{M})-\beta_1(\mathcal{M})} = t$ for any $\alpha_1 \geq \beta_1 \geq \alpha_2$.

Hence, $t(\alpha) \leq t = \min_{\alpha_1 \geq \gamma \geq \alpha_2} \frac{h(\alpha_1,\gamma)}{\alpha_1 - \gamma}$. The mechanism ϕ^t transmit no less than any h-SP mechanism allocating to intermediary with highest aggregate intermediation quality.

Moreover, ϕ^t is h^t-SP and allocates to intermediary with highest aggregate intermediation quality. Then ϕ^t is h^t-optimal among mechanisms allocate to intermediary with highest aggregate intermediation quality.

ii. By proposition 3, the punishment function h, which guarantees mechanism ϕ^t to be h-SP, satisfies: $h \geq h^{\min}(\alpha_i, \beta_i) = \max_{\beta_{-i}}[\frac{v_i(\alpha_i,\beta)-v_i(\alpha_i,\alpha_i,\beta_{-i})}{x_i(\beta)}]$, for any $\beta_i, \alpha_i, \beta_{-i}$, such that $x_i(\beta) > 0$. For β_{-i}, with $\alpha_i(\mathcal{M}) \geq \beta_i(\mathcal{M}) \geq \max_{-i} \beta_{-i}(\mathcal{M})$, $\frac{v_i(\alpha_i,\beta)-v_i(\alpha_i,\alpha_i,\beta_{-i})}{x_i(\beta)} = t|\alpha_i(\mathcal{M}) - \beta_i(\mathcal{M})|$. So h^t is the minimal punishment function of the mechanism ϕ^t. ■

Proof of Corollary 5

Proof.

From Example 2, the minimal punishment function for the first price mechanism is $h(a, b) = \max\{a(\mathcal{M}) - b(\mathcal{M}), 0\}$. ■

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