Sharing sequential values in a network*

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Abstract

Consider a sequential process where agents have individual values at every possible step. A planner is in charge of selecting steps and distributing the accumulated aggregate values among a number of agents. We model this process by a directed network, whereby each edge is associated with a vector of individual values. This model applies to several new and existing problems, e.g. developing a connected public facility and distributing total values received by surrounding districts, selecting a long-term production project and sharing final profits among partners of a firm, or choosing a machine schedule to serve different tasks and distributing total benefits among task owners.

Herein, we provide the first axiomatic study on path selection and value-sharing in networks. We consider four sets of axioms from different perspectives, including those related to (1) the sequential consistency of value-sharing; (2) the monotonicity of value-sharing with respect to technology improvements; (3) the independence of value-sharing with respect to certain network transformations; and (4) the robust implementation of the efficient path selection when the planner has no information about network configuration. Surprisingly, these four disparate sets of axioms characterize similar classes of solutions, namely selecting an efficient path(s) and assigning to each agent a share of total values that is independent of individual values. Furthermore, we characterize more general solutions that depend on individual values.

Keywords: *Sequential values, Sharing, Network, Redistribution*

***JEL* classification:** C72, D44, D71, D82.

1 Introduction

The axiomatic division of costs and benefits has been studied extensively over the past 60 years, starting with bargaining (Nash [55]) and cooperative games (Shapley [58]), and followed by applications to problems such as rationing and bankruptcy (O'Neill [57], Aumann and Maschler [3], Thomson [62], Moulin [50, 51]), airport cost-sharing (Littlechild and Owen [43], Thomson [64]), hierarchical ventures (Hougaard et al. [27]), and more general cost-sharing problems (e.g., Sprumont [59, 60], Moulin [49], Friedman and Moulin [23], Moulin and Sprumont [53], Moulin and Shenker [52]). Such studies have characterized a wide variety of sharing rules using axioms motivated by positive and normative perspectives. However, they are largely limited to scenarios with a fixed resource and little is known regarding scenarios that are more general in two respects: (1) the amount of the resource may not be fixed but can be chosen, and (2) the resource may be generated in a sequence of steps, where the amount in future steps depends on the choices made in previous steps. Such a dynamic problem requires resource-generating steps to be determined together with the allocation. This "two-tiered" approach not only expands the range of problems, but also gives rise to a new question on the interdependence of the step selection and sharing rules.

To illustrate our problem better, consider a planner in charge of developing a connected public facility (e.g. a highway, a railroads, or an irrigation canal). The project might be developed in different steps, each of which might produce different benefits to the agents in a given society. The planner is in charge of choosing the steps and redistributing the benefits of the project among the agents. After proceeding along each step, the planner faces a new problem which is different from the original one and might depend on the preceding steps (Section 1.1 discusses other applications).

Formally, a finite number of agents are facing a sequential process generating individual values that can be redistributed among themselves. A sequential process, or simply **process**, consists of a network and a value function associated with the network. A network is an acyclic and connected finite directed graph with a unique source and possibly multiple sinks. Each node in the network represents a stage at which (1) a step to continue should be chosen (except for a sink) and (2) individual values generated until then could be redistributed. Each edge in the network represents a feasible step to continue. A value function associated with the network assigns to each edge a value vector that specifies for each agent his individual value generated at the step.

Given a process, a planner is in charge of (1) selecting a way of proceeding from the source to a sink which is called a **path**, and (2) redistributing the sum of individual values accumulated over all edges along the path which is called **the value of the path**. A **solution** consists of a path selection rule and a sharing rule which respectively recommends for each process a set of paths with the same value and an allocation of the value among agents.

We provide the first systematic and comprehensive study of this problem by considering axioms appropriate to a wide range of scenarios. Surprisingly, our four sets of axioms from different perspectives characterize similar classes of solutions — selecting the path(s) with the highest value (hereafter referred to as the efficient path(s)) and assigning to each agent a share of the value of the path(s) that is independent of network configuration and individual values. Moreover, we show the richness of suitable solutions in different scenarios; for instance, we characterize a large class of solutions that depend on individual values in a "rationalizable" way. For an overview of our results, see Section 1.2.

1.1 Applications and Solutions

Our model can be applied to the following situations.

Sharing benefits of connected public facilities. Imagine a planner who is in charge of selecting a route for a connected public facility such as highway, rail-road or irrigation canal. The set of potential routes can be modeled by a network in which each node is a geographical end point of a location, and each edge represents a feasible section of the public facility connecting two locations. Agents are the surrounding districts and they receive different individual benefits at an edge depending on the convenience of their access to the section of the public facility represented by the edge. Imagine that the construction of the connected public facility is conducted section by section. When the construction of the facility is completed up to a certain location represented by a node, a further section connecting to the next feasible location needs to be chosen and the benefits brought by the constructed sections can be redistributed among agents.

Profit sharing in companies and joint-business ventures. Imagine that a manager can complete a project by having employees execute a sequence of profit-generating tasks. Different tasks bring different individual profits to employees. The manager is in charge of choosing a sequence of tasks and redistributing profits among employees. The set of possible sequences of tasks can be modeled by a network. Each node represents a stage at which a previous task is completed, the next task needs to be chosen, and individual profits generated by the completion of all previous tasks can be redistributed. Each edge represents a feasible task. Some tasks may be feasible only after the completion of some other tasks, so the choice of earlier tasks give rise to different further tasks.

Sharing benefits in dynamic machine allocation. Imagine a planner who is in charge of allocating a set of machines among a group of users repeatedly over time. At each point in time, the planner assigns to each user a subset of machines for him to use and decides the length of service time that is assumed to be uniform for all users for simplicity. When the service time ends, all users return their machines. Then the planner may redistribute individual benefits of users generated from using their machines, and reallocate the machines among users for another service period forward. The set of possible allocations can be modeled by a network. Each node represents a point in time where machines are to be reallocated, and individual benefits generated in earlier periods can be redistributed. Each edge represents a feasible allocation of machines among users from a certain point in time to another. Individual benefits of users are generated at each edge by the service of their machines. A queuing problem (Chun [16]), where there is only one machine serving only one agent at a time, can be modeled by a particular network with parallel paths representing different service orders.

1.1.1 Solutions

For the public facilities process, the government typically selects the efficient path, and performs no redistributions so that each agent receives his individual benefits. This efficient-path-selection and no-redistribution solution is denoted by **EFF-NR**.[1] Another solution could have been for the government to select the efficient path and assign to each agent an equal share of the total benefits (denoted by **EFF-ES**). Such redistribution can be achieved via a lump-sum tax and subsidy on the agents.

For the case of profit-sharing in companies, agents are often rewarded with bonuses that are tied to the profits that they have generated.[2] EFF-NR is a typical solution that rewards agents for the profit they have contributed. Alternatively, EFF-ES rewards each agent an equal share of the total profit. [3]

[1]When there are multiple efficient paths, we do an average of all the efficient paths.

[2]For partnerships between professionals, such as a group of lawyers in a law firm, redistribution among partners is typically 100% of the total profit (Juarez, Nitta and Vargas [39]). Our model can also be applied to the case of other for-profit companies, where employees are rewarded with a fixed percentage of the total profit, such as Chobani which commits to redistribute 10% of the profits to its employees.

[3]Equal sharing is often used in professional partnership (Encinosa et al. [21], and Farrell and Sctochmer

For the case of dynamic machine allocation, myopically selecting an assignment at each point in time that maximize the total benefits over all allocations available at that point in time is simple to implement, especially for problems with a large number of agents. For instance, supercomputers are allocated using the available information at a particular point in time, disregarding information at subsequent points in time. Selecting the myopic path with no redistribution among agents is denoted by **MYO-NR**, while selecting the myopic path with equal sharing is denoted by **MYO-ES**.

All four solution above divide the value of the path disregarding the information of the values outside the selected path(s). Our model, however, allows for more general solutions that may take into consideration values outside the selected path(s). For instance, consider the rule that selects the efficient path and re-distributes its value in proportion to the sum of individual values at every edge in a process (**EFF-PR**). This solution may be relevant for a planner who wants to compensate the agents for opportunities that were not realized.

1.2 Overview of the Results

We study two versions of the problem in relation to the planner's information. For the first part of the paper, the planner has complete information about the process. The planner is interested in systematically selecting a path(s) and sharing the value of the selected path(s). We provide three axiomatic characterizations in this case.

Our first characterization relates to the independence of the timing of redistribution. Loosely speaking, given any node of a selected path, agents could be paid first based on the subprocess from the source to the node and then based on the other subprocess from the node to the original sinks. We require that the two allocations from the two subprocesses add up to the allocation chosen for the original process. Hence, agents are indifferent between receiving a lump-sum payment at the end of the process or receiving installments step by step (**sequential composition**). This rules out renegotiations of agents at intermediate stages. Besides, we impose two basic axioms. First, in each process with a path of positive value, at least one agent should receive a positive share (**non-triviality**). Second, a small change in the individual values should have a small impact on the allocation (**continuity**). These three axioms characterize the class of solutions that selects efficient path(s) and assigns to each agent a proportion of the value of the selected path(s) where the proportion is constant over all processes. For example, in a two-agent case, for each process, 10% is always assigned to

[22]). See Bartling and von Siemens [6], Bose et al. [12], and Kobayashi et al. [42] for justifications under various situations.

agent 1, and 90% to agent 2.

Our second characterization imposes a monotonicity axiom. It requires that no agent should get worse off after a technology improvement which brings a new value-generating step at some stage in a process leading to either another stage in the process or a new ending stage (**technology monotonicity**). This single axiom characterizes the class of solutions that select efficient path(s) and assigns to each agent a proportion of the value of the selected path(s) where the proportion depends only on this value. For example, in a two-agent case, for each process, equal sharing between agents 1 and 2 whenever the total value is no less than 100, and for any incremental value above 100, 10% is assigned to agent 1 and 90% to agent 2.

Our third characterization relates to several independence principles with respect to certain network transformations. Our first independence axiom relates to splitting a step in a process into two consecutive sub-steps. Imagine that due to a refined examination of the value-generating process, individual values generated at one step in a process are found to be accumulated through two consecutive substeps. Thus, the original step can be divided into two, and at the end of the first one, a new stage at which individual values can be redistributed becomes available. The axiom requires that the refinement of the value-generating process have no impact on value allocation, so that agents would not dispute whether such a refinement should be adopted (**split invariance**). Our second axiom relates to path elimination justified by a basic efficiency criterion. Consider a process with two subnetworks intersecting only at the source. Each path in one subnetwork is step-wise Pareto dominated by some other path in the other subnetwork. Then removing this subnetwork from the network should not affect the allocation (**irrelevance of dominated paths**). Our third axiom supposes that after solving a process, an undiscovered disjoint subprocess connecting to the source is found to be available. To deal with this issue, one procedure is to cancel the initial allocation and select an allocation from the complete process that augments the original process with the new subprocess. An alternative procedure is to select an allocation from the simplified process that augments an edge associated with the initial allocation with the new subprocess. We require that the two procedures lead to the same allocation to avoid the dispute of agents (**parallel composition**). The three axioms, together with continuity, characterize a general class of "rationalizable" solutions. A planner who adopts such a solution selects an efficient path(s) and divides the value in two steps. In each process, the planner first redistributes the value of each path based on individual values at the path. This gives a set of potential redistributions. Second, the planner selects an optimal allocation based on the set of potential redistributions according to a partial order. To understand the partial order, imagine that the planner has some selection criteria such as Pareto dominance and "fairness". If there is one potential redistribution that dominates every other potential redistribution in the set by some criterion, then the planner selects this potential redistribution (the maximum redistribution). Otherwise, the planner selects one outside allocation that dominates each potential redistribution in the set and has the minimum "departure" from the set (the least upper bound). Thus, the sharing rule is rationalizable by this partial order. This order is incomplete since some redistributions may not be comparable by either criterion. When the partial order is complete, the sharing depends only on the value of the selected path(s). This subclass of rules is characterized by an additional axiom. Loosely speaking, it requires that in each process with a generic "parallel network," the allocation depends only on the selected path (**irrelevance of parallel outside options**). The solutions obtained in the first two characterizations are special cases of this subclass.

Although the classes of solutions above are characterized from three different perspectives, they all reduce to EFF-ES if we further impose a basic fairness requirement.[4] That is, agents having the same individual values should receive the same shares (**equal treatment of equals**).

For the second part of the paper, we relax the assumption that a planner has complete information about a process including network configuration and individual values of agents. Instead, we consider a situation in which agents but the planner have complete information about a process, and the planner can only observe the generated individual values along a realized path.

In this incomplete information setting, we assume that the planner's objective is to implement an efficient path, which is well defined since agents have complete information. We depart from the traditional literature and assume that the planner has **limited** power in designing such a game. In particular, we consider a situation in which agents collectively select a path through some **fixed** procedure, and the only way for the planner to influence agents' choices is to redistribute their individual values realized along the chosen path. We characterize axiomatically a class of sharing rules that incentivize the agents to collectively select an efficient path for a large class of voting procedures.

Our first axiom requires that, for any two paths, the sharing rule assign weakly larger shares to at least M agents at the path with a larger value (***M*-majority**), where M is larger than half the number of agents. This is a stability notion since it guarantees an efficient path

[4]In the third characterization, the subclass of rationalizable solutions satisfying irrelevance of parallel outside options, rather than the general class, reduces to EFF-ES.

to be a Condorcet winner when agents vote for paths. Moreover, when agents sequentially vote at each node for edges to continue, it guarantees an efficient path to be chosen as a subgame perfect equilibrium under the M-majority rule. Second, we require that the identity of agents should not matter (**anonymity**). These two axioms imply that a sharing rule assigns the average value of a path to at least M agents. Furthermore, the equal sharing rule is characterized by adding one of several axioms ranging from sequential composition to other monotonicity axioms.

1.3 Literature review

While the axiomatic study of sharing rules has been widely discussed and applied in many settings, our general two-tiered framework, which selects the path along with the sharing rule, has not received much attention in the literature. Our model provides an abstract framework for more stylistic two-step problems such as the queuing problem (Chun [16]), the minimal cost spanning tree (Kar [41], Dutta and Kar [19], Bergantiños and Vidal-Puga [8], Hougaard et al. [28], Claus and Kleitman [17]) and other cost-sharing models (Juarez [36], Juarez [37], Juarez and Kumar [38]). In such problems, an ordering of agents (queuing), a network meeting certain conditions (spanning tree), or other decisions (selection of a group or a path) must be made and its benefit/cost divided among agents. In contrast with this literature, we do not assume that the most efficient path (subnetwork, subgraph, ordering or group of agents) is selected, but instead its selection is axiomatized along with a sharing rule.

The division of benefits/costs under exogenous network structures has been recently studied. For instance, allocations in linear river problems are studied by Ambec and Sprumont [2], Ni and Wang [56], and Ambec and Ehlers [1]. More complex river network problems are studied by van den Brink et al. [11], and Dong et al. [18]. The allocation of benefits in hierarchical ventures is studied by Hougaard et al. [27]. Values of cooperative game under permissible structures are studied by van den Brink [9], van den Brink et al. [10] and Gilles et al. [25]. Unlike our paper, this literature does not study the selection of the path, but instead assume that it is given.

The second part of the paper relates to the recent literature on the implementation of the efficient subgraphs in networks. For instance, Juarez and Kumar [38] implement the efficient subgraph in connection networks, Hougaard and Tvede [29, 30] implement the minimal cost spanning tree, and Juarez and Nitta [39] implement the efficient time allocation in production economies. Similar to our paper, the main objective of this body of literature is to select an "efficient path". The main difference is that we adopt an axiomatic approach that works for a variety of games, including sequential voting for a path.

Our model is the first to address jointly the issue of selecting paths and sharing the total value axiomatically for sequential processes where information on the individual values of agents is available at every step.

2 The model under complete information

A group of agents, $\mathcal{N} := \{1, ..., N\}$, is facing a "sequential process" that generates individual values which can be redistributed among themselves. We model a sequential process using a network and a value function associated with the network. In this paper, a **network** is a finite directed graph that is acyclic and connected and has a unique source (possibly multiple sinks). We denote a network by $G := (D, E)$, where D is a set of nodes and $E \subseteq D \times D$ a set of edges, a generic element of D by d, and that of E by either e or (d_1, d_2), where $d_1, d_2 \in D$. Let $\mathcal{G}$ be the set of all networks. Given $G = (D, E) \in \mathcal{G}$, a **value function** $v : E \to \mathbb{R}^N_+$ associated with G specifies for each $e \in E$ a value vector $v(e)$ whose n-th coordinate, $v_n(e)$, is agent n's value at edge e. For each $G \in \mathcal{G}$, let $\mathcal{V}^G$ be the set of all value functions associated with G. We define a sequential process, or simply a **process**, to be a couple (G, v) where $G \in \mathcal{G}$ and $v \in \mathcal{V}^G$. It represents a series of value-generating choices. Specifically, each node in a network is a stage at which a step to continue should be chosen (except for a sink) and individual values accumulated until then could be redistributed, each edge is a feasible step to continue, and a value function specifies at each feasible step a value generated for each agent. Let $\mathcal{P}$ be the set of all processes.

To facilitate the discussion about the aggregate values generated by different choices in a process, we need some notations. Given a network $G = (D, E) \in \mathcal{G}$, a **path** in G is a sequence $\{d_j\}_{j=1}^J$, $J \geq 2$, of elements of D such that d_1 is the source, d_J a sink, and for each $j \in \{1, \cdots, J-1\}$, $(d_j, d_{j+1}) \in E$ a feasible edge.[5] It describes a way of proceeding from the source to a sink. Alternatively, given a network $G = (D, E) \in \mathcal{G}$, a path in G can be defined as a sequence $\{e_j\}_{j=1}^J$, $J \geq 1$, of elements of E such that there is a sequence $\{d_j\}_{j=1}^{J+1}$ of elements of D with d_1 being the source, d_{J+1} a sink, and for each $j \in \{1, \cdots, J\}$, $e_j = (d_j, d_{j+1})$. We will interchangeably refer to a path as a sequence of nodes or a sequence of edges. For each $G \in \mathcal{G}$, let $\mathcal{L}^G$ denote the set of all paths in G, and let L be a generic element of $\mathcal{L}^G$. Given a process $(G, v) \in \mathcal{P}$ with $G = (D, E)$, for each $e \in E$, we define $v_e := \sum_{n \in \mathcal{N}} v_n(e)$ to be the **value**

[5] The usual definition of a path does not require d_1 to be a source and d_J a sink. We impose the requirement since we are only interested in the paths going from the source to a sink.

of *e*, for each $L \in \mathcal{L}^G$ with $L = \{e_j\}_{j=1}^J$, $v_L := \sum_{j=1}^{J} v_{e_j}$ the **value of *L***, and $v_G := \max_{L \in \mathcal{L}^G} v_L$ the **value of *G***. In addition, we say that $L \in \mathcal{L}^G$ is **efficient** if $v_L = v_G$.

We are interested in systematic solutions on how to select paths and divide their values in all processes. Formally, a **solution** is a pair (φ, μ) where $\varphi : \mathcal{P} \rightrightarrows \bigcup_{G \in \mathcal{G}} \mathcal{L}^G$ and $\mu : \mathcal{P} \to \mathbb{R}^N_+$ are such that (1) for each $(G, v) \in \mathcal{P}$, $\varphi(G, v) \subseteq \mathcal{L}^G$ is nonempty, and (2) for each $L \in \varphi(G, v)$, $\sum_{n \in N} \mu_n(G, v) = v_L$, where $\mu_n(G, v)$, the n-th coordinate of $\mu(G, v)$, is the value assigned to agent n. We call φ a **path selection rule** and μ a **sharing rule**. For each process, φ recommends which path(s) to select, and μ recommends how to share the value among the agents. We allow multiple paths to be recommended for generality, while requiring them to have the same value (condition (2)). In a generic case where all paths have different values, a unique path would be recommended.[6]

Throughout the paper, we assume that nodes and edges have no identity, so our solutions do not depend on their labels. Moreover, for each process with only one edge, we shall use the value vector associated with the edge to denote the process for simplicity — for each $x \in \mathbb{R}^N_+$, let $\varphi(x), \mu(x)$ denote, respectively, $\varphi(G, v), \mu(G, v)$ where $(G, v) \in \mathcal{P}$ with $G = (D, E)$ is such that $E = \{e\}$ and $v(e) = x$. Lastly, for each $c \in \mathbb{R}$, let $c\mathbf{1}$ denote the constant vector $(c, \cdots, c) \in \mathbb{R}^N_+$, and when $c = 1$ or $c = 0$, we simply write $\mathbf{1}$ or $\mathbf{0}$ respectively.

Both network configurations and value-generation dynamics may matter for a solution, as illustrated in Examples 1 and 2. Moreover, a path selection rule and a sharing rule can be interdependent, as illustrated in Example 3.

Example 1. *[Consecutive edges may not be treated as a collapsed edge.]*

Imagine three public projects with two agents modeled as single-path processes shown in Figure 1. On one hand, the projects differ in either the number of phases represented by edges, or individual values generated at each phase shown by the value vector attached above each edge. On the other hand, all the projects generate for both agents the same total individual values of 2 *and* 8 *respectively, so that collapsing all edges to one in the later two processes yields exactly the first one.*

Imagine a progressive planner, redistributing individual values phase by phase, who wants to exempt small individual values from redistribution. For example, at the end of each phase of a project, the part of individual value that is below 10% of the project's value is awarded entirely to each agent, whereas any excess is divided equally between the two. This

[6] In case multiple paths are recommended, condition (2) implicitly requires that the sharing rule is path-independent. This requirement makes sharing simple. It is also reasonable when the choice of a path from the recommended ones is made by some outside force that should have no influence on sharing.

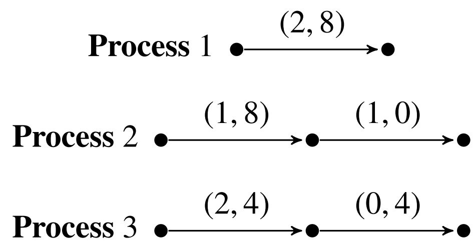

Figure 1: Three public projects modeled as three processes

sharing rule yields different value allocations among agents for different projects. Indeed, the allocation for process 1 is $(1+\frac{2-1+8-1}{2}, 1+\frac{2-1+8-1}{2}) = (5,5)$*, for process 2* $(1+\frac{8-1}{2}, 1+\frac{8-1}{2})+(1,0) = (5.5, 4.5)$*, and for process 3* $(1+\frac{2-1+4-1}{2}, 1+\frac{2-1+4-1}{2})+(0+\frac{4-1}{2}, 1+\frac{4-1}{2}) = (4.5, 5.5)$.

Example 2. *[The solution may depend on network configuration.]*

Consider two sets of possible routes for a connected public facility modeled by the processes in Figure 2. Observe that both processes have two routes, generating the same individual values at each section of the connected public facility. The difference is that the two routes in the left process share a common section, while the two routes in the right process do not intersect. Even if the same route is selected in both processes, the difference in network configuration may lead to different redistribution outcomes.

For instance, a social planner who decides to select an efficient route may be willing to redistribute benefits taking into account unselected opportunities. Specifically, the planner may assign to each agent a share in proportion to the average of his individual benefits over all potential sections of the public facility. Then the allocation in the left process is $(\frac{10+10+10+40+0}{71} \times 61, \frac{10+10+10+0+1}{71} \times 61) = (\frac{70}{101} \times 61, \frac{31}{101} \times 61)$*, and the allocation in the right process is* $(\frac{10+10+10+40+0+10}{121} \times 61, \frac{10+10+10+0+1+10}{71} \times 61) = (\frac{80}{121} \times 61, \frac{41}{121} \times 61)$.

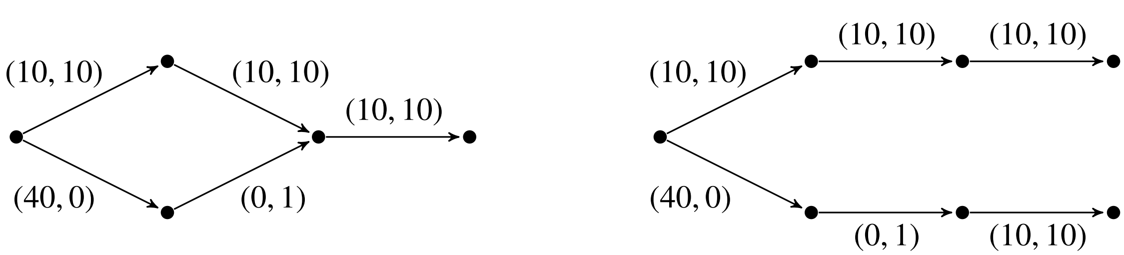

Figure 2: Two processes with identical paths and different network configuration

Example 3. *[Path selection and sharing can be interdependent.]*

Consider the process in Figure 3. Imagine two planners both of whom are both interested in equalizing agents' benefits, while they have different opinions on sharing. The first planner is more libertarian and would like to equalize as much as possible agents' shares without making any redistribution among the agents. The planner would then select the top path and allocate $(10, 10) + (10, 10) = (20, 20)$ *although the top path is not efficient. The second planner is more parsimonious and is willing to equalize agents' benefits with all means of redistribution. The planner would then select the bottom (efficient) path and allocate* $\frac{(40,0)+(0,1)}{2} = (20.5, 20.5)$*. This shows that the choice of a sharing rule may well affect the path selection.*

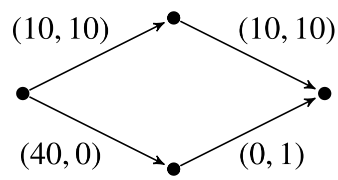

Figure 3: A public project with two paths and two different processes on each path.

We now introduce two parametric families of path selection rules followed by four familiar sharing rules in the literature.

Definition 1. *[Path selection rules] Let* $u : \mathbb{R}^N_+ \to \mathbb{R}$ *be an arbitrary function. For each value vector* $x \in \mathbb{R}^N$, $u(x)$ *is interpreted as a planner's utility of choosing a step that generates* x *in a process. Let* $u^0 : \mathbb{R}^N_+ \to \mathbb{R}$ *be such that for each* $x \in \mathbb{R}^N_+$, $u^0(x) = \sum_{n \in \mathcal{N}} x_n$.

i. [Additively separable] For each $(G, v) \in \mathcal{P}$*, let a preference ordering* $\succsim_u$ *over* $\mathcal{L}^G$ *be such that for each pair* $L, L' \in \mathcal{L}^G$ *with* $L = \{e_j\}_{j=1}^J$ *and* $L' = \{e'_j\}_{j=1}^{J'}$, $L \succsim_u L' \Leftrightarrow \sum_{j=1}^{J} u(v(e_j)) \geq \sum_{j=1}^{J'} u(v(e'_j))$*. This preference ordering is additively separable across value vectors generated along paths.*

An ***additively separable path selection rule*** *associated with* $u : \mathbb{R}^N_+ \to \mathbb{R}$ *recommends for each* $(G, v) \in \mathcal{P}$ *a subset of* $\{L \in \mathcal{L}^G : \text{for each } L' \in \mathcal{L}^G, L \succsim_u L'\}$ *in which all paths have the same value. A planner who adopts such a rule selects the path(s) to maximize the sum of the utilities of all steps. In particular, selecting all efficient paths in each process is an additively separable path selection rule associated with* u^0*. We denote this rule by* ***EFF****.*

ii. *[Myopic] For each $(G, v) \in \mathcal{P}$, let a preference ordering $\succsim^u$ over $\mathcal{L}^G$ be such that for each pair $L, L' \in \mathcal{L}^G$ with $L = \{e_j\}_{j=1}^J$ and $L' = \{e'_j\}_{j=1}^{J'}$, $L \succsim^u L'$ if and only if for $z, z' \in \mathbb{R}_+^{\max\{J,J'\}}$ defined by*

$$z_j := \begin{cases} u(v(e_j)) & \text{if } j \in \{1, \cdots, J\}, \\ 0 & \text{if } j \in \{J+1, \cdots, \max\{J, J'\}\}, \end{cases}$$

and

$$z'_j := \begin{cases} u(v(e'_j)) & \text{if } j \in \{1, \ldots, J'\}, \\ 0 & \text{if } j \in \{J'+1, \cdots, \max\{J, J'\}\}, \end{cases}$$

either z lexicographically dominates z' or $z = z'$. This preference ordering ranks paths by myopically comparing the utilities of immediate steps, and only when immediate steps have the same utility, it compares the utilities of further steps.

*A **myopic path selection rule** associated with $u : \mathbb{R}_+^N \to \mathbb{R}$ recommends for each $(G, v) \in \mathcal{P}$ a subset of $\{L \in \mathcal{L}^G : \text{for each } L' \in \mathcal{L}^G, L \succsim^u L'\}$ in which all paths have the same value. A planner who adopts such a rule discounts future utilities lexicographically and selects the path(s) to maximize the immediate utility. In particular, selecting all paths that maximize the sum of individual values over all immediate steps lexicographically is a myopic path selection rule associated with u^0. We denote this rule by **MYO**.*

Definition 2. *[Sharing rules] Let (φ, μ) be a solution.*

i. *[Equal sharing] We call μ the equal sharing rule, denoted by **ES**, if for each $(G, v) \in \mathcal{P}$, $\mu(G, v) = \frac{v_L}{N}\boldsymbol{1}$ where $L \in \varphi(G, v)$. It divides the value of a selected path equally among agents.*

ii. *[No redistribution] We call μ the no redistribution sharing rule, denoted by **NR**, if for each $(G, v) \in \mathcal{P}$, $\mu(G, v) = \dfrac{1}{|\varphi(G, v)|} \sum\limits_{L \in \varphi(G,v)} x^L$ where for each $L \in \varphi(G, v)$ with $L = \{e_j\}_{j=1}^J$, $x^L \in \mathbb{R}_+^N$ is given by $x^L = \sum\limits_{j=1}^J v(e_j)$. It assigns to each agent his average total individual value generated along all selected paths. In a generic case where a unique path is selected, it simply assigns total individual values to agents without redistribution.*

*iii. [Proportional] We call μ the proportional sharing rule, denoted by **PR**, if for each $(G, v) \in \mathcal{P}$ with $G = (D, E)$, $\mu(G, v) = v_L \alpha$ where $L \in \varphi(G, v)$ and $\alpha \in [0, 1]^N$ is given by*

$$\alpha = \begin{cases} \dfrac{1}{\sum\limits_{e \in E} v_e} \sum\limits_{e \in E} v(e) & \text{if } \sum\limits_{e \in E} v_e > 0, \\ \boldsymbol{0} & \text{if } \sum\limits_{e \in E} v_e = 0. \end{cases}$$

It divides the value of the selected path(s) in proportion to the sum of individual values of agents over all feasible steps in a process.

It is readily evident that any combination of the path selection rules and sharing rules defined above constitutes a solution. We use EFF-ES to denote a combination of an EFF path selection rule and the ES sharing rule, and use EFF-NR, EFF-PR, MYO-ES, MYO-NR, and MYO-PR to denote analogous combinations.

To find desirable solutions, we adopt an axiomatic approach. We propose three different types of axioms on a solution (φ, μ) respectively from the perspectives of sequential value redistribution, technology improvement, and network transformations.

2.1 Sequential Composition

We start with two basic axioms before introducing the main one regarding sequential value redistribution. The first basic axiom states that each process with a positive value should positively benefit at least one agent. This is a minimal efficiency property ruling out that all agents get nothing when there is something to distribute.

Non-triviality: For each $(G, v) \in \mathcal{P}$ such that $v_G > 0$, there is $n \in \mathcal{N}$ such that $\mu_n(G, v) > 0$.

While *non-triviality* imposes a requirement on the sharing rule μ, it is equivalent to the requirement on the path selection rule φ that for each process with a positive value, φ select positive-value path(s). It is readily evident that *non-triviality* holds for a large class of solutions. For example, it holds when φ is either an additively separable or a myopic path selection rule associated with $u : \mathbb{R}^N_+ \to \mathbb{R}$ such that for each $x \gneq \mathbf{0}$, $u(x) > u(\mathbf{0})$ (see Definition 1). In particular, it holds for EFF-ES, EFF-NR, EFF-PR, MYO-ES, MYO-NR, and MYO-PR.

The second basic axiom is a continuity property, which states that in each process, small changes in individual values should have a small impact on the value allocation. Such small changes often happen due to measurement errors, and we therefore require that the solution is robust with respect to such errors.

Continuity: For each $(G, v) \in \mathcal{P}$ and a sequence $\{v^k\}_{k=1}^{\infty}$ of elements of $\mathcal{V}^G$, if for each e in G, $\lim_{k\to\infty} v^k(e) = v(e)$, then $\lim_{k\to\infty} \mu(G, v^k) = \mu(G, v)$.

Continuity is standard in the fair allocation literature, and in our model, it has strong implications for both the path selection rule and the sharing rule. Both MYO-ES and EFF-NR violate *continuity*, which can be seen by considering the processes in Figure 4. In the left process, for each $\epsilon > 0$, MYO-ES selects the top path and allocates $(2 + \frac{\epsilon}{2}, 2 + \frac{\epsilon}{2})$, while for each $\epsilon < 0$, it selects the bottom path and allocates $(1, 1)$. In the right process, for each $\epsilon > 0$, EFF-NR selects the top path and allocates $(2 + \epsilon, 0)$, and for each $\epsilon < 0$, it selects the lower path and allocates $(0, 2)$.

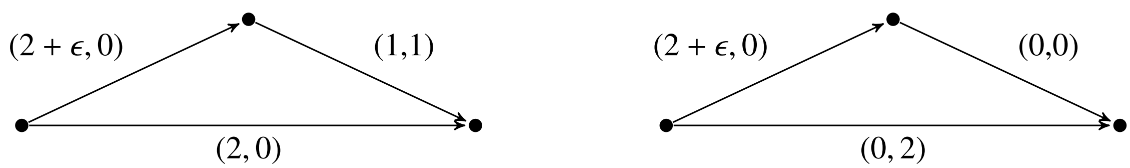

Figure 4: Processes illustrating that MYO-ES and EFF-NR violate *continuity*.

It can also be readily seen that MYO-NR and MYO-PR violate *continuity*. On the other hand, there is a large class of solutions satisfying *continuity*. The following example provides an interesting class of such solutions.

Example 4 (Solutions satisfying *continuity*)**.** *EFF-PR and EFF-ES satisfy continuity. More generally, there is a large class of path selection rules that in combination with the sharing rules PR or ES satisfy continuity. For instance, consider a path selection rule such that for each process with K paths, the path with the* $\gamma(K)$*-th largest value is selected, where* $\gamma(K) \in \{1, \ldots, K\}$ *for each* $K \in \mathbb{N}$*. When* $\gamma(K) = 1$ *for all K, an efficient path is selected. When* $\gamma(K) = K$ *for all K, a least efficient path is selected. When* $\gamma(K) = \lfloor \frac{K+1}{2} \rfloor$*,*[7] *a path with the median value is selected.*

Continuity is satisfied by either of these path selection rules and sharing rule PR (or ES), because the value of the $\gamma(K)$*-th largest path is a continuous function on the individual values of each process.*

[7] For each $c \in \mathbb{R}$, $\lfloor c \rfloor$ is the largest integer no more than c.

Our main axiom in this subsection relates to the independence of value redistribution timing. Imagine that a path involving at least two steps is selected in a process. The accounting practice may require that an interim payment be made in the middle of implementing the path, in which case agents receive two instalments, where the first of which is based on the "backward subprocess" from the beginning to the middle of the path, and the second one is based on the "forward subprocess" from the middle of the path to the end. The axiom requires that receiving a lump-sum payment at the end of the process be the same as receiving two instalments, and it rules out the possibility that some agents, after receiving their interim payments, initiate renegotiation for the forward subprocess in order to get better overall payments than initially agreed.

This requirement naturally arises in the applications discussed in the Introduction. For example, imagine that a planner selects the route of a connected public facility. The construction is usually done section by section, while the realized interim benefits are redistributed after each section of the public facility is completed. Similarly, imagine that a manager selects a sequence of profit-generating tasks. The realized interim profits are typically redistributed after the completion of each task. In either situation, *sequential composition* requires that interim redistribution be "dynamically consistent" with the initially planned redistribution.

To define subprocesses formally, let $(G, v) \in \mathcal{P}$ with $G = (D, E)$ and $d \in D$ be given such that d is neither the source nor a sink. Let $D|_d := \{d' \in D :$ there are $\{d_j\}_{j=1}^J \in \mathcal{L}^G$ and $j, j' \in \mathbb{N}$ with $1 \leq j' \leq j \leq J$ such that $d_{j'} = d'$ and $d_j = d\}$ be the set of nodes preceding d in some path going through d. Let $E|_d := E \cap (D|_d \times D|_d)$ be the set of feasible edges whose end points precede d. Let $G|_d := (D|_d, E|_d)$ be the maximum sub-network with d being the sink, and $v|_d : E|_d \to \mathbb{R}^N_+$ the restriction of v to $E|_d$. We call $(G|_d, v|_d)$ the **backward process** at d.

Analogously, Let $D|^d := \{d' \in D :$ there are $\{d_j\}_{j=1}^J \in \mathcal{L}^G$ and $j, j' \in \mathbb{N}$ with $1 \leq j \leq j' \leq J$ such that $d_{j'} = d'$ and $d_j = d\}$ be the set of nodes following d in some path going through d. Let $E|_d := E \cap (D|^d \times D|^d)$ be the set of feasible edges whose end points follow d. Let $G|^d := (D|^d, E|^d)$ be maximum sub-network with d being the source, and $v|^d : E|^d \to \mathbb{R}^N_+$ the restriction of v to $E|^d$. We call $(G|^d, v|^d)$ be **forward process** at d.

Sequential composition: For each $(G, v) \in \mathcal{P}$ with $G = (D, E)$ and each $d \in D$ which is neither a source nor a sink, if all paths in $\varphi(G, v)$ pass d, then $\mu(G, v) = \mu(G|_d, v|_d) + \mu(G|^d, v|^d)$.

For a generic process in which all paths have different values, a unique path is selected. In this case, *sequential composition* requires that at every intermediate node in the selected path, the allocations for the backward and forward subprocesses add up to the one for the original

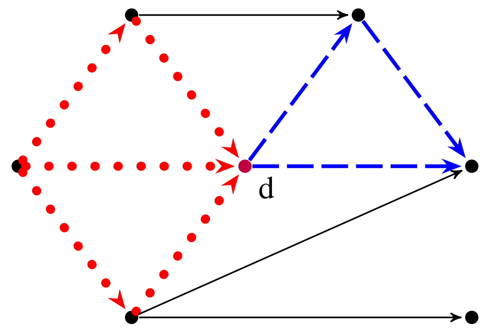

Figure 5: The dotted and dashed sub-networks illustrate respectively the maximum sub-networks with d being the sink and the source.

process. In a non-generic case where multiple paths are selected, *sequential composition* only imposes the requirement at a common intermediate node belonging to all selected paths. A stronger version of *sequential composition* may require at every intermediate node of any selected path the same to happen. Our results hold no matter which version of *sequential composition* we consider.

While *sequential composition* imposes a dynamic consistency requirement on the sharing rule μ, it implies an analogous requirement on the path selection rule φ. For a generic process (G, v) in which all paths have different values including when restricted to subprocesses, *sequential composition* implies that if path L is selected in (G, v) and d is a node in L which is neither the source nor a sink, then the restriction of L to $(G|_d, v|_d)$ and $(G|^d, v|^d)$ should continue to be selected in the respective subprocesses.

A solution combining any path selection rule with PR violates *sequential composition*. To see this, recall that PR assigns to each agent a share in proportion to the sum of his individual values across all edges in a process. When restricted to each subprocess, the edges connecting two subprocesses become irrelevant (see the solid edges in Figure 5), so that the individual values associated with these edges are irrelevant in calculating agents' proportional shares in the subprocess. Thus, the PR allocations for the two subprocesses may not add up to the PR allocation for the initial process.

On the other hand, *sequential composition* holds for a large class of rules. For example, it holds for an additively separable or a myopic path selection rule combined with ES or NR, as long as in non-generic cases where there are multiple optimal paths with the same value, the path selection rule consistently selects the same path(s) for subprocesses as for the initial process. In particular, it holds for EFF-ES, EFF-NR, MYO-ES, and MYO-NR.

Although each of the three axioms above holds for a large class of solutions, together they narrow down a very specific class.

Theorem 1. *A solution* (φ, μ) *satisfies* non-triviality, continuity, *and* sequential composition *if and only if the path(s) selected by* φ *is (are) efficient, and there is* $\alpha \in [0,1]^N$ *with* $\sum_{n \in \mathcal{N}} \alpha_n = 1$ *such that for each* $(G, v) \in \mathcal{P}$, $\mu(G, v) = v_G \alpha$.

Surprisingly, Theorem 1 shows that efficient path selection is guaranteed with the three axioms that are seemingly far from efficiency. More interestingly, it shows that these three axioms jointly rules out the dependence of sharing on network configurations and individual values — each agent receives a constant proportion of the value of each process.

The characterization is tight. Dropping *sequential composition*, a class of solutions satisfying *non-triviality* and *continuity* will be discussed after introducing Theorem 2. Dropping *continuity*, EFF-NR, MYO-ES, and MYO-NR satisfy *non-triviality* and *sequential composition*. Dropping *non-triviality*, the solution combining the path selection rule that chooses all the least efficient paths in each process with ES satisfies *continuity* and *sequential composition*.

It follows readily from Theorem 1 that if the solution satisfies an additional fundamental fairness requirement that agents having the same individual values at all steps are assigned the same amount, then μ is the ES sharing rule.

Equal treatment of equals: For each $(G, v) \in \mathcal{P}$ with $G = (D, E)$ and each pair $n, n' \in \mathcal{N}$, if for each $e \in E$, $v_n(e) = v_{n'}(e)$, then $\mu_n(G, v) = \mu_{n'}(G, v)$.

Corollary 1. *A solution* (φ, μ) *satisfies* non-triviality, continuity, sequential composition, *and* equal treatment of equals *if and only if the path(s) selected by* φ *is (are) efficient, and for each* $(G, v) \in \mathcal{P}$, $\mu(G, v) = \frac{v_G}{N}\boldsymbol{1}$.

2.2 Technology Monotonicity

We now turn to a different perspective and propose a monotonicity axiom regarding technology improvement. It says that no agent should get worse off after a technology improvement which brings a new value-generating step at some stage in a process leading to either another stage in the process or a new ending stage. For example, after a planner selects a railway route, a new section of route connecting either two existing stations or one existing station and a new destination may become available. Alternatively, after a manager selects a sequential series of profit-generating tasks in a long-term project, a new task may become feasible at some phase of the project resulting in either another phase facing the same continuation tasks forward or a new ending phase. In either situation, we require the improvement to hurt no agent.

To define a technology improvement formally, let $(G, v) \in \mathcal{P}$ with $G = (D, E)$ and $(G', v') \in \mathcal{P}$ with $G' = (D', E')$ be given. We say that (G', v') is a **technology improvement** of (G, v) if (1) there are $d_1 \in D$ and $d_2 \in D'$ such that $D' = D \cup \{d_2\}$ and $E' = E \cup \{(d_1, d_2)\}$, and (2) for each $e \in E$, $v'(e) = v(e)$. Note that condition (1) includes two cases. First, G' is formed by adding a new edge connecting two unconnected nodes in G, i.e., $d_2 \in D$ and $(d_1, d_2) \notin E$. Second, G' is formed by adding a new sink and a new edge connecting a node in G to it, i.e., $d_2 \notin D$.

Technology monotonicity: For each $(G, v), (G', v') \in \mathcal{P}$, if (G', v') is a technology improvement of (G, v), then $\mu(G', v') \geq \mu(G, v)$.

Theorem 2. *A solution (φ, μ) satisfies technology monotonicity if and only if the path(s) selected by φ is (are) efficient, and there is a non-decreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+^N$ such that for each $(G, v) \in \mathcal{P}$, $\mu(G, v) = f(v_G)$.*

To compare this characterization with the one in the last subsection, we first observe that by Theorem 2, *technology monotonicity* implies efficient path selection and thus *non-triviality*. Second, *technology monotonicity* also implies *continuity*. To see this, observe that by the definition of a solution, the non-decreasing function f in Theorem 2 satisfies that for each $c \in \mathbb{R}_+$, $\sum_{n \in \mathcal{N}} f_n(c) = c$. Hence, f is continuous and so is μ. Third, it can be readily seen with single-path processes that if f is not additive, then (φ, μ) does not satisfy *sequential composition*, which completes the discussion on the tightness of Theorem 1. On the other hand, if f is additive, then (φ, μ) satisfies *sequential composition*. Since a continuous function is additive if and only if it is linear,[8] the class of solutions characterized in Theorem 1 is also characterized by *technology improvement* and *sequential composition*.

Corollary 2. *A solution (φ, μ) satisfies technology improvement and sequential composition if and only if the path(s) selected by φ is (are) efficient, and there is $\alpha \in [0, 1]^N$ with $\sum_{n \in \mathcal{N}} \alpha_n = 1$ such that for each $(G, v) \in \mathcal{P}$, $\mu(G, v) = v_G \alpha$.*

Lastly, it is apparent that if the *equal treatment of equals* is imposed additionally in Theorem 2, then the function f satisfies that for each $c \in \mathbb{R}_+$, $f(c) = \mu(\frac{c}{N}\mathbf{1}) = \frac{c}{N}\mathbf{1}$, and thus μ is the ES sharing rule.

[8] See Step 4 in the proof of Theorem 1.

Corollary 3. *A solution* (φ, μ) *satisfies* technology monotonicity *and* equal treatment of equals *if and only if the path(s) selected by* φ *is (are) efficient, and for each* $(G, v) \in \mathcal{P}$, $\mu(G, v) = \frac{v_G}{N}\boldsymbol{1}$.

2.3 Independence with respect to network transformations

Turning to a third different perspective, we now propose several independence axioms with respect to various types of network transformations.

Our first independence axiom relates to splitting a step in a process into two consecutive substeps. Imagine that due to a refined examination of the value-generating process, individual values generated at one step in a process are found to be accumulated through two consecutive substeps. Thus, the original step can be divided into two, and at the end of the first one, a new stage at which individual values can be redistributed becomes available. The axiom requires that the refinement of the value-generating process have no impact on value allocation, so that agents would not dispute about whether such a refinement should be adopted.

This situation naturally appears in our applications. For example, after choosing a route for a connected public facility and a benefit allocation, a planner may find that individual benefits at one potential section of the facility are accumulated through two adjacent subsections and redistribution of benefits can be made after the construction of each subsection.[9] Alternatively, after choosing a sequential series of tasks for a long-term project and a profit allocation, a manager may find that individual profits at one possible task are accumulated through two consecutive subtasks and redistribution of profits can be made after the completion of each subtask. In either situation, our axiom require that the initial allocation of benefits or profits be still chosen after the refinement of the value-generating process.

To state the axiom formally, let $(G, v) \in \mathcal{P}$ with $G = (D, E)$ and $(G', v') \in \mathcal{P}$ with $G' = (D', E')$ be given. We say that (G', v') is a **split** of (G, v) if (1) there are $d_1, d_2 \in D$ and $d_0 \in D'$ such that for $e_1 := (d_1, d_2)$, $e_0 := (d_1, d_0)$, and $e_2 := (d_0, d_2)$, $E \setminus E' = \{e_1\}$, $E' \setminus E = \{e_0, e_2\}$, and (2) $v'(e_0) + v'(e_2) = v(e_1)$ and for each $e \in E \cap E'$, $v'(e) = v(e)$. See Figure 6. Note that e_0, e_2 are two consecutive steps in (G', v') that together yield the same individual values as e_1 does in (G, v), and d_0 is a new stage at which individual values can be redistributed.

[9] For instance, it may be found that individual benefits at a canal section are generated by connecting its two end locations with some intermediate location in an additive way.

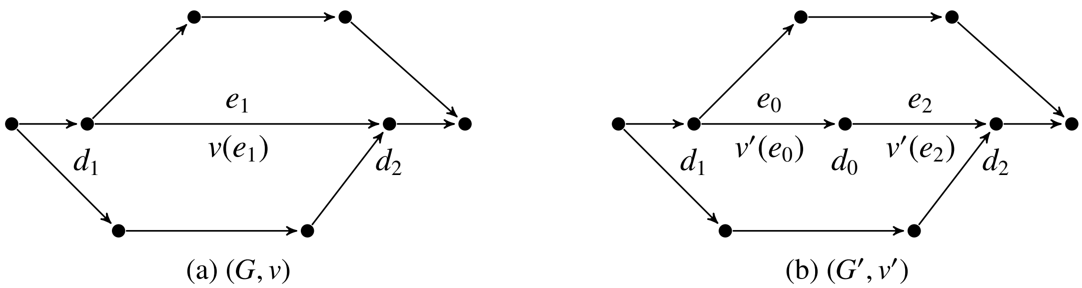

Figure 6: (G', v') is a split of (G, v): $v(e_1) = v'(e_0) + v'(e_2)$

Split invariance: For each pair $(G, v), (G', v') \in \mathcal{P}$, if (G', v') is a split of (G, v), then $\mu(G, v) = \mu(G', v')$.

It is interesting to compare *sequential composition* with *split invariance*. The former requires the independence of the timing of redistribution, and the latter the independence of the refinement of a process. The two types of independence are out of different concerns, while they have exactly opposite implications. To see this, consider a single-path process. *Sequential composition* requires the allocation selected for this process be given by adding up stepwise allocations. In contrast, *split invariance* requires that the allocation for the process only depend on the sum of the value vectors over all steps along the path. While the former accommodates the choice of a progressive planner as illustrated in Example 1, the latter simplifies the choice and would be accepted by a reductionist.

Our *split invariance* is different from the split (or merge) proofness axiom in the fair allocation literature (Banker [5], Moulin [46], de Frutos [24], Ju [33], Sprumont [61], and Chun [15]).[10] There the amount of a resource is fixed, and a sharing rule is required to be immune to the split of an agent into several participation units by dividing his endowment or other divisible personal characteristics.

The next independence axiom relates to path elimination justified by a basic efficiency criterion. Imagine that for each path in one "component" of a process, there is another path outside the component having the same number of steps and generating the value vectors that Pareto dominate those generated by the former path step by step. In this situation, we require that removing the dominated component from the process not affect the allocation.

To formally describe a component of a process, let $(G, v) \in \mathcal{P}$ with $G = (D, E)$ and $(G', v') \in \mathcal{P}$ with $G' = (D', E')$ be given. Since nodes and edges have no identity, we can relabel the nodes in D' so that $D \cap D' = \{d\}$ where d is the source in both G and G'. Abusing

[10] *Split (or merge) invariance* is investigated in a unified framework of allocation problems by Ju, Miyagawa, and Sakai [35].

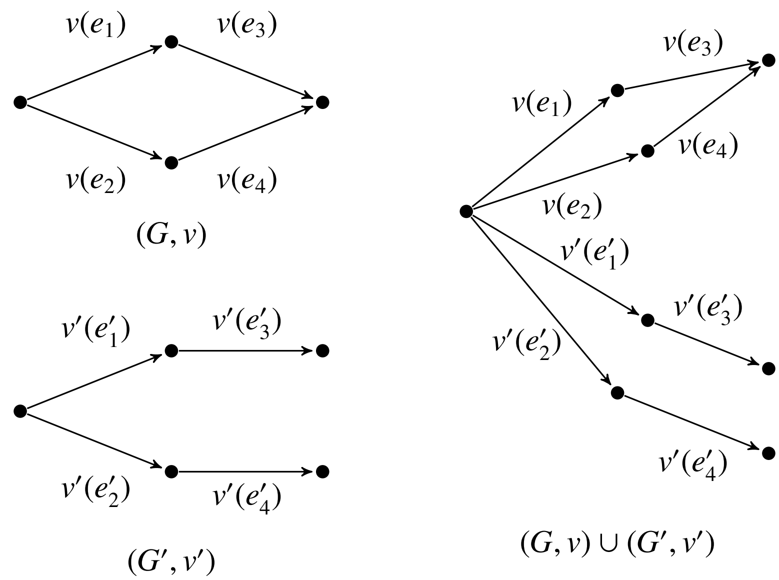

Figure 7: Union of two processes

notation, let D' continue to denote the relabeled set of nodes in G'. Let $(G, v) \cup (G', v') := (G'', v'') \in \mathcal{P}$ in which $G'' = (D \cup D', E \cup E')$, for each $e \in E$, $v''(e) = v(e)$, and for each $e \in E'$, $v''(e) = v'(e)$. See Figure 7. Since our solutions do not depend on labels of nodes and edges, the choice of new labels of nodes in D' does not matter for the definition of $(G, v) \cup (G', v')$. Either (G, v) or (G', v') is regarded as a **component** of $(G, v) \cup (G', v')$. It can also be readily seen that $(G, v) \cup (G', v') = (G', v') \cup (G, v)$.

Recall that a single-edge process is simply denoted by the value vector associated with the edge. For each $x \in \mathbb{R}^N_+$ and each $(G, v) \in \mathcal{P}$, let $(G, v) \cup x$ (or $x \cup (G, v)$) denote $(G, v) \cup (G', v')$ where $(G', v') \in \mathcal{P}$ with $G' = (D', E')$ is such that $E' = \{e\}$ and $v(e) = x$. Similarly, for each pair $x, x' \in \mathbb{R}^N_+$, let $x \cup x'$ denote the union of two single-edge processes with the associated value vectors being x and x' respectively.

For each pair $(G, v), (G', v') \in \mathcal{P}$, we say that a path $\{e_j\}_{j=1}^J \in \mathcal{L}^G$ is **stepwise dominated** by a path $\{e'_j\}_{j=1}^{J'} \in \mathcal{L}^{G'}$ if $J = J'$ and for each $j \in \{1, \cdots, J\}$, $v(e_j) \lneq v'(e'_j)$.

Irrelevance of dominated paths: For each pair $(G, v), (G', v') \in \mathcal{P}$, if each $L \in \mathcal{L}^G$ is stepwise dominated by some $L' \in \mathcal{L}^{G'}$, then $\mu((G, v) \cup (G', v')) = \mu(G', v')$.

The third independence axiom pertains to the possibility that after a value allocation has been chosen for a process, a new component is found to be available. Then the initial allocation can be canceled and a new allocation selected based on the augmented process that combines the new component with the initial process. Alternatively, the initial allocation can be preserved, and a new allocation selected based on the process that combines the new component with the reduced initial process consisting of a single edge and taking the initial allocation as the associated value vector. Both approaches are reasonable and we require that they yield the same allocation in the end.

Parallel composition: For each pair $(G, v), (G', v') \in \mathcal{P}$, $\mu((G, v) \cup (G', v')) = \mu(\mu(G, v) \cup (G', v'))$.

Parallel composition basically allows a process to be solved component by component. This is reminiscent of the "lower composition" axiom in the rationing problem (Young [65]) and the "step by step negotiation" property in the axiomatic bargaining problem (Kalai [40]).

The solutions that we characterized in the previous subsections trivially satisfy the three independence axioms, since all the sharing rules disregard network configuration and individual values. We now provide an example of a solution satisfying the three axioms and depending on individual values in a desirable way. For the ease of presentation, we assume that $N = 2$ in the example. The proofs involved are provided in Appendix A.2 with a general N.

Example 5. *[A solution satisfying the three axioms and depending on individual values] Imagine a planner who selects every efficient path in a process and divides the value of the process in the following way. First, for each path in the process, the planner finds the no-redistribution allocation induced by the path — the sum of the value vectors over all steps along the path — as a potential allocation. Second, based on the set of potential allocations induced by all paths, the planner selects the optimal allocation according to two criteria.*

To illustrate the two criteria, let $x, y \in \mathbb{R}^2_+$ be two potential allocations. The first criterion is Pareto efficiency — if x Pareto dominates y, then x is preferred to y. The second criterion is about fairness. If x is a convex combination of y and the equal-sharing allocation $\frac{y_1+y_2}{2}\boldsymbol{1}$, i.e., x and y allocate the same aggregate value and x is closer to equal sharing, then x is preferred to y.

The transitive closure of the two binary relations given by the two criteria is a partial order $\succsim$ over $\mathbb{R}^2_+$, defined by setting for each pair $x, y \in \mathbb{R}^2_+$, $x \succsim y$ if there is $\lambda \in [0, 1]$ such that

$$x \geq \lambda y + (1 - \lambda)\frac{y_1 + y_2}{2}\boldsymbol{1}. \tag{1}$$

Moreover, it can be shown that the set $\mathbb{R}^2_+$ of potential allocations, equipped with $\succsim$, is a *join-semilattice. Thus, precisely speaking, the optimal allocation selected by the planner for a process is a join of the set of no-redistribution allocations induced by all paths.*

Formally, define $\varphi : \mathcal{P} \rightrightarrows \bigcup_{G \in \mathcal{G}} \mathcal{L}^G$ *and* $\mu : \mathcal{P} \to \mathbb{R}^2_+$ *by setting for each* $(G, v) \in \mathcal{P}$, $\varphi(G, v) := \{L \in \mathcal{L}^G : v_L = v_G\}$ *and* $\mu(G, v) := \max_{\succsim}\{\sum_{j=1}^{J} v(e_j) : \{e_j\}_{j=1}^{J} \in \mathcal{L}^G\}$, *where* $\max_{\succsim}$ *is the join operator on the set* $\mathbb{R}^2_+$ *partially ordered by* $\succsim$.

Proposition 1. *The couple* (φ, μ) *is a well-defined solution. It satisfies* split invariance*,* irrelevance of dominated paths*,* parallel composition*, and* continuity*.*

To see how the sharing rule depends on individual values at all paths (including inefficient ones), consider the three processes shown in Figure 8. The process in (a) has a single edge. Thus, the no-redistribution allocation $(0, 2)$*, as a trivially optimal allocation, is selected. The process in (b) is given by adding to (a) a parallel edge at which the individual values of agents are switched. Then instead of selecting a no-redistribution allocation favoring a particular agent, the solution selects the equal-sharing allocation* $(1, 1)$ *as a compromise. Even when the additional path is inefficient, as shown in (c), the solution still depends on the individual values at the path and selects* $(1 - \frac{\epsilon}{2}, 1 + \frac{\epsilon}{2})$*. Figure 9 illustrates why* $(1 - \frac{\epsilon}{2}, 1 - \frac{\epsilon}{2})$ *is the join of* $(0, 2)$ *and* $(2 - \epsilon, 0)$*, and it can also be seen that when* ϵ *goes to* 0*,* $(1, 1)$ *is the join of* $(0, 2)$ *and* $(2, 0)$*.*

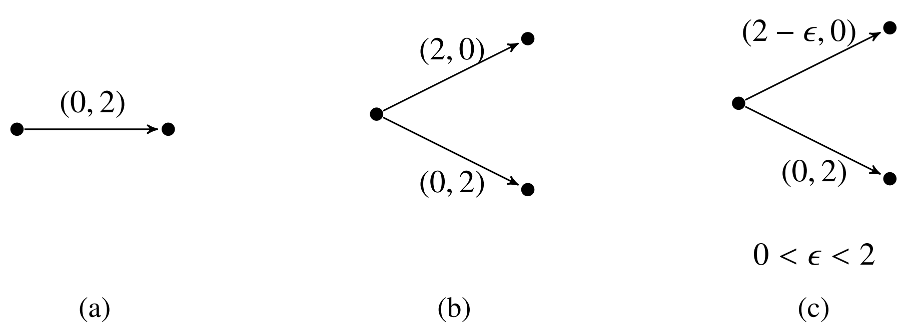

Figure 8: The sharing rule depends on individual values at all paths.

Unlike the axioms in the previous subsections, the three independence axioms together with *continuity* actually characterize a general class of solutions that takes into account individual values in all paths of a process. Similar to the solution in Example 5, each solution

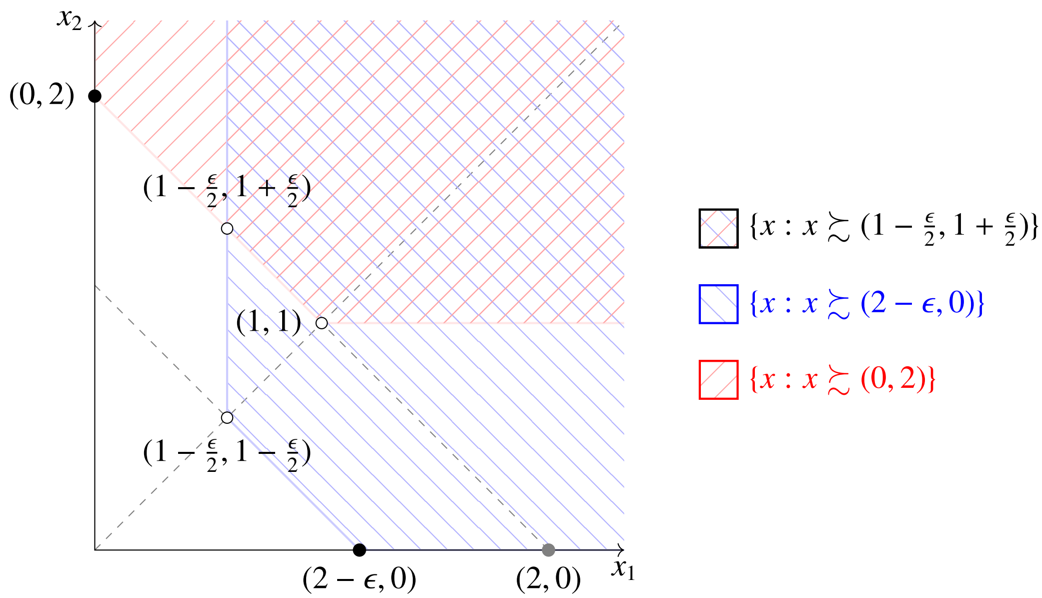

Figure 9: The upper contour sets of $(2-\epsilon, 0)$, $(0,2)$, and $(1-\frac{\epsilon}{2}, 1+\frac{\epsilon}{2})$.

in this class makes efficient path selection and has a sharing rule that is "rationalizable" by some (partial) preference order. More precisely, given a process, the sharing rule selects an allocation in the following way. First, it collapses each path into a single step and sums up value vectors over all steps along the path. Second, it conducts a redistribution for the sum of value vectors associated with each collapsed path. Lastly, it selects an optimal allocation based on the redistributed allocations induced by all paths according to some partial order. The solution in Example 5 is a special case that simply conducts no redistribution for collapsed paths.

Formally, a **redistribution scheme** is a pair $(r, \succsim)$ where $r : \mathbb{R}^N_+ \to \mathbb{R}^N_+$ is a function such that for each $x \in \mathbb{R}^N_+$, $\sum_{n \in \mathcal{N}} r_n(x) = \sum_{n \in \mathcal{N}} x_n$ and $r(r(x)) = r(x)$, and $\succsim$ a partial order over $r(\mathbb{R}^N_+)$ equipped with which $r(\mathbb{R}^N_+)$ is a join-semilattice. Intuitively, for each value vector, r determines a redistribution of individual values which is stable in the sense that no further redistribution will be made when r is applied again. Besides, $\succsim$ provides a partial order over redistributed value vectors so that for each nonempty finite set of redistributed value vectors, there is an optimal one, i.e., a join (or a least upper bound) with respect to $\succsim$. For each finite $X \subseteq r(\mathbb{R}^N_+)$, let $\max_{\succsim} X$ denote the join of X. If $\succsim$ is a complete order, then for each finite $X \subseteq r(\mathbb{R}^N_+)$, $\max_{\succsim} X \in X$.

We say that a redistribution scheme $(r, \succsim)$ is **monotone** if for each pair $x, y \in \mathbb{R}^N_+$, $x \geq$

$y \Rightarrow r(x) \succsim r(y)$, and **continuous** if both r and the join operator $\max_{\succsim}\{\cdot,\cdot\} : r(\mathbb{R}^N_+)^2 \to r(\mathbb{R}^N_+)$ are continuous. It can be readily seen that if $(r, \succsim)$ is continuous, then $\succsim$ is continuous. Given a redistribution scheme $(r, \succsim)$ and a solution (φ, μ), we say that μ is **$(r, \succsim)$ - rationalizable** if for each $(G, v) \in \mathcal{P}$, $\mu(G, v) = \max_{\succsim}\{r(\sum_{j=1}^{J} v(e_j)) : \{e_j\}_{j=1}^J \in \mathcal{L}^G\}$.

Theorem 3. *A solution (φ, μ) satisfies split invariance, irrelevance of dominated paths, parallel composition, and continuity if and only if the path(s) selected by φ is (are) efficient, and there is a monotone and continuous redistribution scheme $(r, \succsim)$ such that μ is $(r, \succsim)$ - rationalizable.*

In particular, the solutions characterized in Theorem 2 (and thus in Theorem 1) are associated with some redistribution schemes. To see this, let (φ, μ) be a solution in Theorem 2 defined in terms of a non-decreasing function $f : \mathbb{R}_+ \to \mathbb{R}^N_+$. Let $r : \mathbb{R}^N_+ \to \mathbb{R}^N_+$ be such that for each $x \in \mathbb{R}^N_+$, $r(x) = f(\sum_{n \in \mathcal{N}} x_n)$. Let $\succsim$ be a complete order on $r(\mathbb{R}^N_+)$ such that for each pair $x, y \in r(\mathbb{R}^N_+)$, $x \succsim y$ if $\sum_{n \in \mathcal{N}} x_n \geq \sum_{n \in \mathcal{N}} y_n$. It can be readily seen that $(r, \succsim)$ is a monotone and continuous redistribution scheme and μ is $(r, \succsim)$ - rationalizable.

Indeed, whenever the partial order in Theorem 3 is complete, the corresponding solution disregards individual values in each process and selects an allocation based only on the value of the process, like the solutions in Theorem 2. Such solutions are singled out by an additional axiom which essentially implies the completeness of associated partial orders. This axiom says that in each process containing only parallel paths, the allocation depends only on one of the paths.

Irrelevance of parallel outside options: For each $K \in \mathbb{N}$ and each set $\{(G^k, v^k) \in \mathcal{P} : \mathcal{L}^{G^k}$ is a singleton, $k = 1, \cdots, K\}$, there is $k' \in \{1, \cdots, K\}$ such that $\mu(\bigcup_{k=1}^{K}(G^k, v^k)) = \mu(G^{k'}, v^{k'})$.

Generically, if no two paths in a process have the same value, the allocation depends only on the selected path. When this axiom is imposed in addition to those in Theorem 3, *split invariance* and *parallel composition* become redundant.

Theorem 4. *A solution (φ, μ) satisfies irrelevance of dominated paths, irrelevance of parallel outside options, and continuity if and only if the path(s) selected by φ is (are) efficient, and there is a continuous function $f : \mathbb{R}_+ \to \mathbb{R}^N_+$ such that for each $(G, v) \in \mathcal{P}$, $\mu(G, v) = f(v_G)$.*

The characterizations in both Theorem 3 and 4 are tight. Dropping *split invariance*, consider the solution in Appendix A.2 (an N-agent version of Example 5) with r depending on not only the value vector but also on the number of edges in the path that generate the value vector. That is, for each $(G, v) \in \mathcal{P}$ and each $L \in \mathcal{L}^G$, if L has two edges, then r divides equally among agents in group $\mathcal{S}_1$ ($\mathcal{S}_1 \subseteq \mathcal{N}$) the aggregate values generated by agents of the other group $\mathcal{S}_2$ ($\mathcal{S}_2 = \mathcal{N} \setminus \mathcal{S}_1$) along the path, and among agents in $\mathcal{S}_2$ those generated by agents of $\mathcal{S}_1$; if L does not have two edges, then r agrees with the redistribution function in Appendix A.2. The solution with modified redistribution function satisfies *irrelevance of dominated paths*, *parallel composition*, and *continuity*. Dropping *irrelevance of dominated paths*, the solution that selects all paths with the smallest values and divides the value equally among agents satisfies *split invariance*, *parallel composition*, *irrelevance of parallel outside options*, and *continuity*. Dropping *parallel composition*, consider the solution that selects all the efficient paths and divides the value to each agent in proportion to the maximum aggregate values he can generate over all the paths in a process. This solution satisfies *split invariance*, *irrelevance of dominated paths*, and *continuity*. Dropping *continuity*, consider a path selection rule that picks in the first round the paths among all the efficient ones that maximizes the total individual value of agent 1. Then, it picks among the selected ones in the first round those maximizing the total individual value of agent 2, and so on and so forth, to the N-th round. The solution that adopts this path selection rule and assigns to the agents their individual aggregate values along the selected path(s) satisfies *split invariance*, *irrelevance of dominated paths*, *parallel composition*, and *irrelevance of parallel outside options*. Dropping *irrelevance of parallel outside options*, the solution in Example 5 satisfies *irrelevance of dominated paths* and *continuity*.

Lastly, as in the previous subsections, *equal treatment of equals* singles out the ES sharing rule.

Corollary 4. *A solution (φ, μ) satisfies irrelevance of dominated paths, irrelevance of parallel outside options, continuity, and equal treatment of equals if and only if the path(s) selected by φ is (are) efficient, and for each $(G, v) \in \mathcal{P}$, $\mu(G, v) = \frac{v_G}{N}\boldsymbol{1}$.*

3 The model under incomplete information

Our study so far has been based on the underlying assumption that a planner has complete information about a process including network configuration and individual values of agents. We relax the assumption in this section. In particular, we consider a situation in which agents but the planner have complete information about a process, and the planner can only observe the generated individual values along a realized path.

In this incomplete information setting, we directly assume that the planner's objective is to implement an efficient path. On the one hand, efficient path selection stands out in the complete information setting in all our axiomatizations (Theorems 1-4) from different perspectives. On the other hand, it also naturally arises in the profit sharing context where the goal of a manager is to maximize the total profit in a project process. Note that although the planner is uninformed about the details of a process, efficiency is a well-defined objective since agents have full information and are able to find an efficient path.

The traditional mechanism design literature is centralized in designing a game to incentivize agents to either reveal their information or select an efficient path. We depart from the traditional literature and assume that the planner has **limited** power in designing such a game.[11] In particular, we consider a situation in which agents collectively select a path through some **fixed** procedure, and the only way for the planner to influence agents' choices is to redistribute their individual values realized along the chosen path.

For example, imagine that a planner does not fully know about agents' benefits from each potential section of a connected public facility, and the choice of its route is determined through some political procedure such as step-by-step voting. In such a situation, the planner may not be free to change the political procedure to influence the outcome. But the planner may find a way of sharing realized benefits to align her objective with agents' incentives.

Alternatively, imagine that a manager has limited information about the exact profits brought by employees from completing each potential task, and a sequence of tasks is collectively chosen by employees. The manager may have little control of the way of making a collective decision by employees. But the manager may set a profit sharing scheme to incentive them to choose a profit-maximizing sequence.

Instead of finding a sharing rule that implements efficient path selection for each specific collective decision rule of agents, we are interested in finding a sharing rule that works for a large class of common collective decision rules. For this propose, we follow an axiomatic approach and the key axiom in this section captures this robustness idea.[12]

We need to modify our model primitives since the planner can only choose a sharing rule depending on sequences of individual values generated along realized paths. Let a **path** be a finite sequence $\{x^j\}_{j=1}^J$ of elements of $\mathbb{R}^N_+$, where J is the number of edges in the path and x^j is the vector of individual values of the j-th edge. We denote a typical path by ℓ, and the

[11] Section 3.1 briefly discusses a more traditional approach regarding eliciting private information of agents.

[12] This is related to recent works on robust mechanism design in which the designer selects a mechanism that works well for a variety of scenarios. Jackson and Moulin [32] and Bag [4] study a related problem where the planner can control the voting rule but not the sharing rule.

set of all paths by $\mathcal{L}$. For each path $\ell \in \mathcal{L}$ with $\ell = \{x^j\}_{j=1}^J$, let $v_\ell := \sum_{j=1}^{J} \sum_{n\in\mathcal{N}} x_n^j$ be the **value of $\boldsymbol{\ell}$**. Abusing notation, a **sharing rule** is a function $\mu : \mathcal{L} \to \mathbb{R}_+^N$ such that for each $\ell \in \mathcal{L}$, $\sum_{n\in\mathcal{N}} \mu_n(\ell) = v_\ell$. For each path with only one edge, we simply use the value vector associated with the edge to denote the path when we study a sharing rule. Precisely, for each $x \in \mathbb{R}_+^N$, we shall write $\mu(x)$ instead of $\mu(\{x\})$.

As mentioned above, our key axiom requires that the sharing rule implement efficient path selection for common collective decision rules of agents. It says that for each pair of paths, at least M agents should receive weakly more at the path with a larger value, where M is more than half of the number of the agents. This is a robustness requirement which is necessary to guarantee efficient path selection when agents are delegated to make a decision using a Condorcet social choice function and other M-majoritarian voting rules, as we will see in Applications 1 and 2.

***M*-majority** ($\frac{N}{2} < M \leq N$): For each pair $\ell, \ell' \in \mathcal{L}$, if $v_\ell \geq v_{\ell'}$, then there is $\mathcal{S} \subseteq \mathcal{N}$ such that $|\mathcal{S}| \geq M$ and for each $n \in \mathcal{S}$, $\mu_n(\ell) \geq \mu_n(\ell')$.

A sharing rule satisfying this axiom will be referred to as ***M*-majoritarian**. When $M = \lfloor \frac{N}{2} \rfloor + 1$, a majority of agents always prefer a more efficient path. When $M = N$, all agents prefer a more efficient path.[13]

Application 1 (Path selection using a Condorcet Social Choice Function)**.** *Suppose that a path is selected using a Social Choice Function (SCF) that satisfies the Condorcet property. That is, the SCF elects a Condorcet winner when available.*[14]

For each agent, a sharing rule determines an allocation for each path, which induces an ordinal ranking over paths. Each $M-$majoritarian sharing rule guarantees that an efficient path is a Condorcet winner regardless of the network and individual values. Therefore, each SCF that meets the Condorcet property picks an efficient path. Conversely, if a rule is not $M-$majoritarian for each $M > \frac{N}{2}$, then a SCF that meets the Condocret property may pick an inefficient path.

[13]When $M = N$, we have an *N-majority*. It is reminiscent to the property of Pareto Nash Implementation in connection networks by Juarez and Kumar [38], that requires that the efficient Nash equilibrium be preferred by all the agents over any other equilibrium.

[14]Formally, given the set of objects $\mathcal{L}$, let $\mathcal{R}$ be the set of ordinal preferences over $\mathcal{L}$. A social choice function $\Psi : \mathcal{R}^N \to \mathcal{L}$ meets the Condorcet property if for the preference profile $\succeq = (\succeq_1, \ldots, \succeq_N) \in \mathcal{R}^N$ there exists $\ell^* \in \mathcal{L}$ such that for any $\ell \in \mathcal{L}$, $|\{n \in \mathcal{N} | \ell^* \succ_n \ell\}| > \frac{N}{2}$, then $\Psi(\succeq) = \{\ell^*\}$. A large class of SCFs that satisfy this property are discussed in Moulin [48].

Application 2 (Sequential voting)**.** *Suppose that agents vote step by step to decide a path. More precisely, consider the dynamic game of complete information where at every node agents vote on the direction to continue. A path is selected using an M-majoritarian voting rule at every node; that is, if a direction receives at least M-votes, then it is chosen. The payoff of the agents is given by applying the sharing rule to the realized path.*

A M-majoritarian sharing rule will always implement an efficient path as a strong subgame perfect Nash equilibrium. Such an equilibrium is unique when there are no two paths with the same values.

Finally, the implementation of the efficient path(s) is robust to voting only at a subset of nodes. It is also robust to incomplete information of the agents about at which nodes voting will occur.[15]

The following example shows some sharing rules satisfying M-majority.

Example 6. *[Sharing rules satisfying M-majority]*

i. *Fix a priority group $\mathcal{S} \subseteq \mathcal{N}$ with at least M agents. Let μ divide the value of each path equally among the agents in the priority group and assign nothing to the others. That is, for each $\ell \in \mathcal{L}$,*

$$\mu_n(\ell) = \begin{cases} \frac{v_\ell}{|\mathcal{S}|} & \text{if } n \in \mathcal{S}, \\ 0 & \text{if } n \in \mathcal{N} \setminus \mathcal{S}. \end{cases}$$

Since all agents in $\mathcal{S}$ prefer a path with a larger value and $|\mathcal{S}| \geq M$, μ satisfies M-majority.

ii. *Consider a sharing rule that depends on agents' individual values. For each realized path, agents are classified into two groups: (1) the top group with the top M agents who have the highest M accumulated individual values along the path, and (2) the bottom group with the remaining $N - M$ agents. Each top group member shares equally the value of the realized path. The bottom group receives nothing. Formally, let $K \in \mathbb{N}$ be such that $M \leq K \leq N$. Given $\ell = \{x^j\}_{j=1}^J$ and $n \in \mathcal{N}$, let $v_{\ell_n} := \sum_{j=1}^J x_n^j$ be the accumulated value of agent n at the path ℓ. For each path ℓ with $v_{\ell_{n_1}} \geq v_{\ell_{n_2}} \geq \cdots \geq v_{\ell_{n_N}}$ (a tie-breaking rule is needed when there is indifference) and each $n_i \in \mathcal{N}$,*

[15] This problem often occurs when voters want to re-evaluate a chosen route after it has been partially built, for instance in projects that take several years to construct, like the rail in Honolulu, from Ewa side to Waikiki via Downtown. While the decision to build the rail from Honolulu to Ewa was approved by voters, their construction stopped in the middle to re-evaluate the route chosen and to be confirmed by the voters before further spending on the project occurs.

$$\mu_{n_i}(\ell) = \begin{cases} \frac{v_\ell}{K} & \text{if } 1 \leq i \leq K, \\ 0 & \text{if } K+1 \leq i \leq N. \end{cases}$$

The sharing rule μ satisfies M-majority. To see this, let ℓ, ℓ' be two paths such that $v_\ell \geq v_{\ell'}$. Then, the K agents in the top group at ℓ is assigned no less than at ℓ'.

We focus on sharing rules that do not discriminate agents by their names. Formally, for permutation π of $\mathcal{N}$ and each vector $x \in \mathbb{R}^N_+$, let $x^\pi \in \mathbb{R}^N_+$ be such that for each $n \in \mathcal{N}$, $x^\pi_{\pi(n)} = x_n$. For permutation π of $\mathcal{N}$ and each path $\ell \in \mathcal{L}$ with $\ell = \{x^j\}_{j=1}^J$, let $\ell^\pi \in \mathcal{L}$ be such that $\ell^\pi = \{x^{j\pi}\}_{j=1}^J$.

Anonymity: For each permutation π of N and each $\ell \in \mathcal{L}$, $\mu(\ell^\pi) = \mu(\ell)^\pi$.

The combination of the above two axioms lead to an important restriction on the sharing rules: at least M agents should be allocated the average of the value of the path.

Proposition 2. *If μ satisfies M-majority and anonymity, then for each $\ell \in \mathcal{L}$, there is $\mathcal{S} \subseteq \mathcal{N}$ such that $|\mathcal{S}| \geq M$ and for each $n \in \mathcal{S}$, $\mu_n(\ell) = \frac{v_\ell}{N}$.*

When the population is small, in particular, when $N = 3$ or $N = 4$, Proposition 2 implies that the equal sharing rule is the only rule meeting M-majority and anonymity. When the population is large, the equal sharing rule can be characterized by adding either one of the axioms stated in the following theorem. These axioms include a version of sequential composition in Section 2.1, a fairness axiom that requires no less egalitarian sharing at a more egalitarian path, and a basic monotonicity of a sharing rule with respect to transfers between agents.

Theorem 5. *A rule μ is the equal sharing rule if and only if it satisfies M-majority, anonymity, and either of the following axioms:*

i. Sequential composition: For each $\ell \in \mathcal{L}$ with $\ell = \{x^j\}_{j=1}^J$, $\mu(l) = \sum_{j=1}^{J} \mu(x^j)$.

ii. Lorenz monotonicity: For each pair $\ell, \ell' \in \mathcal{L}$ with $\ell = \{x^j\}_{j=1}^J$ and $\ell' = \{x'^j\}_{j=1}^J$, if for each $j \in \{1, \cdots, J\}$, $x^j \succ_{Lorenz} x'^j$, then $\mu(\ell') \not\succ_{Lorenz} \mu(\ell)$.[16]

[16]To define $\succ_{Lorenz}$, for each $x \in \mathbb{R}^N_+$, let $x^* \in \mathbb{R}^N_+$ be such that for each $n \in \{1, \cdots, N\}$, x^*_N is the n-th smallest number of $x_1, \cdots, x_N$. For each pair $x, y \in \mathbb{R}^N_+$, we denote by $x \succ_{Lorenz} y$ if for each $m \in \{1, \cdots, N\}$, $\sum_{n=1}^{m} x^*_n \geq \sum_{n=1}^{m} y^*_n$, with at least one inequality being strict. See Hougaard [26] for recent applications of Lorenz monotonicity to allocation problems.

iii. Transfer monotonicity: For each pair $n, n' \in \mathcal{N}$, and each pair $\ell, \ell' \in \mathcal{L}$ with $\ell = \{x^j\}_{j=1}^J$ and $\ell' = \{x'^j\}_{j=1}^J$, if for each $j \in \{1, \cdots, J\}$, $x_n'^j - x_n^j = x_{n'}^j - x_{n'}'^j \geq 0$ and for each $m \in \mathcal{N} \setminus \{n, n'\}$, $x_m^j = x_m'^j$, then $\mu_n(\ell') \geq \mu_n(l)$.

3.1 Remarks about Information

So far, we have studied the case of complete information among agents. We can alternatively consider the case where only a few of the agents have complete information about the network and individual values. A natural mechanism to select a path is one that delegates the more informed agents to decide which direction to continue. Consider a rule that incentivizes the delegates to make the efficient decision regardless of the network and individual values. It is easy to show that such a rule should allocate the delegates a share depending only on the value of a path, while non-delegates can be given arbitrary shares. Moreover, we can show that the equal sharing rule is the only anonymous rule in this class.

We can alternatively consider the problem where there is no agent with complete information about the network and individual values. For instance, consider the case where agents only know the network and their own values. In this setting, a more traditional approach from the mechanism design literature would require agents to report their information to the planner, who will use this information to make an estimation of the values in the network and select an efficient path —see Hougaard and Tvede [29, 30] for a related study of implementation of the efficient path in minimal cost spanning trees. A natural issue in this setting is to find the mechanisms and sharing rules that incentivize agents to report their true information. When agents are critical, that is, when they have information about the values at some edges that no one else has, it is easy to prove that every mechanism is manipulable. On the other hand, when agents are not critical, in particular, when for every edge there are at least three agents who have information about all agent's values on that edge, several mechanisms and sharing rules can achieve truth telling as an equilibrium.

4 Conclusion

We have introduced the problem of division of sequential values and provided a comprehensive study of it. In particular, we have addressed the problem from different perspectives, including the complete and incomplete information case, and used old and new axioms from other strands in the literature to characterize several classes of solutions not uncovered elsewhere. An advantage of covering this problem from a wide variety of angles is that axioms can be chosen according to the relevancy of the application in mind. Our analysis highlights the robustness of the EFF-ES solution, in both the complete and incomplete information settings. This rule, however, is by no means the only rule when less stringent axioms are imposed. Indeed, we have also characterized a class of solutions that take into account individual values accumulated along all paths in a rationalizable way.

We see this work as opening up new avenues of research in the distribution of sequential costs and benefits, especially from the normative and positive angles. New classes of solutions, like the ones uncovered in Examples 1 and 4, require further study.

Several extensions of our characterizations to variants of sequential processes remain an open question. One such variant of sequential processes occurs when values are realized at nodes instead of edges, where the graph only describes the transition between states and each state would be associated to one particular vector.[17] Another variant of a sequential process is where collective values (instead of individual values) are observed.

A Appendix

A.1 Proofs of results

Proof of Theorem 1. The "if" direction is readily verified, so the proof is omitted. To show the "only if" direction, let (φ, μ) be a solution satisfying *non-triviality*, *continuity*, and *sequential composition*.

Step 1. If $(G, v) \in \mathcal{P}$ has a unique efficient path, the value of each edge along the path is positive, and the value of each other edge is zero, then $\varphi(G, v)$ selects the efficient path.

Let $(G, v) \in \mathcal{P}$ with $G = (D, E)$ have a unique efficient path $L := \{d_j\}_{j=1}^{J} \in \mathcal{L}^G$. For each $j \in \{1, \cdots, J-1\}$, let $e_j := (d_j, d_{j+1})$. Suppose that for each $j \in \{1, \cdots, J-1\}$, $v_{e_j} > 0$, and for each $e \in E \setminus \{e_1, \cdots, e_{J-1}\}$, $v_e = 0$. We want to show that $\varphi(G, v) = \{L\}$. When $J = 2$, there is a unique edge with a positive value, so *non-triviality* implies that $\varphi(G, v) = \{L\}$. In the following, we assume $J > 2$.

For each $\lambda \in [0, 1]$, let $v^\lambda \in \mathcal{V}^G$ be such that $v^\lambda(e_1) = v(e_1)$, $v^\lambda(e_2) = \lambda v(e_2) + (1-\lambda)\mathbf{0}$, and for each $e \in E \setminus \{e_1, e_2\}$, $v^\lambda(e) = \mathbf{0}$. Since $v^0_G \geq v^0_{e_1} = v_{e_1} > 0$, by *non-triviality*, $\sum_{n \in \mathcal{N}} \mu_n(G, v^0) > 0$. Since $\sum_{n \in \mathcal{N}} \mu_n(G, v^0) > 0$ and for each $e \in E \setminus \{e_1\}$, $v^0(e) = \mathbf{0}$, by the definition of a solution, $\sum_{n \in \mathcal{N}} \mu_n(G, v^0) = v_{e_1}$. Since for each $e \in E$, $\lim_{\lambda \to 0} v^\lambda(e) = v^0(e)$, by

[17] We thank one of the referees for suggesting this variation.

continuity, $\lim_{\lambda\to 0}\sum_{n\in\mathcal{N}}\mu_n(G,v^\lambda)=\sum_{n\in\mathcal{N}}\mu_n(G,v^0)=v_{e_1}$. Then there is $\lambda'\in(0,1]$ such that for each $\lambda\in[0,\lambda']$, $\sum_{n\in\mathcal{N}}\mu_n(G,v^\lambda)>\frac{v_{e_1}}{2}$ and $v^\lambda_{e_2}<\frac{v_{e_1}}{2}$, so each path in $\varphi(G,v^\lambda)$ passes $e_1=(d_1,d_2)$, and since d_2 is neither the source nor a sink in G, by *sequential composition*, $\sum_{n\in\mathcal{N}}\mu_n(G,v^\lambda)=\sum_{n\in\mathcal{N}}\mu_n(G|_{d_2},v^\lambda|_{d_2})+\sum_{n\in\mathcal{N}}\mu_n(G|^{d_2},v^\lambda|^{d_2})$. Since for each $\lambda\in(0,\lambda']$, $v^\lambda_{e_1},v^\lambda_{e_2}>0$ and for each $e\in E\setminus\{e_1,e_2\}$, $v^\lambda_e=0$, by *non-triviality* and the definition of a solution, $\sum_{n\in\mathcal{N}}\mu_n(G|_{d_2},v^\lambda|_{d_2})=v^\lambda_{e_1}$ and $\sum_{n\in\mathcal{N}}\mu_n(G|^{d_2},v^\lambda|^{d_2})=v^\lambda_{e_2}$, and thus $\sum_{n\in\mathcal{N}}\mu_n(G,v^\lambda)=v^\lambda_{e_1}+v^\lambda_{e_2}$, i.e., each path in $\varphi(G,v^\lambda)$ passes e_1,e_2.

Let $\overline{\lambda}:=\sup\{\lambda'\in(0,1]:$ for each $\lambda\in(0,\lambda']$, each path in $\varphi(G,v^\lambda)$ passes $e_1,e_2\}$. By the analysis in the previous paragraph, $\overline{\lambda}>0$. By the definition of λ, for each $\lambda\in[0,\overline{\lambda})$, $\sum_{n\in\mathcal{N}}\mu_n(G,v^\lambda)=v^\lambda_{e_1}+v^\lambda_{e_2}$. Then $\lim_{\lambda\uparrow\overline{\lambda}}\sum_{n\in\mathcal{N}}\mu_n(G,v^\lambda)=\lim_{\lambda\uparrow\overline{\lambda}}(v^\lambda_{e_1}+v^\lambda_{e_2})=v_{e_1}+\overline{\lambda}v_{e_2}$. Since for each $e\in E$, $\lim_{\lambda\to\overline{\lambda}}v^\lambda(e)=v^{\overline{\lambda}}(e)$, by *continuity*, $\lim_{\lambda\downarrow\overline{\lambda}}\sum_{n\in\mathcal{N}}\mu_n(G,v^\lambda)=\sum_{n\in\mathcal{N}}\mu_n(G,v^{\overline{\lambda}})=\lim_{\lambda\uparrow\overline{\lambda}}\sum_{n\in\mathcal{N}}\mu_n(G,v^\lambda)=v_{e_1}+\overline{\lambda}v_{e_2}$. Thus, each path in $\varphi(G,v^{\overline{\lambda}})$ passes e_1,e_2.

We claim that $\overline{\lambda}=1$. Suppose to the contrary that $\overline{\lambda}<1$. By the definition of $\overline{\lambda}$, for each $\epsilon>0$ satisfying $\overline{\lambda}+\epsilon\le 1$ and $\epsilon v_{e_2}<\frac{1}{2}v_{e_1}$, there is $\lambda\in(\overline{\lambda},\overline{\lambda}+\epsilon]$ such that some path in $\varphi(G,v^\lambda)$ does not pass either e_1 or e_2, and thus either

$$\sum_{n\in\mathcal{N}}\mu_n(G,v^\lambda)\le v^\lambda_{e_2}\le(\overline{\lambda}+\epsilon)v_{e_2}<\overline{\lambda}v_{e_2}+\frac{1}{2}v_{e_1}\le v_{e_1}+\overline{\lambda}v_{e_2}-\min\{\frac{1}{2}v_{e_1},\overline{\lambda}v_{e_2}\}<v_{e_1}+\overline{\lambda}v_{e_2},$$

$$\text{or }\sum_{n\in\mathcal{N}}\mu_n(G,v^\lambda)\le v^\lambda_{e_1}=v_{e_1}\le v_{e_1}+\overline{\lambda}v_{e_2}-\min\{\frac{1}{2}v_{e_1},\overline{\lambda}v_{e_2}\}<v_{e_1}+\overline{\lambda}v_{e_2},$$

which contradicts that $\lim_{\lambda\downarrow\overline{\lambda}}\sum_{n\in\mathcal{N}}\mu_n(G,v^\lambda)=v_{e_1}+\overline{\lambda}v_{e_2}$. Hence, $\overline{\lambda}=1$, and thus each path in $\varphi(G,v^1)$ passes e_1,e_2.

If $J=3$, then L passes edges e_1,e_2 only, and thus $\varphi(G,v)=\varphi(G,v^1)=\{L\}$. If $J>3$, then by repeating the analogous argument as above, it can be seen that each path in $\varphi(G,v)$ passes $e_1,\cdots,e_{J-1}$, and thus $\varphi(G,v)=\{L\}$.

Step 2. For each $(G,v)\in\mathcal{P}$, $\sum_{n\in\mathcal{N}}\mu_n(G,v)=v_G$, i.e., the path(s) selected by φ is (are) efficient.

Let $(G,v)\in\mathcal{P}$ with $G=(D,E)$. Let $L:=\{d_j\}_{j=1}^J\in\mathcal{L}^G$ be an efficient path in (G,v). For each $j\in\{1,\cdots,J-1\}$, let $e_j:=(d_j,d_{j+1})$. For each $\lambda\in[0,1]$, let $v^\lambda\in\mathcal{V}^G$ be such that for each $e\in\{e_1,\cdots,e_{J-1}\}$, $v^\lambda(e)=v(e)+(1-\lambda)\mathbf{1}$, and for each $e\in E\setminus\{e_1,\cdots,e_{J-1}\}$, $v^\lambda(e)=\lambda v(e)+(1-\lambda)\mathbf{0}$. Note that $v^1=v$, and for each $\lambda\in[0,1]$ and each $L'\in\mathcal{L}^G\setminus\{L\}$, $v_{L'}+(1-\lambda)N(J-2)\ge v^\lambda_{L'}$. Since L is an efficient path in (G,v), $v_L\ge\max_{L'\in\mathcal{L}^G\setminus\{L\}}v_{L'}$, and thus for each $\lambda \in [0,1)$, $v^\lambda_L = v_L + (1-\lambda)N(J-1) > \max_{L' \in \mathcal{L}^G \setminus \{L\}} [v_{L'} + (1-\lambda)N(J-2)] \geq \max_{L' \in \mathcal{L}^G \setminus \{L\}} v^\lambda_{L'}$. In particular, $v^0_L > \max_{L' \in \mathcal{L}^G \setminus \{L\}} v^0_{L'}$, so L is the unique efficient path in (G, v^0).

Since L is the unique efficient path in (G, v^0), for each $e \in \{e_1, \cdots, e_{J-1}\}$, $v^0_e > 0$, and for each $e \in E \setminus \{e_1, \cdots, e_{J-1}\}$, $v^0_e = 0$, by Step 1, $\varphi(G, v^0) = \{L\}$, and thus by the definition of a solution, $\sum_{n \in \mathcal{N}} \mu_n(G, v^0) = v^0_L$. Since for each $e \in E$, $\lim_{\lambda \to 0} v^\lambda(e) = v^0(e)$, by *continuity*, $\lim_{\lambda \to 0} \sum_{n \in \mathcal{N}} \mu_n(G, v^\lambda) = \sum_{n \in \mathcal{N}} \mu_n(G, v^0) = v^0_L$. Since $\lim_{\lambda \to 0} \sum_{n \in \mathcal{N}} \mu_n(G, v^\lambda) = v^0_L > \max_{L' \in \mathcal{L}^G \setminus \{L\}} v^0_{L'} = \lim_{\lambda \to 0} \max_{L' \in \mathcal{L}^G \setminus \{L\}} v^\lambda_{L'}$, there is $\lambda' \in (0,1]$ such that for each $\lambda \in [0, \lambda']$, $\sum_{n \in \mathcal{N}} \mu_n(G, v^\lambda) > \max_{L' \in \mathcal{L}^G \setminus \{L\}} v^\lambda_{L'}$, and thus by the definition of a solution, $\sum_{n \in \mathcal{N}} \mu_n(G, v^\lambda) = v^\lambda_L$.

Let $\overline{\lambda} := \{\lambda' \in [0,1] : \text{for each } \lambda \in [0, \lambda'], \sum_{n \in \mathcal{N}} \mu_n(G, v^\lambda) = v^\lambda_L\}$. By the analysis in the previous paragraph, $\overline{\lambda} > 0$. By the definition of $\overline{\lambda}$, for each $\lambda \in [0, \overline{\lambda})$, $\sum_{n \in \mathcal{N}} \mu_n(G, v^\lambda) = v^\lambda_L$. Then $\lim_{\lambda \uparrow \overline{\lambda}} \sum_{n \in \mathcal{N}} \mu_n(G, v^\lambda) = \lim_{\lambda \uparrow \overline{\lambda}} v^\lambda_L = v_L + (1-\overline{\lambda})N(J-1)$. Since for each $e \in E$, $\lim_{\lambda \to \overline{\lambda}} v^\lambda(e) = v^{\overline{\lambda}}(e)$, by *continuity*, $\lim_{\lambda \downarrow \overline{\lambda}} \sum_{n \in \mathcal{N}} \mu_n(G, v^\lambda) = \sum_{n \in \mathcal{N}} \mu_n(G, v^{\overline{\lambda}}) = \lim_{\lambda \uparrow \overline{\lambda}} \sum_{n \in \mathcal{N}} \mu_n(G, v^\lambda) = v_L + (1-\overline{\lambda})N(J-1)$.

We claim that $\overline{\lambda} = 1$. Suppose to the contrary that $\overline{\lambda} < 1$. By the definition of $\overline{\lambda}$, for each $\epsilon > 0$ satisfying $\overline{\lambda} + \epsilon \leq 1$, there is $\lambda \in (\overline{\lambda}, \overline{\lambda} + \epsilon)$ such that $\sum_{n \in \mathcal{N}} \mu_n(G, v^\lambda) \neq v^\lambda_L$, and thus

$$\begin{aligned}\sum_{n \in \mathcal{N}} \mu_n(G, v^\lambda) &\leq \max_{L' \in \mathcal{L}^G \setminus \{L\}} v^\lambda_{L'} \leq \max_{L' \in \mathcal{L}^G \setminus \{L\}} [v_{L'} + (1-\lambda)N(J-2)] \\ &\leq \max_{L' \in \mathcal{L}^G \setminus \{L\}} [v_{L'} + (1-\overline{\lambda})N(J-2)] < v_L + (1-\overline{\lambda})N(J-1),\end{aligned}$$

which contradicts that $\lim_{\lambda \downarrow \overline{\lambda}} \sum_{n \in \mathcal{N}} \mu_n(G, v^\lambda) = v_L + (1-\overline{\lambda})N(J-1)$. Hence, $\overline{\lambda} = 1$, and thus $\sum_{n \in \mathcal{N}} \mu_n(G, v) = \sum_{n \in \mathcal{N}} \mu_n(G, v^1) = v_L + (1-1)N(J-1) = v_G$, as desired.

Step 3. For each $(G, v) \in \mathcal{P}$, $\mu(G, v) = \mu(\frac{v_G}{N}\mathbf{1})$.

Let $(G, v) \in \mathcal{P}$ with $G = (D, E)$ and $d_1 \in D$ being the source in G. We enlarge G by adding a new source and a new path as depicted in Figure 10. Precisely, let $G' := (D', E')$ be such that $D' = D \cup \{d_0, d_2, d_3\}$, $E' = E \cup \{(d_0, d_1), (d_0, d_2), (d_2, d_3)\}$, where $d_0, d_2, d_3 \notin D$. Note that $G' \in \mathcal{G}$, and d_0 is the new source and d_3 a new sink. Let $e_1 := (d_0, d_1)$, $e_2 := (d_0, d_2)$, $e_3 := (d_2, d_3)$, and $L := \{d_0, d_2, d_3\}$.

For each $\lambda \in [0,1]$, let $v^\lambda \in \mathcal{V}^{G'}$ be such that $v^\lambda(e_1) = \frac{\lambda}{N}\mathbf{1}$, $v^\lambda(e_2) = \frac{v_G}{N}\mathbf{1}$, $v^\lambda(e_3) = \mathbf{0}$, and for each $e \in E$, $v^\lambda(e) = v(e)$. For each $\lambda \in (0,1]$, observe that $v^\lambda_{G'} = \lambda + v_G$, so by Step 2, $\sum_{n \in \mathcal{N}} \mu_n(G', v^\lambda) = \lambda + v_G$. For each $\lambda \in (0,1]$, since $\sum_{n \in \mathcal{N}} \mu_n(G', v^\lambda) = \lambda + v_G > v^\lambda_L$, then every

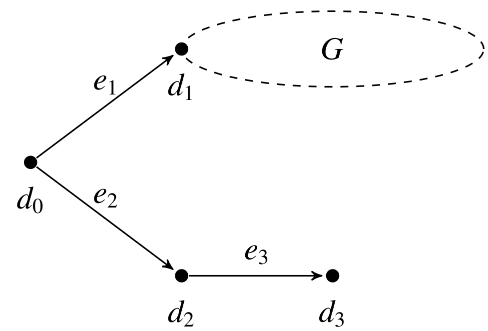

Figure 10: Enlarged network G' based on G

path in $\varphi(G', v^\lambda)$ passes d_1, and since d_1 is neither the source nor a sink in G', by *sequential composition*, $\mu(G', v^\lambda) = \mu(G'|_{d_1}, v^\lambda|_{d_1}) + \mu(G'|^{d_1}, v^\lambda|^{d_1}) = \mu(\frac{\lambda}{N}\mathbf{1}) + \mu(G, v)$. Since for each $e \in E'$, $\lim\limits_{\lambda\downarrow 0} v^\lambda(e) = v^0(e)$, and $\lim\limits_{\lambda\downarrow 0} \frac{\lambda}{N}\mathbf{1} = \mathbf{0}$, by *continuity*,

$$\mu(G', v^0) = \lim_{\lambda\downarrow 0}\mu(G', v^\lambda) = \lim_{\lambda\downarrow 0}[\mu(\frac{\lambda}{N}\mathbf{1}) + \mu(G, v)] = \mu(\mathbf{0}) + \mu(G, v) = \mu(G, v), \tag{2}$$

where the last equality is because $\mu(\mathbf{0}) = \mathbf{0}$ by the definition of a solution.

Analogously, for each $\lambda \in [0, 1]$, let $v'^\lambda \in \mathcal{V}^{G'}$ be such that $v'^\lambda(e_1) = v'^\lambda(e_3) = \mathbf{0}$, $v'^\lambda(e_2) = \frac{\lambda+v_G}{N}\mathbf{1}$, and for each $e \in E$, $v'^\lambda(e) = v(e)$. For each $\lambda \in (0, 1]$, observe that $v'^\lambda_{G'} = \lambda + v_G$, so by Step 2, $\sum\limits_{n\in\mathcal{N}} \mu_n(G', v'^\lambda) = \lambda + v_G$. For each $\lambda \in (0, 1]$, since $\sum\limits_{n\in\mathcal{N}} \mu_n(G', v'^\lambda) = \lambda + v_G > v'^\lambda_{e_1} + v_G$, then $\varphi(G', v'^\lambda) = \{L\}$, and since $d_2 \in L$ is neither the source nor a sink in G', by *sequential composition*, $\mu(G', v'^\lambda) = \mu(G'|_{d_2}, v'^\lambda|_{d_2}) + \mu(G'|^{d_2}, v'^\lambda|^{d_2}) = \mu((\frac{\lambda+v_G}{N})\mathbf{1}) + \mu(\mathbf{0}) = \mu((\frac{\lambda+v_G}{N})\mathbf{1})$, where the last equality is because $\mu(\mathbf{0}) = \mathbf{0}$ by the definition of a solution. Since for each $e \in E'$, $\lim\limits_{\lambda\downarrow 0} v'^\lambda(e) = v'^0(e)$, and $\lim\limits_{\lambda\downarrow 0}(\frac{\lambda+v_G}{N})\mathbf{1} = \frac{v_G}{N}\mathbf{1}$, by *continuity*,

$$\mu(G', v'^0) = \lim_{\lambda\downarrow 0}\mu(G', v'^\lambda) = \lim_{\lambda\downarrow 0}\mu((\frac{\lambda + v_G}{N})\mathbf{1}) = \mu(\frac{v_G}{N}\mathbf{1}). \tag{3}$$

Since $v^0 = v'^0$, by (2) and (3), $\mu(G, v) = \mu(\frac{v_G}{N}\mathbf{1})$.

Step 4. There is $\alpha \in \mathbb{R}^N_+$ with $\sum\limits_{n\in\mathcal{N}} \alpha_n = 1$ such that for each $(G, v) \in \mathcal{P}$, $\mu(G, v) = v_G\alpha$.

By Step 3, it is sufficient to show that there is $\alpha \in \mathbb{R}^N_+$ with $\sum\limits_{n\in\mathcal{N}} \alpha_n = 1$ such that for each $c \in \mathbb{R}_+$, $\mu(\frac{c}{N}\mathbf{1}) = c\alpha$. To show it, we first check that for each pair $c, c' \in \mathbb{R}_+$, $\mu(\frac{c+c'}{N}\mathbf{1}) = \mu(\frac{c}{N}\mathbf{1}) + \mu(\frac{c'}{N}\mathbf{1})$. Let $c, c' \in \mathbb{R}_+$. Let $G := (D, E) \in \mathcal{G}$ be such that $D = \{d_1, d_2, d_3\}$ and $E = \{(d_1, d_2), (d_2, d_3)\}$ as depicted in Figure 11. Let $e_1 := (d_1, d_2)$ and $e_2 := (d_2, d_3)$. Let

$v \in \mathcal{V}^G$ be such that $v(e_1) = \frac{c}{N}\mathbf{1}$ and $v(e_2) = \frac{c'}{N}\mathbf{1}$. Since $\mathcal{L}^G$ contains the single path $\{d_j\}_{j=1}^3$, it is uniquely selected by φ, and thus by *sequential composition*, $\mu(G, v) = \mu(G|_{d_2}, v|_{d_2}) + \mu(G|^{d_2}, v|^{d_2}) = \mu(\frac{c}{N}\mathbf{1}) + \mu(\frac{c'}{N}\mathbf{1})$. By Step 3, $\mu(G, v) = \mu(\frac{v_G}{N}\mathbf{1}) = \mu(\frac{c+c'}{N}\mathbf{1})$. Hence, $\mu(\frac{c+c'}{N}\mathbf{1}) = \mu(G, v) = \mu(\frac{c}{N}\mathbf{1}) + \mu(\frac{c'}{N}\mathbf{1})$.

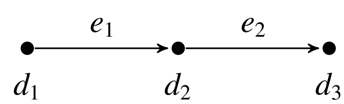

Figure 11: Network G

For each $n \in \mathcal{N}$, define $f_n : \mathbb{R}_+ \to \mathbb{R}_+$ by setting for each $c \in \mathbb{R}_+$, $f_n(c) = \mu_n(\frac{c}{N}\mathbf{1})$. For each $n \in \mathcal{N}$, by *continuity*, f_n is continuous, and since for each pair $c, c' \in \mathbb{R}_+$, $\mu(\frac{c+c'}{N}\mathbf{1}) = \mu(\frac{c}{N}\mathbf{1}) + \mu(\frac{c'}{N}\mathbf{1})$, f_n is additive. For each $n \in \mathcal{N}$, since f_n is additive and continuous, there is $\alpha_n \in \mathbb{R}$ such that for each $c \in \mathbb{R}_+$, $f_n(c) = \alpha_n c$, and by the definition of a solution, $\alpha_n \geq 0$ and $\sum\limits_{n \in \mathcal{N}} \alpha_n = 1$, as desired. □

Proof of Theorem 2. The "if" direction is readily verified, so the proof is omitted. To show the "only if" direction, let (φ, μ) be a solution satisfying *technology monotonicity*. Define $f : \mathbb{R}_+ \to \mathbb{R}_+^N$ by setting for each $c \in \mathbb{R}_+$, $f(c) = \mu(\frac{c}{N}\mathbf{1})$. Let $(G, v) \in \mathcal{P}$ with $G = (D, E)$. To show that $\mu(G, v) = f(v_G)$, it is equivalent to show that $\mu(G, v) = \mu(\frac{v_G}{N}\mathbf{1})$. This also implies that the path(s) in $\varphi(G, v)$ is (are) efficient.

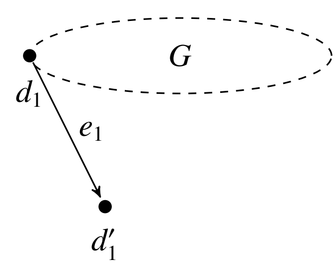

Figure 12: Enlarged network (G^K, v^K) based on G

First, we show that the solution to a process (G, v) coincides with the solution to the process restricted to one of its efficient path. To see this, let $L := \{d_j\}_{j=1}^J \in \mathcal{L}^G$ be an efficient path in (G, v). Construct a sequence $\{(G^k, v^k)\}_{k=1}^K$ of processes in the following way. First, (G^1, v^1) is the restriction of (G, v) to the path L. Precisely, $G^1 = (D^1, E^1)$ is such that $D^1 = \{d_1, \cdots, d_J\}$ and $E^1 = \{(d_j, d_{j+1}) : j = 1, \cdots, J-1\}$, and for each $e \in E^1$, $v^1(e) = v(e)$. Second, for each $k \in \{2, \cdots, K-1\}$, (G^k, v^k) is a technological improvement of (G^{k-1}, v^{k-1}), and $(G^{K-1}, v^{K-1}) := (G, v)$. It can be readily seen that there is a sequence of gradually expanding processes from (G^1, v^1) to (G, v) in which each process is a technological improvement of the previous one. Lastly, (G^K, v^K) is a technological improvement of $(G^{K-1}, v^{K-1}) = (G, v)$, with a sink $d'_1 \notin D$ and an edge $e_1 := (d_1, d'_1)$ being added (see Figure 12). Precisely, $G^K = (D^K, E^K)$ is such that $D^K = D \cup \{d'_1\}$, $E^K = E \cup \{e_1\}$, $v^K(e_1) = \frac{v_G}{N}\mathbf{1}$, and for each $e \in E$, $v^K(e) = v(e)$.

Since each process in $\{(G^k, v^k)\}_{k=1}^K$ is a technological improvement of the previous one, by *technology monotonicity*, $\mu(G^K, v^K) \geq \mu(G^{K-1}, v^{K-1}) \geq \mu(G^1, v^1)$, and thus $\sum\limits_{n \in \mathcal{N}} \mu_n(G^K, v^K) \geq \sum\limits_{n \in \mathcal{N}} \mu_n(G, v) \geq \sum\limits_{n \in \mathcal{N}} \mu_n(G^1, v^1)$. Since $\mathcal{L}^{G^1} = \{L\}$, by the definition of a solution, $\sum\limits_{n \in \mathcal{N}} \mu_n(G^1, v^1) = v^1_L$. Since a technological improvement does not increase the value of the efficient path, $v_G \geq \sum\limits_{n \in \mathcal{N}} \mu_n(G^K, v^K) \geq \sum\limits_{n \in \mathcal{N}} \mu_n(G, v) \geq \sum\limits_{n \in \mathcal{N}} \mu_n(G^1, v^1) = v^1_L = v_G$. Thus, $\sum\limits_{n \in \mathcal{N}} \mu_n(G^K, v^K) = \sum\limits_{n \in \mathcal{N}} \mu_n(G, v)$. Since $\mu(G^K, v^K) \geq \mu(G^{K-1}, v^{K-1}) = \mu(G, v)$ and $\sum\limits_{n \in \mathcal{N}} \mu_n(G^K, v^K) = \sum\limits_{n \in \mathcal{N}} \mu_n(G, v)$, $\mu(G^K, v^K) = \mu(G, v) = \mu(G^1, v^1)$.

Next, we note that the process (G^K, v^K) has two efficient paths, (G^1, v^1) and $(\frac{v_G}{N}\mathbf{1})$. Therefore, by repeating the above argument starting from $(\frac{v_G}{N}\mathbf{1})$ instead of (G^1, v^1), we have that $\mu(G^K, v^K) = \mu(\frac{v_G}{N}\mathbf{1})$. Since $\mu(G^K, v^K) = \mu(G, v) = \mu(G^1, v^1)$, $\mu(G, v) = \mu(\frac{v_G}{N}\mathbf{1})$ holds. □

Proof of Theorem 3. The "if" direction is readily verified, so the proof is omitted. To show the "only if" direction, let (φ, μ) be a solution satisfying *split invariance*, *irrelevance of dominated paths*, *parallel composition*, and *continuity*.

Step 1. For each $x \in \mathbb{R}^N_+$, $\mu(x \cup x) = \mu(x)$.

Let $x \in \mathbb{R}^N_+$. For each $k \in \mathbb{N}$, by *irrelevance of dominated paths*, $\mu((x + \frac{1}{k}\mathbf{1}) \cup x) = \mu(x + \frac{1}{k}\mathbf{1})$, and thus $\lim\limits_{k\to\infty} \mu((x + \frac{1}{k}\mathbf{1}) \cup x) = \lim\limits_{k\to\infty} \mu(x + \frac{1}{k}\mathbf{1})$. Since $\lim\limits_{k\to\infty}(x + \frac{1}{k}\mathbf{1}) = x$, by *continuity*, $\lim\limits_{k\to\infty} \mu((x + \frac{1}{k}\mathbf{1}) \cup x) = \mu(x \cup x)$ and $\lim\limits_{k\to\infty} \mu(x + \frac{1}{k}\mathbf{1}) = \mu(x)$. Hence, $\mu(x \cup x) = \mu(x)$.

Step 2. Define $r : \mathbb{R}^N_+ \to \mathbb{R}^N_+$ by setting for each $x \in \mathbb{R}^N_+$, $r(x) = \mu(x)$. Then for each $x \in \mathbb{R}^N_+$, $\sum\limits_{n \in \mathcal{N}} r_n(x) = \sum\limits_{n \in \mathcal{N}} x_n$ and $r(r(x)) = r(x)$.

Let $x \in \mathbb{R}^N_+$. By definition of a solution, $\sum\limits_{n \in \mathcal{N}} \mu_n(x) = \sum\limits_{n \in \mathcal{N}} x_n$, and thus $\sum\limits_{n \in \mathcal{N}} r_n(x) = \sum\limits_{n \in \mathcal{N}} x_n$. To see that $r(r(x)) = r(x)$, let $y := r(x)$. Thus, $\mu(x) = y$, and by *parallel composition*, $\mu(x \cup x) = \mu(\mu(x) \cup x) = \mu(\mu(x) \cup \mu(x)) = \mu(y \cup y)$. By Step 1, $\mu(x \cup x) = \mu(x)$ and $\mu(y \cup y) = \mu(y)$. Then $r(y) = \mu(y) = \mu(x) = y$, as desired.

Step 3. Define a binary relation $\succsim$ over $r(\mathbb{R}^N_+)$ by setting for each pair $x, y \in r(\mathbb{R}^N_+)$, $y \succsim x$ if $\mu(x \cup y) = y$. Then $\succsim$ is a partial order.

By Step 1, for each $x \in r(\mathbb{R}^N_+)$, $\mu(x \cup x) = x$, so $\succsim$ is reflexive. By definition of $\succsim$, it is antisymmetric. To show that $\succsim$ is transitive, let $x, y, z \in r(\mathbb{R}^N_+)$ be such that $y \succsim x$ and $z \succsim y$, and $z' := \mu(x \cup z)$. We want to show that $z \succsim x$. Since $y \succsim x$ and $z \succsim y$, $\mu(x \cup y) = y$ and $\mu(y \cup z) = z$. By *parallel composition* and Step 1, on the one hand,

$$\mu(z \cup x \cup y \cup z) = \mu((x \cup y) \cup (z \cup z)) = \mu(\mu(x \cup y) \cup \mu(z \cup z)) = \mu(y \cup \mu(z)) = \mu(y \cup z) = z,$$

and on the other hand,

$$\begin{aligned} &\mu(z \cup x \cup y \cup z) = \mu(z \cup x \cup (y \cup z)) = \mu(z \cup x \cup \mu(y \cup z)) = \mu(z \cup x \cup z) \\ =&\mu(x \cup (z \cup z)) = \mu(x \cup \mu(z \cup z)) = \mu(x \cup \mu(z)) = \mu(x \cup z) = z', \end{aligned}$$

Hence, $z = z' = \mu(x \cup z)$, and thus $z \succsim x$, as desired.

Step 4. Let $K \in \mathbb{N}$, and for each $k \in \{1, \cdots, K\}$, $x^k \in r(\mathbb{R}^N_+)$. Then $\mu(\bigcup\limits_{k=1}^{K} x^k) = \max\limits_{\succsim}\{x^k : k = 1, \cdots, K\}$, i.e., $\mu(\bigcup\limits_{k=1}^{K} x^k) \in r(\mathbb{R}^N_+)$, for each $k \in \{1, \cdots, K\}$, $z \succsim x^k$, and whenever there is $z' \in r(\mathbb{R}^N_+)$ such that for each $k \in \{1, \cdots, K\}$, $z' \succsim x^k$, $z' \succsim z$.

Let $z := \mu(\bigcup\limits_{k=1}^{K} x^k)$. By definition of a solution, $z \in \mathbb{R}^N_+$, and then by *parallel composition* and Step 1,

$$\begin{aligned} &\mu(z) = \mu(z \cup z) = \mu(\mu(\bigcup_{k=1}^{K} x^k) \cup \mu(\bigcup_{k=1}^{K} x^k)) = \mu((\bigcup_{k=1}^{K} x^k) \cup \bigcup_{k=1}^{K} x^k) \\ =&\mu(\bigcup_{k=1}^{K}(x^k \cup x^k)) = \mu(\bigcup_{k=1}^{K}\mu(x^k \cup x^k)) = \mu(\bigcup_{k=1}^{K}\mu(x^k)) = \mu(\bigcup_{k=1}^{K} x^k) = z. \end{aligned}$$

Thus, by definition of r, $z \in r(\mathbb{R}^N_+)$.

For each $k' \in \{1, \cdots, K\}$, since $z = \mu(\bigcup_{k=1}^{K} x^k)$, by *parallel composition* and Step 1,

$$\begin{aligned}
&\mu(x^{k'} \cup z) = \mu(x^{k'} \cup \mu(\bigcup_{k=1}^{K} x^k)) = \mu(x^{k'} \cup \bigcup_{k=1}^{K} x^k) = \mu((x^{k'} \cup x^{k'}) \cup \bigcup_{k\in\{1,\cdots,K\}\setminus\{k'\}} x^k) \\
=&\mu(\mu(x^{k'} \cup x^{k'}) \cup \bigcup_{k\in\{1,\cdots,K\}\setminus\{k'\}} x^k) = \mu(\mu(x^{k'}) \cup \bigcup_{k\in\{1,\cdots,K\}\setminus\{k'\}} x^k) \\
=&\mu(x^{k'} \cup \bigcup_{k\in\{1,\cdots,K\}\setminus\{k'\}} x^k) = \mu(\bigcup_{k=1}^{K} x^k) = z,
\end{aligned}$$

Thus, by definition of $\succsim$, $z \succsim x^{k'}$.

Let $z' \in r(\mathbb{R}^N_+)$ be such that for each $k \in \{1, \cdots, K\}$, $z' \succsim x^k$, i.e., $z' = \mu(x^k \cup z')$. Since $z = \mu(\bigcup_{k=1}^{K} x^k)$ and for each $k \in \{1, \cdots, K\}$, $z' = \mu(x^k \cup z')$, by *parallel composition*,

$$\begin{aligned}
&\mu(z \cup z') \\
=&\mu(\mu(\bigcup_{k=1}^{K} x^k) \cup z') = \mu((\bigcup_{k=1}^{K} x^k) \cup z') \\
=&\mu((\bigcup_{k=2}^{K} x^k) \cup (x^1 \cup z')) = \mu((\bigcup_{k=2}^{K} x^k) \cup \mu(x^1 \cup z')) = \mu((\bigcup_{k=2}^{K} x^k) \cup z') \\
=&\mu((\bigcup_{k=3}^{K} x^k) \cup (x^2 \cup z')) = \mu((\bigcup_{k=3}^{K} x^k) \cup \mu(x^2 \cup z')) = \mu((\bigcup_{k=3}^{K} x^k) \cup z') \\
&\cdots \\
=&\mu(x^K \cup \mu(x^{K-1} \cup z')) = \mu(x^K \cup z') \\
=&z'.
\end{aligned}$$

Thus, $z' \succsim z$.

Step 5. The pair $(r, \succsim)$ is a monotone and continuous redistribution scheme.

By Steps 3 and 4, $r(\mathbb{R}^N_+)$ equipped with the partial order $\succsim$ is a join-semilattice, and together by Step 2, $(r, \succsim)$ is a redistribution scheme.

Next, we show that $(r, \succsim)$ is continuous. By *continuity* and the definition of r, r is continuous. To show that the join operator $\max_{\succsim}\{\cdot, \cdot\} : r(\mathbb{R}^N_+)^2 \to r(\mathbb{R}^N_+)$ is continuous, let $x, y \in r(\mathbb{R}^N_+)$, $z := \max_{\succsim}\{x, y\}$, and for each $k \in \mathbb{N}$, $x^k, y^k \in r(\mathbb{R}^N_+)$, $z^k := \max_{\succsim}\{x^k, y^k\}$ be such that $\{(x^k, y^k)\}_{k=1}^{\infty}$ converges to (x, y). Then $\lim_{k\to\infty} x^k = x$ and $\lim_{k\to\infty} y^k = y$. By Step 4, $z = \mu(x \cup y)$, and for each $k \in \mathbb{N}$, $z^k = \mu(x^k \cup y^k)$. Since $\lim_{k\to\infty} x^k = x$ and $\lim_{k\to\infty} y^k = y$, by *continuity*, $\lim_{k\to\infty} \mu(x^k \cup y^k) = \mu(x \cup y)$. Thus, $\lim_{k\to\infty} z^k = z$.

Lastly, we show that $(r, \succsim)$ is monotone. Let $x, y \in \mathbb{R}^N_+$ be such that $x \geq y$. For each $k \in \mathbb{N}$, let $x^k := x + \frac{1}{k}\mathbf{1}$ so that $x^k > y$, and thus by *irrelevance of dominated paths*, $\mu(x^k \cup y) = \mu(x^k)$. Since $\lim_{k\to\infty} x^k = x$, by *continuity*, $\mu(x \cup y) = \lim_{k\to\infty} \mu(x^k \cup y) = \lim_{k\to\infty} \mu(x^k) = \mu(x)$. By *parallel composition*, $\mu(x \cup y) = \mu(\mu(x) \cup \mu(y))$. Thus, $\mu(\mu(x) \cup \mu(y)) = \mu(x)$. By definition of r, $\mu(r(x) \cup r(y)) = r(x)$, and by definition of $\succsim$, $r(x) \succsim r(y)$.

Step 6. Let $(G, v) \in \mathcal{P}$. For each $L \in \mathcal{L}^G$ with $L = \{d_j\}_{j=1}^J$, let $(G^L, v^L) \in \mathcal{P}$ with $G^L = (D^L, E^L)$ consist of the single path L, i.e., $D^L = \{d_1, \cdots, d_J\}$, $E^L = \{(d_j, d_{j+1}) : j = 1, \cdots, J-1\}$, and for each $e \in E^L$, $v^L(e) = v(e)$. Then $\mu(G, v) = \mu(\bigcup_{L\in\mathcal{L}^G} (G^L, v^L))$.

Let E be the set of edges in G. For each $k \in \mathbb{N}$, let $(G, v^k) \in \mathcal{P}$ be such that for each $e \in E$, $v^k(e) = v(e) + \frac{1}{k}\mathbf{1}$, so that for each $L \in \mathcal{L}^G$ and each $e \in E^L$, $v^L(e) = v(e) < v^k(e)$. Then for each $k \in \mathbb{N}$, each path in $\bigcup_{L\in\mathcal{L}^G} (G^L, v^L)$ is stepwise dominated by a path in (G, v^k), so by *irrelevance of dominated paths*, $\mu((G, v^k) \cup \bigcup_{L\in\mathcal{L}^G} (G^L, v^L)) = \mu(G, v^k)$. Since for each $e \in E$, $\lim_{k\to\infty} v^k(e) = v(e)$, by *continuity*, $\mu((G, v) \cup \bigcup_{L\in\mathcal{L}^G} (G^L, v^L)) = \lim_{k\to\infty} \mu((G, v^k) \cup \bigcup_{L\in\mathcal{L}^G} (G^L, v^L)) = \lim_{k\to\infty} \mu(G, v^k) = \mu(G, v)$.

Similarly, for each $k \in \mathbb{N}$ and each $L \in \mathcal{L}^G$, let $(G^L, v^{Lk}) \in \mathcal{P}$ be such that for each $e \in E^L$, $v^{Lk}(e) = v(e) + \frac{1}{k}\mathbf{1}$, so that $v(e) < v^{Lk}(e)$. Then for each $k \in \mathbb{N}$, each path in (G, v) is stepwise dominated by a path in $\bigcup_{L\in\mathcal{L}^G} (G^L, v^{Lk})$, so by *irrelevance of dominated paths*, $\mu((G, v) \cup \bigcup_{L\in\mathcal{L}^G} (G^L, v^{Lk})) = \mu(\bigcup_{L\in\mathcal{L}^G} (G^L, v^{Lk}))$. Since for each $L \in \mathcal{L}^G$ and each $e \in E^L$, $\lim_{k\to\infty} v^{Lk}(e) = v^L(e)$, by *continuity*, $\mu((G, v) \cup \bigcup_{L\in\mathcal{L}^G} (G^L, v^L)) = \lim_{k\to\infty} \mu((G, v) \cup \bigcup_{L\in\mathcal{L}^G} (G^L, v^{Lk})) = \lim_{k\to\infty} \mu(\bigcup_{L\in\mathcal{L}^G} (G^L, v^{Lk})) = \mu(\bigcup_{L\in\mathcal{L}^G} (G^L, v^L))$.

Combining the results in the two paragraphs above, $\mu(G, v) = \mu((G, v) \cup \bigcup_{L\in\mathcal{L}^G} (G^L, v^L)) = \mu(\bigcup_{L\in\mathcal{L}^G} (G^L, v^L))$, as desired.

Step 7. The path(s) selected by φ is (are) efficient.

Let $(G, v) \in \mathcal{P}$ with $G = (D, E)$. It suffices to show that $\sum_{n\in\mathcal{N}} \mu_n(G, v) = v_G$. This is true if for each $L \in \mathcal{L}^G$, $v_L = v_G$. Suppose that there is $L \in \mathcal{L}^G$ such that $v_L < v_G$. Then $v_G > 0$. For each $L \in \mathcal{L}^G$, let $(G^L, v^L) \in \mathcal{P}$ with $G^L = (D^L, E^L)$ be defined as in Step 6. By Step 6, it is equivalent to show that $\sum_{n\in\mathcal{N}} \mu_n(\bigcup_{L\in\mathcal{L}^G} (G^L, v^L)) = v_G$.

Let $L^* \in \mathcal{L}^G$ be an efficient path. For each $\lambda \in [0, 1]$, let for each $e \in E^{L^*}$, $v^{L^*\lambda}(e) :=$

$\lambda v^{L^*}(e)+2v_G(1-\lambda)\mathbf{1}$, and for each $L \in \mathcal{L}^G \setminus \{L^*\}$ and each $e \in E^L$, $v^{L\lambda}(e) := v^L(e)$. Since $v_G > 0$, $2v_G\mathbf{1} > v_G\mathbf{1}$. Hence, for each $L \in \mathcal{L}^G \setminus \{L^*\}$ and each $e \in E^L$, $v^{L^*0}(e) = 2v_G\mathbf{1} > v_G\mathbf{1} \geq v_L\mathbf{1} \geq v^L(e) = v^{L0}(e)$. Then by *irrelevance of dominated paths*, $\mu(\bigcup_{L\in\mathcal{L}^G}(G^L, v^{L0})) = \mu((G^{L^*}, v^{L^*0}))$. Thus, $\sum_{n\in\mathcal{N}}\mu_n(\bigcup_{L\in\mathcal{L}^G}(G^L, v^{L0})) = \sum_{n\in\mathcal{N}}\mu_n((G^{L^*}, v^{L^*0})) \geq 2v_G N > v_G$. Since for each $L \in \mathcal{L}^G$ and each $e \in E^L$, $\lim_{\lambda\to 0} v^{L\lambda}(e) = v^{L0}(e)$, by *continuity*, $\lim_{\lambda\to 0}\mu(\bigcup_{L\in\mathcal{L}^G}(G^L, v^{L\lambda})) = \mu(\bigcup_{L\in\mathcal{L}^G}(G^L, v^{L0}))$. Thus, there is $\lambda' \in (0,1]$ such that for each $\lambda \in [0,\lambda']$, $\sum_{n\in\mathcal{N}}\mu_n(\bigcup_{L\in\mathcal{L}^G}(G^L, v^{L\lambda})) > v_G$.

Let $\overline{\lambda} := \sup\{\lambda' \in [0,1] :$ for each $\lambda \in [0,\lambda']$, $\sum_{n\in\mathcal{N}}\mu_n(\bigcup_{L\in\mathcal{L}^G}(G^L, v^{L\lambda})) \geq v_G\}$. By analysis in the previous paragraph, $\overline{\lambda} > 0$. By definition of $\overline{\lambda}$, for each $\lambda \in [0,\overline{\lambda})$, $\sum_{n\in\mathcal{N}}\mu_n(\bigcup_{L\in\mathcal{L}^G}(G^L, v^{L\lambda})) \geq v_G$. Then $\sup_{\lambda\in[0,\overline{\lambda})}\sum_{n\in\mathcal{N}}\mu_n(\bigcup_{L\in\mathcal{L}^G}(G^L, v^{L\lambda})) \geq v_G > \max_{L\in\mathcal{L}^G: v_L<v_G} v_L$. Since for each $L \in \mathcal{L}^G$ and each $e \in E^L$, $\lim_{\lambda\to\overline{\lambda}} v^{L\lambda}(e) = v^{L\overline{\lambda}}(e)$, by *continuity*, $\lim_{\lambda\downarrow\overline{\lambda}}\sum_{n\in\mathcal{N}}\mu_n(\bigcup_{L\in\mathcal{L}^G}(G^L, v^{L\lambda})) = \sum_{n\in\mathcal{N}}\mu_n(\bigcup_{L\in\mathcal{L}^G}(G^L, v^{L\overline{\lambda}})) = \lim_{\lambda\uparrow\overline{\lambda}}\sum_{n\in\mathcal{N}}\mu_n(\bigcup_{L\in\mathcal{L}^G}(G^L, v^{L\lambda})) = \sup_{\lambda\in[0,\overline{\lambda})}\sum_{n\in\mathcal{N}}\mu_n(\bigcup_{L\in\mathcal{L}^G}(G^L, v^{L\lambda})) > \max_{L\in\mathcal{L}^G: v_L<v_G} v_L$.

We claim that $\overline{\lambda} = 1$. Suppose on the contrary that $\overline{\lambda} < 1$. By definition of $\overline{\lambda}$, for each $\epsilon > 0$ satisfying $\overline{\lambda} + \epsilon \leq 1$, there is $\lambda \in (\overline{\lambda}, \overline{\lambda}+\epsilon)$ such that $\sum_{n\in\mathcal{N}}\mu_n(\bigcup_{L\in\mathcal{L}^G}(G^L, v^{L\lambda})) < v_G$, and since $v^{L^*\lambda}_{L^*} \geq v_{L^*} = v_G$ and for each $L \in \mathcal{L}^G \setminus \{L\}$, $v^{L\lambda}_L = v^L_L = v_L$,

$$\sum_{n\in\mathcal{N}}\mu_n(\bigcup_{L\in\mathcal{L}^G}(G^L, v^{L\lambda})) \leq \max_{L\in\mathcal{L}^G: v_L<v_G} v_L,$$

which contradicts that $\lim_{\lambda\downarrow\overline{\lambda}}\sum_{n\in\mathcal{N}}\mu_n(\bigcup_{L\in\mathcal{L}^G}(G^L, v^{L\lambda})) > \max_{L\in\mathcal{L}^G: v_L<v_G} v_L$. Hence, $\overline{\lambda} = 1$, and thus

$$\sum_{n\in\mathcal{N}}\mu_n(\bigcup_{L\in\mathcal{L}^G}(G^L, v^L)) = \sum_{n\in\mathcal{N}}\mu_n(\bigcup_{L\in\mathcal{L}^G}(G^L, v^{L1})) > \max_{L\in\mathcal{L}^G: v_L<v_G} v_L.$$

Since $\sum_{n\in\mathcal{N}}\mu_n(\bigcup_{L\in\mathcal{L}^G}(G^L, v^L)) > \max_{L\in\mathcal{L}^G: v_L<v_G} v_L$, and for each $L \in \mathcal{L}^G$, $v^L_L = v_L$, by definition of a solution, $\sum_{n\in\mathcal{N}}\mu_n(\bigcup_{L\in\mathcal{L}^G}(G^L, v^L)) = v_G$.

Step 8. The sharing rule μ is $(r, \succsim)$ - rationalizable.

Let $(G, v) \in \mathcal{P}$. For each $L \in \mathcal{L}^G$, let $(G^L, v^L) \in \mathcal{P}$ with $G^L = (D^L, E^L)$ be defined as in Step 6. By Step 6, $\mu(G, v) = \mu(\bigcup_{L\in\mathcal{L}^G}(G^L, v^L))$. By *parallel composition*, $\mu(\bigcup_{L\in\mathcal{L}^G}(G^L, v^L)) = \mu(\bigcup_{L\in\mathcal{L}^G}\mu(G^L, v^L))$. By *split invariance*, for each $L \in \mathcal{L}^G$, $\mu(G^L, v^L) = \mu(\sum_{e\in E^L} v^L(e))$. By definition of r and Step 5,

$$\mu(\bigcup_{L\in\mathcal{L}^G}\mu(\sum_{e\in E^L}v^L(e))) = \mu(\bigcup_{L\in\mathcal{L}^G}r(\sum_{e\in E^L}v^L(e))) = \max_{\succsim}\{r(\sum_{e\in E^L}v^L(e)) : L\in\mathcal{L}^G\}.$$

Thus, $\mu(G,v) = \max_{\succsim}\{r(\sum_{e\in E^L}v^L(e)) : L\in\mathcal{L}^G\} = \max_{\succsim}\{r(\sum_{j=1}^{J-1}v(e_j)) : \{d_j\}_{j=1}^J \in \mathcal{L}^G$ and for each $j\in\{1,\cdots,J-1\}, e_j = (d_j, d_{j+1})\}$. □

Proof of Theorem 4. The "if" direction is readily verified, so the proof is omitted. To show the "only if" direction, let (φ,μ) be a solution satisfying *irrelevance of dominated paths*, *irrelevance of parallel outside options*, and *continuity*.

Step 1. Let $(G,v)\in\mathcal{P}$. For each $L\in\mathcal{L}^G$ with $L=\{d_j\}_{j=1}^J$, let $(G^L,v^L)\in\mathcal{P}$ with $G^L=(D^L,E^L)$ consist of the single path L, i.e., $D^L=\{d_1,\cdots,d_J\}$, $E^L=\{(d_j,d_{j+1}) : j=1,\cdots,J-1\}$, and for each $e\in E^L$, $v^L(e)=v(e)$. Then $\mu(G,v)=\mu(\bigcup_{L\in\mathcal{L}^G}(G^L,v^L))$. Moreover, the path(s) selected by φ is (are) efficient.

Since Steps 6 and 7 in the proof of Theorem 3 use only *irrelevance of dominated paths* and *continuity*, the arguments can be applied here to show the same results.

Step 2. For each $(G,v)\in\mathcal{P}$, if $L^*\in\mathcal{L}^G$ is efficient, then $\mu(G,v)=\mu(G^{L^*},v^{L^*})$, where (G^{L^*},v^{L^*}) is defined as in Step 1.

Let $(G,v)\in\mathcal{P}$. For each $L\in\mathcal{L}^G$, let $(G^L,v^L)\in\mathcal{P}$ with $G^L=(D^L,E^L)$ be defined as in Step 1. By Step 1, $\mu(G,v)=\mu(\bigcup_{L\in\mathcal{L}^G}(G^L,v^L))$. Suppose that $L^*\in\mathcal{L}^G$ is efficient. Then for each $L\in\mathcal{L}^G\setminus\{L^*\}$, $v^{L^*}_{L^*}=v_{L^*}\geq v_L=v^L_L$. For each $k\in\mathbb{N}$, let $v^{L^*k}\in\mathcal{V}^{G^{L^*}}$ be such that for each $e\in E^{L^*}$, $v^{L^*k}(e)=v^{L^*}(e)+\frac{1}{k}\mathbf{1}$. Thus, for each $k\in\mathbb{N}$ and each $L\in\mathcal{L}^G\setminus\{L^*\}$, $v^{L^*k}_{L^*}>v^{L^*}_{L^*}\geq v^L_L$, i.e., L^* is the unique efficient path in $(G^{L^*},v^{L^*k})\cup\bigcup_{L\in\mathcal{L}^G\setminus\{L^*\}}(G^L,v^L)$, so by Step 1, $\varphi((G^{L^*},v^{L^*k})\cup\bigcup_{L\in\mathcal{L}^G\setminus\{L^*\}}(G^L,v^L))=\{L^*\}$. Then for each $k\in\mathbb{N}$, $\sum_{n\in\mathcal{N}}\mu_n((G^{L^*},v^{L^*k})\cup\bigcup_{L\in\mathcal{L}^G\setminus\{L^*\}}(G^L,v^L))=\sum_{n\in\mathcal{N}}\mu_n(G^{L^*},v^{L^*k})>\max_{L\in\mathcal{L}^G\setminus\{L^*\}}v^L_L$, and thus by *irrelevance of parallel outside options*, $\mu((G^{L^*},v^{L^*k})\cup\bigcup_{L\in\mathcal{L}^G\setminus\{L^*\}}(G^L,v^L))=\mu(G^{L^*},v^{L^*k})$. Since for each $e\in E^{L^*}$, $\lim_{k\to\infty}v^{L^*k}(e)=v^{L^*}(e)$, by *continuity*, $\mu(\bigcup_{L\in\mathcal{L}^G}(G^L,v^L))=\lim_{k\to\infty}\mu((G^{L^*},v^{L^*k})\cup\bigcup_{L\in\mathcal{L}^G\setminus\{L^*\}}(G^L,v^L))=\lim_{k\to\infty}\mu(G^{L^*},v^{L^*k})=\mu(G^{L^*},v^{L^*})$. Hence, $\mu(G,v)=\mu(\bigcup_{L\in\mathcal{L}^G}(G^L,v^L))=\mu(G^{L^*},v^{L^*})$, as desired.

Step 3. For each pair $(G,v),(G',v')\in\mathcal{P}$, if $v_G=v'_{G'}$, then $\mu(G,v)=\mu(G',v')$.

Let $(G, v), (G', v') \in \mathcal{P}$. Let $L \in \mathcal{L}^G$ be an efficient path in (G, v), and $L' \in \mathcal{L}^{G'}$ an efficient path in (G', v'). Let (G^L, v^L) and $(G'^{L'}, v'^{L'})$ be defined as in Step 1. By Step 2, $\mu(G, v) = \mu(G^L, v^L)$ and $\mu(G', v') = \mu(G'^{L'}, v'^{L'})$. Since $v_G = v'_{G'}$, both L and L' are efficient paths in $(G, v) \cup (G', v')$. Then by Step 2, $\mu((G, v) \cup (G', v')) = \mu(G^L, v^L)$ and $\mu((G, v) \cup (G', v')) = \mu(G'^{L'}, v'^{L'})$, so that $\mu(G^L, v^L) = \mu(G'^{L'}, v'^{L'})$. Hence, $\mu(G, v) = \mu(G^L, v^L) = \mu(G'^{L'}, v'^{L'}) = \mu(G', v')$, as desired.

Step 4. Define $f : \mathbb{R}_+ \to \mathbb{R}^N_+$ by setting for each $c \in \mathbb{R}_+$, $f(c) = \mu(\frac{c}{N}\mathbf{1})$. Then f is continuous, and for each $(G, v) \in \mathcal{P}$, $\mu(G, v) = f(v_G)$.

It can be readily seen that by *continuity*, f is continuous. Let $(G, v) \in \mathcal{P}$. Let $L^* \in \mathcal{L}^G$ be an efficient path and (G^{L^*}, v^{L^*}) be defined as in Step 1. By Step 2, $\mu(G, v) = \mu(G^{L^*}, v^{L^*})$. Since $v^{L^*}_{G^{L^*}} = v^{L^*}_{L^*} = v_{L^*} = v_G$, by Step 3, $\mu(G^{L^*}, v^{L^*}) = \mu(\frac{v_G}{N}\mathbf{1})$. Thus, $\mu(G, v) = \mu(\frac{v_G}{N}\mathbf{1}) = f(v_G)$. □

Before proceeding to prove Proposition 2 and Theorem 5, we first prove two useful lemmas.

Lemma 1. *Let $c > 0$ and $X \subseteq \{x \in \mathbb{R}^N_+ : \sum_{n \in \mathcal{N}} x_n = c\}$. Assume that (1) $\frac{c}{N}\mathbf{1} \in X$, (2) for each pair $x, y \in X$, there is $\mathcal{S} \subseteq \mathcal{N}$ such that $|\mathcal{S}| \geq M$ and for each $n \in \mathcal{S}$, $x_n \geq y_n$, and (3) for each permutation π of $\mathcal{N}$ and each $x \in X$, $x^\pi \in X$. Then for each $x \in X$, there is $\mathcal{S} \subseteq \mathcal{N}$ such that $|\mathcal{S}| \geq M$ and for each $n \in \mathcal{S}$, $x_n = \frac{c}{N}$.*

Proof. For each pair $x, y \in X$, let $\mathcal{N}^{xy}_> := \{n \in \mathcal{N} : x_n > y_n\}$, $\mathcal{N}^{xy}_< := \{n \in \mathcal{N} : x_n < y_n\}$, and $\mathcal{N}^{xy}_= := \{n \in \mathcal{N} : x_n = y_n\}$; by assumption (1), $\frac{c}{N}\mathbf{1} \in X$, and when $y = \frac{c}{N}\mathbf{1}$, we simply omit the superscript y in $\mathcal{N}^{xy}_>$, $\mathcal{N}^{xy}_<$, and $\mathcal{N}^{xy}_=$, and write $\mathcal{N}^x_>$, $\mathcal{N}^x_<$, and $\mathcal{N}^x_=$ respectively instead. For each pair $x, y \in X$, by assumption (2), $|\mathcal{N}^{xy}_> \cup \mathcal{N}^{xy}_=| \geq M$ and $|\mathcal{N}^{yx}_> \cup \mathcal{N}^{yx}_=| \geq M$, so that $|\mathcal{N}^{xy}_<| \leq N - M$, $|\mathcal{N}^{xy}_>| = |\mathcal{N}^{yx}_<| \leq N - M$, and

$$|\mathcal{N}^{xy}_=| = N - |\mathcal{N}^{xy}_>| - |\mathcal{N}^{xy}_<| \geq N - 2(N - M) = 2M - N. \tag{4}$$

In particular, for each $x \in X$, $|\mathcal{N}^x_<| \leq N - M$, $|\mathcal{N}^x_>| \leq N - M$, and $|\mathcal{N}^x_=| \geq 2M - N$.

Let $x \in X$. To show that there is $\mathcal{S} \subseteq \mathcal{N}$ such that $|\mathcal{S}| \geq M$ and for each $n \in \mathcal{S}$, $x_n = \frac{c}{N}$, it is equivalent to show that $|\mathcal{N}^x_=| \geq M$. Suppose to the contrary that $|\mathcal{N}^x_=| < M$. We shall find a contradiction in each of the following two cases.

Case 1: $|\mathcal{N}^x_=| \geq \frac{N}{2}$. Since $|\mathcal{N}^x_=| \geq \frac{N}{2} \geq |\mathcal{N}^x_> \cup \mathcal{N}^x_<|$, we can pick a permutation π of $\mathcal{N}$ such that $\pi(\mathcal{N}^x_> \cup \mathcal{N}^x_<) \subseteq \mathcal{N}^x_=$. By assumption (3), $x^\pi \in X$. Since for each $n \in \mathcal{N}$,

$x^\pi_{\pi(n)} = \frac{c}{N} \Leftrightarrow x_n = \frac{c}{N}$, $\mathcal{N}^{x^\pi}_= = \pi(\mathcal{N}^x_=)$. Then

$$\mathcal{N}^{x^\pi}_> \cup \mathcal{N}^{x^\pi}_< = \mathcal{N} \setminus \mathcal{N}^{x^\pi}_= = \mathcal{N} \setminus \pi(\mathcal{N}^x_=) = \pi(\mathcal{N}^x_> \cup \mathcal{N}^x_<) \subseteq \mathcal{N}^x_=. \tag{5}$$

Since $\mathcal{N}^{x^\pi}_> \cup \mathcal{N}^{x^\pi}_< \subseteq \mathcal{N}^x_=$, $\mathcal{N} \setminus \mathcal{N}^x_= \subseteq \mathcal{N} \setminus (\mathcal{N}^{x^\pi}_> \cup \mathcal{N}^{x^\pi}_<)$ and thus $\mathcal{N}^x_> \cup \mathcal{N}^x_< \subseteq \mathcal{N}^{x^\pi}_=$. See Figure 13 for an illustration.

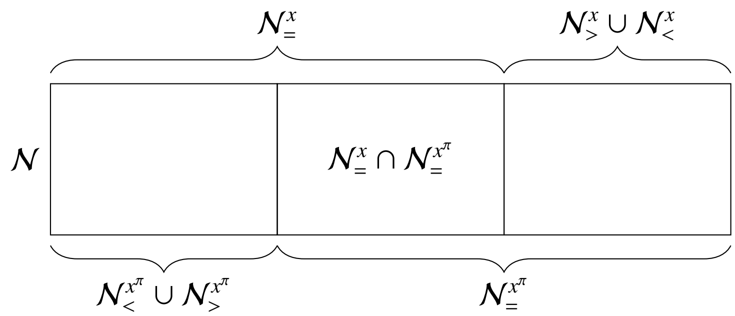

Figure 13: Permutation π of $\mathcal{N}$

We claim that $\mathcal{N}^{xx^\pi}_= = \mathcal{N}^x_= \cap \mathcal{N}^{x^\pi}_=$. Since for each $n \in \mathcal{N}^x_= \cap \mathcal{N}^{x^\pi}_=$, $x_n = \frac{c}{N} = x^\pi_n$, $\mathcal{N}^x_= \cap \mathcal{N}^{x^\pi}_= \subseteq \mathcal{N}^{xx^\pi}_=$. We now show that $\mathcal{N}^{xx^\pi}_= \subseteq \mathcal{N}^x_= \cap \mathcal{N}^{x^\pi}_=$. Since $\mathcal{N}^{x^\pi}_> \cup \mathcal{N}^{x^\pi}_< \subseteq \mathcal{N}^x_=$, for each $n \in \mathcal{N}^{x^\pi}_> \cup \mathcal{N}^{x^\pi}_<$, $x^\pi_n \neq \frac{c}{N} = x_n$. Thus, $\mathcal{N}^{xx^\pi}_= \subseteq \mathcal{N} \setminus (\mathcal{N}^{x^\pi}_> \cup \mathcal{N}^{x^\pi}_<) = \mathcal{N}^{x^\pi}_=$. Similarly, since $\mathcal{N}^x_> \cup \mathcal{N}^x_< \subseteq \mathcal{N}^{x^\pi}_=$, for each $n \in \mathcal{N}^x_> \cup \mathcal{N}^x_<$, $x_n \neq \frac{c}{N} = x^\pi_n$. Thus, $\mathcal{N}^{xx^\pi}_= \subseteq \mathcal{N} \setminus (\mathcal{N}^x_> \cup \mathcal{N}^x_<) = \mathcal{N}^x_=$. Hence, $\mathcal{N}^{xx^\pi}_= \subseteq \mathcal{N}^x_= \cap \mathcal{N}^{x^\pi}_=$.

Since $\mathcal{N}^{xx^\pi}_= = \mathcal{N}^x_= \cap \mathcal{N}^{x^\pi}_= = \mathcal{N}^x_= \setminus (\mathcal{N}^{x^\pi}_> \cup \mathcal{N}^{x^\pi}_<)$, and since (5) gives $\mathcal{N}^{x^\pi}_> \cup \mathcal{N}^{x^\pi}_< = \pi(\mathcal{N}^x_> \cup \mathcal{N}^x_<) \subseteq \mathcal{N}^x_=$,

$$|\mathcal{N}^{xx^\pi}_=| = |\mathcal{N}^x_=| - |\mathcal{N}^{x^\pi}_> \cup \mathcal{N}^{x^\pi}_<| = |\mathcal{N}^x_=| - |\mathcal{N}^x_> \cap \mathcal{N}^x_<| = |\mathcal{N}^x_=| - (N - |\mathcal{N}^x_=|) = 2|\mathcal{N}^x_=| - N. \tag{6}$$

By our hypothesis that $|\mathcal{N}^x_=| < M$, (6) implies $|\mathcal{N}^{xx^\pi}_=| < 2M - N$, which contradicts (4), as desired.

Case 2: $|\mathcal{N}^x_=| < \frac{N}{2}$. Recall that $|\mathcal{N}^x_<| \leq N - M$ and $|\mathcal{N}^x_>| \leq N - M$, and since $M > \frac{N}{2}$, $|\mathcal{N}^x_<| < \frac{N}{2}$ and $|\mathcal{N}^x_>| < \frac{N}{2}$. Assume that $|\mathcal{N}^x_>| \leq |\mathcal{N}^x_<| \leq |\mathcal{N}^x_=|$. Since $|\mathcal{N}^x_=| < \frac{N}{2}$, $|\mathcal{N}^x_=| < N - |\mathcal{N}^x_=| = |\mathcal{N}^x_> \cup \mathcal{N}^x_<|$. Since $|\mathcal{N}^x_>| \leq |\mathcal{N}^x_=| < |\mathcal{N}^x_> \cup \mathcal{N}^x_<|$ and $|\mathcal{N}^x_<| \leq |\mathcal{N}^x_=|$, we can pick a permutation π of N such that $\mathcal{N}^x_> \subseteq \pi(\mathcal{N}^x_=) \subseteq \mathcal{N}^x_> \cup \mathcal{N}^x_<$ and $\pi(\mathcal{N}^x_<) \subseteq \mathcal{N}^x_=$. By assumption (3), $x^\pi \in X$. Since for each $n \in \mathcal{N}$, $x^\pi_{\pi(n)} = \frac{c}{N} \Leftrightarrow x_n = \frac{c}{N}$ and $x^\pi_{\pi(n)} < \frac{c}{N} \Leftrightarrow x_n < \frac{c}{N}$, $\mathcal{N}^{x^\pi}_= = \pi(\mathcal{N}^x_=)$ and $\mathcal{N}^{x^\pi}_< = \pi(\mathcal{N}^x_<)$. Then $\mathcal{N}^x_> \subseteq \mathcal{N}^{x^\pi}_= \subseteq \mathcal{N}^x_> \cup \mathcal{N}^x_<$, $\mathcal{N}^{x^\pi}_< \subseteq \mathcal{N}^x_=$. Since $\mathcal{N}^x_> \subseteq \mathcal{N}^{x^\pi}_=$, $\mathcal{N}^{x^\pi}_> = \mathcal{N} \setminus (\mathcal{N}^{x^\pi}_< \cup \mathcal{N}^{x^\pi}_=) \subseteq \mathcal{N} \setminus \mathcal{N}^x_> = \mathcal{N}^x_< \cup \mathcal{N}^x_=$. See Figure 14 for an illustration.

Since $\mathcal{N}^{x^\pi}_= \subseteq \mathcal{N}^x_> \cup \mathcal{N}^x_<$, for each $n \in \mathcal{N}^{x^\pi}_=$, $x^\pi_n = \frac{c}{N} \neq x_n$. Thus, $\mathcal{N}^{xx^\pi}_= \subseteq \mathcal{N} \setminus \mathcal{N}^{x^\pi}_=$. Since

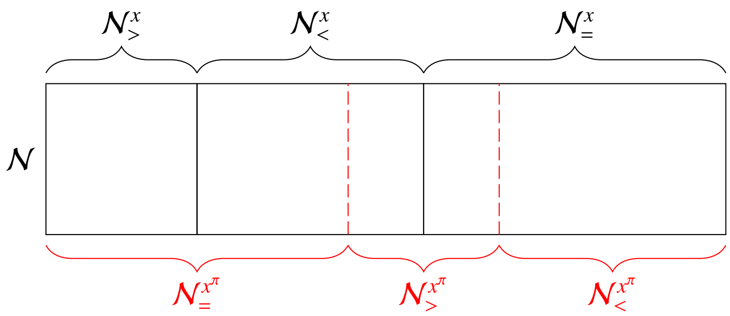

Figure 14: Permutation π of $\mathcal{N}$

$\mathcal{N}^{x^\pi}_< \subseteq \mathcal{N}^x_=$, for each $n \in \mathcal{N}^{x^\pi}_<$, $x^\pi_n < \frac{c}{N} = x_n$. Thus, $\mathcal{N}^{xx^\pi}_= \subseteq \mathcal{N} \setminus \mathcal{N}^{x^\pi}_<$. Since $\mathcal{N}^{x^\pi}_> \subseteq \mathcal{N}^x_< \cup \mathcal{N}^x_=$, for each $n \in \mathcal{N}^{x^\pi}_>$, $x^\pi_n > \frac{c}{N} \geq x_n$. Thus, $\mathcal{N}^{xx^\pi}_= \subseteq \mathcal{N} \setminus \mathcal{N}^{x^\pi}_>$. Since $\mathcal{N}^{xx^\pi}_= \subseteq \mathcal{N} \setminus (\mathcal{N}^{x^\pi}_= \cup \mathcal{N}^{x^\pi}_< \cup \mathcal{N}^{x^\pi}_>)$, $\mathcal{N}^{xx^\pi}_= = \emptyset$. Since $\mathcal{N}^{xx^\pi}_= = \emptyset$ and $M > \frac{N}{2}$, $|\mathcal{N}^{xx^\pi}_=| = 0 < 2M - N$, which contradicts (4), as desired.

Relaxing the assumption that $|\mathcal{N}^x_>| \leq |\mathcal{N}^x_<| \leq |\mathcal{N}^x_=|$, observe that the arguments above essentially rely on the fact that $\max\{|\mathcal{N}^x_=|, |\mathcal{N}^x_=|, |\mathcal{N}^x_=|\} < \frac{N}{2}$. For a different ordering of $|\mathcal{N}^x_=|$, $|\mathcal{N}^x_=|$, and $|\mathcal{N}^x_=|$, an analogous permutation of $\mathcal{N}$ can be constructed and analogous arguments applied to derive a contradiction. □

Lemma 2. *Suppose that μ satisfies anonymity. For each $\ell \in \mathcal{L}$ with $\ell = \{x^j\}_{j=1}^J$ and each pair $n, n' \in \mathcal{N}$, if for each $k \in \{1, \cdots, J\}$, $x^j_n = x^j_{n'}$, then $\mu_n(\ell) = \mu_{n'}(\ell)$.*

Proof. Suppose to the contrary that there are $\ell \in \mathcal{L}$ with $\ell = \{x^j\}_{j=1}^J$ and $n, n' \in \mathcal{N}$ such that for each $j \in \{1, \cdots, J\}$, $x^j_n = x^j_{n'}$, and $\mu_n(\ell) \neq \mu_{n'}(\ell)$. Define $\pi : \mathcal{N} \to \mathcal{N}$ by setting for each $m \in \mathcal{N} \setminus \{n, n'\}$, $\pi(m) = m$, $\pi(n) = n'$, and $\pi(n') = n$. For each $j \in \{1, \cdots, J\}$, Since $x^j_n = x^j_{n'}$, by definition of π, $x^j = x^{j\pi}$, and thus $\ell = \ell^\pi$. By anonymity, $\mu(\ell)^\pi = \mu(\ell^\pi)$. Since $\mu_n(\ell) = \mu_{\pi(n)}(\ell)^\pi = \mu_{n'}(\ell)^\pi$ and $\mu(\ell)^\pi = \mu(\ell^\pi)$, $\mu_n(\ell) = \mu_{n'}(\ell^\pi)$. Since $\ell = \ell^\pi$ and $\mu_n(\ell) = \mu_{n'}(\ell^\pi)$, $\mu_n(\ell) = \mu_{n'}(\ell)$, which contradicts that $\mu_n(\ell) \neq \mu_{n'}(\ell)$, as desired. □

Proof of Proposition 2. The "if" direction is readily verified, so the proof is omitted. To show the "only if" direction, let μ be a sharing rule satisfying *M-majority* and *anonymity*.

Let $\ell \in \mathcal{L}$, $c := v_\ell$, and $X := \{\mu(\ell') : \ell' \in \mathcal{L}, v_{\ell'} = c\}$. We want to show that there is $\mathcal{S} \subseteq \mathcal{N}$ such that $|\mathcal{S}| \geq M$ and for each $n \in \mathcal{S}$, $\mu_n(\ell) = \frac{v_\ell}{N}$. Note that $\mu(\ell) \in X \subseteq \{x \in \mathbb{R}^N_+ : \sum_{n \in \mathcal{N}} x_n = c\}$. Thus by Lemma 1, it suffices to show that X satisfies assumptions (1) - (3) in Lemma 1. By Lemma 2, $\mu(\frac{c}{N}\mathbf{1}) = \frac{c}{N}\mathbf{1}$, so assumption (1) is satisfied. By *M-majority*, assumption (2) is also satisfied, and by *anonymity*, so is assumption (3). □

Proof of Theorem 5. The "if" direction is readily verified, so the proof is omitted. To show the "only if" direction, let μ be a sharing rule satisfying *M-majority* and *anonymity*. We want to show that if μ satisfies either of the following three additional axioms, then for each $\ell \in \mathcal{L}$, $\mu(\ell) = \frac{v_\ell}{N}\mathbf{1}$.

Suppose that μ satisfies *sequential composition*. Suppose to the contrary that there is $\ell \in \mathcal{L}$ with $\ell = \{x^j\}_{j=1}^J$ such that $\mu(\ell) \neq \frac{v_\ell}{N}\mathbf{1}$. For each $\ell' \in \mathcal{L}$, let $\mathcal{N}_=^{\ell'} := \{n \in \mathcal{N} : \mu_n(\ell') = \frac{v_{\ell'}}{N}\}$ and $\mathcal{N}_{\neq}^{\ell'} := \mathcal{N} \setminus \mathcal{N}_=^{\ell'}$. Since $\mu(\ell) \neq \frac{v_\ell}{N}\mathbf{1}$, $\mathcal{N}_{\neq}^{\ell} \neq \emptyset$. By Proposition 2, $|\mathcal{N}_=^{\ell}| \geq M$. Since $|\mathcal{N}_=^{\ell}| \geq M > \frac{N}{2}$ and $|\mathcal{N}_{\neq}^{\ell}| = N - |\mathcal{N}_=^{\ell}| \leq N - M < \frac{N}{2}$, $|\mathcal{N}_=^{\ell}| > |\mathcal{N}_{\neq}^{\ell}|$. Thus, we can pick a permutation π of $\mathcal{N}$ such that $\pi(\mathcal{N}_{\neq}^{\ell}) \subseteq \mathcal{N}_=^{\ell}$. By *anonymity*, $\mu(\ell^\pi) = \mu(\ell)^\pi$. Since for each $n \in \mathcal{N}$, $\mu_{\pi(n)}(\ell^\pi) \neq \frac{v_\ell}{N} \Leftrightarrow \mu_n(\ell) \neq \frac{v_\ell}{N}$, $\mathcal{N}_{\neq}^{\ell^\pi} = \pi(\mathcal{N}_{\neq}^{\ell})$. Since $\mathcal{N}_{\neq}^{\ell^\pi} = \pi(\mathcal{N}_{\neq}^{\ell}) \subseteq \mathcal{N}_=^{\ell}$ and $\mathcal{N}_{\neq}^{\ell} \neq \emptyset$, $\emptyset \neq \mathcal{N}_{\neq}^{\ell^\pi} \subseteq \mathcal{N}_=^{\ell}$ and thus $\mathcal{N}_{\neq}^{\ell} = \mathcal{N} \setminus \mathcal{N}_=^{\ell} \subseteq \mathcal{N} \setminus \mathcal{N}_{\neq}^{\ell^\pi} = \mathcal{N}_=^{\ell^\pi}$. See Figure 15(a) for an illustration.

Let $\ell^1 := \{y^j\}_{j=1}^{2J}$ where for each $j \in \{1, \cdots, J\}$, $y^j = x^j$ and $y^{J+j} = x^{j\pi}$. We shall show that $\mathcal{N}_=^{\ell^1} \subsetneq \mathcal{N}_=^{\ell}$. By *sequential composition*, $\mu(\ell^1) = \sum_{j=1}^{2J} \mu(y^j) = \sum_{j=1}^{J} \mu(y^j) + \sum_{j=1}^{J} \mu(y^{J+j}) = \sum_{j=1}^{J} \mu(x^j) + \sum_{j=1}^{J} \mu(x^{j\pi}) = \mu(\ell) + \mu(\ell^\pi)$. Since $\mathcal{N}_{\neq}^{\ell^\pi} \subseteq \mathcal{N}_=^{\ell}$, for each $n \in \mathcal{N}_{\neq}^{\ell^\pi}$, $\mu_n(\ell^\pi) \neq \frac{v_{\ell^\pi}}{N}$ and $\mu_n(\ell) = \frac{v_\ell}{N}$, and thus $\mu_n(\ell^1) = \mu_n(\ell) + \mu_n(\ell^\pi) \neq \frac{v_\ell + v_{\ell^\pi}}{N} = \frac{v_{\ell^1}}{N}$. Thus, $\mathcal{N}_=^{\ell^1} \subseteq \mathcal{N} \setminus \mathcal{N}_{\neq}^{\ell^\pi}$. Since $\mathcal{N}_{\neq}^{\ell} \subseteq \mathcal{N}_=^{\ell^\pi}$, for each $n \in \mathcal{N}_{\neq}^{\ell}$, $\mu_n(\ell) \neq \frac{v_\ell}{N}$ and $\mu_n(\ell^\pi) = \frac{v_{\ell^\pi}}{N}$, and thus $\mu_n(\ell^1) = \mu_n(\ell) + \mu_n(\ell^\pi) \neq \frac{v_\ell + v_{\ell^\pi}}{N} = \frac{v_{\ell^1}}{N}$. Thus, $\mathcal{N}_=^{\ell^1} \subseteq \mathcal{N} \setminus \mathcal{N}_{\neq}^{\ell} = \mathcal{N}_=^{\ell}$. Since $\mathcal{N}_=^{\ell^1} \subseteq \mathcal{N}_=^{\ell} \setminus \mathcal{N}_{\neq}^{\ell^\pi}$ and $\emptyset \neq \mathcal{N}_{\neq}^{\ell^\pi} \subseteq \mathcal{N}_=^{\ell}$, $\mathcal{N}_=^{\ell^1} \subsetneq \mathcal{N}_=^{\ell}$. Thus, $|\mathcal{N}_=^{\ell^1}| \leq N - 1$ and $\emptyset \neq \mathcal{N}_=^{\ell} \setminus \mathcal{N}_=^{\ell^1} \subseteq \mathcal{N} \setminus \mathcal{N}_=^{\ell^1} = \mathcal{N}_{\neq}^{\ell^1}$. See Figure 15(b).

Since $\mathcal{N}_{\neq}^{\ell^1} \neq \emptyset$, we can apply the analogous arguments as in the last two paragraphs to show that there is $\ell^2 \in \mathcal{L}$ such that $|\mathcal{N}_=^{\ell^2}| \leq N - 2$ and $\mathcal{N}_{\neq}^{\ell^2} \neq \emptyset$. Repeating the arguments, eventually we can find $\ell^K \in \mathcal{L}$, where $K \in \{1, \cdots, N - M + 1\}$, such that $|\mathcal{N}_=^{\ell^K}| < M$, which contradicts Proposition 2, as desired.

Suppose that μ satisfies *Lorenz monotonicity*. Let $\ell \in \mathcal{L}$ with $\ell = \{x^j\}_{j=1}^J$. If $N = 1$, by definition of a sharing rule, $\mu(\ell) = \frac{v_\ell}{N}\mathbf{1}$. Suppose that $N \geq 2$. Let $\ell' := \{x'^j\}_{j=1}^J$ be such that for each $j \in \{1, \cdots, J\}$, $x_1'^j = \sum_{n \in \mathcal{N}} x_n^j$ and for each $n \in \mathcal{N} \setminus \{1\}$, $x_n'^j = 0$. Thus, $v_\ell = v_{\ell'}$, and for each $j \in \{1, \cdots, J\}$, $x^j \succ_{Lorenz} x'^j$. By *Lorenz monotonicity*, $\mu(\ell') \not\succ_{Lorenz} \mu(\ell)$. Since for each $n \in \mathcal{N} \setminus \{1\}$ and each $j \in \{1, \cdots, J\}$, $x_n'^j = x_N'^j$, by Lemma 2, $\mu_n(\ell') = \mu_N(\ell')$. Suppose that $\mu_N(\ell') \neq \frac{v_{\ell'}}{N}$. Then for each $n \in \mathcal{N}$, $\mu_n(\ell') \neq \frac{v_{\ell'}}{N}$, which contradicts Proposition 2. Hence, $\mu_N(\ell') = \frac{v_{\ell'}}{N}$, and thus $\mu(\ell') = \frac{v_{\ell'}}{N}\mathbf{1} = \frac{v_\ell}{N}\mathbf{1}$. Hence, $\mu(\ell') = \frac{v_\ell}{N}\mathbf{1}$. Since $\mu(\ell') = \frac{v_\ell}{N}\mathbf{1}$ and $\mu(\ell') \not\succ_{Lorenz} \mu(\ell)$, $\frac{v_\ell}{N}\mathbf{1} \not\succ_{Lorenz} \mu(\ell)$ and thus $\mu(\ell) = \frac{v_\ell}{N}\mathbf{1}$.

Suppose that μ satisfies *transfer monotonicity*. Let $\ell \in \mathcal{L}$ with $\ell = \{x^j\}_{j=1}^J$. If $N = 1$, by definition of a sharing rule, $\mu(\ell) = \frac{v_\ell}{N}\mathbf{1}$. Suppose that $N \geq 2$. Fix $n \in \mathcal{N}$. For each

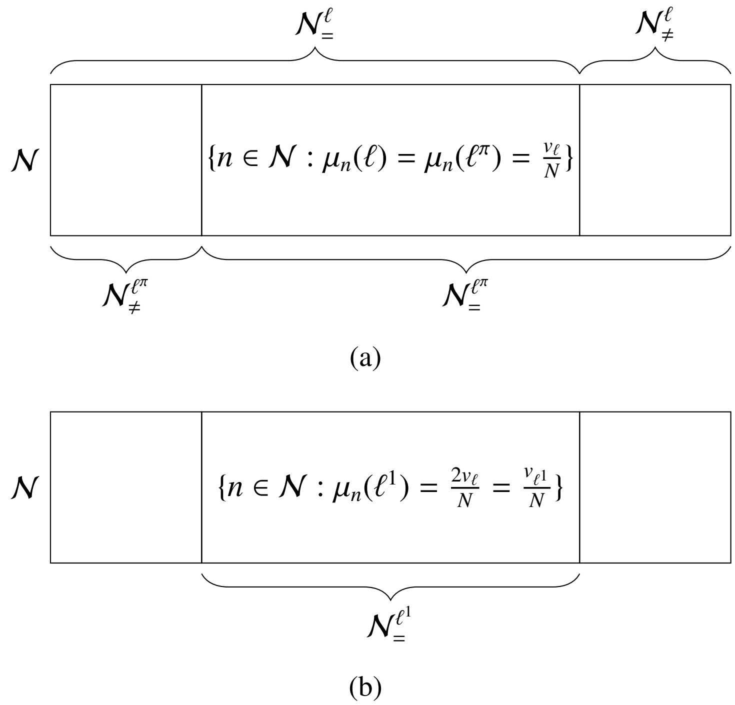

Figure 15: Permutation π of $\mathcal{N}$

$m \in \mathcal{N}$, let $\ell^m := \{x^{mj}\}_{j=1}^{J}$, where for each $j \in \{1, \cdots, J\}$, $x_n^{mj} = x_n^j + \sum_{n' \in \{1,\cdots,m\}\setminus\{n\}} x_{n'}^j$, for each $n' \in \{1, \cdots, m\} \setminus \{n\}$, $x_{n'}^{mj} = 0$ and for each $n' \in \mathcal{N} \setminus (\{1, \cdots, m\} \cup \{n\})$, $x_{n'}^{mj} = x_{n'}$. Thus, for each $j \in \{1, \cdots, J\}$, $x_n^{1j} - x_n^j = x_1^j - x_1^{1j} \geq 0$ and for each $n' \in \mathcal{N} \setminus \{n, 1\}$, $x_{n'}^j = x_{n'}^{1j}$; for each $m \in \mathcal{N} \setminus \{1\}$ and each $j \in \{1, \cdots, J\}$, $x_n^{mj} - x_n^{(m-1)j} = x_m^{(m-1)j} - x_m^{mj} \geq 0$ and for each $n' \in \mathcal{N} \setminus \{n, m\}$, $x_{n'}^{(m-1)j} = x_{n'}^{mj}$. Hence, $v_\ell = v_{\ell^1} = \cdots = v_{\ell^N}$, and by *transfer monotonicity*, $\mu_n(\ell) \leq \mu_n(\ell^1) \leq \cdots \leq \mu_n(\ell^N)$. Since for each pair $n', n'' \in \mathcal{N} \setminus \{n\}$ and each $j \in \{1, \cdots, J\}$, $x_{n'}^{Nj} = x_{n''}^{Nj}$, by Lemma 2, $\mu_{n'}(\ell^N) = \mu_{n''}(\ell^N)$. Then $\mu_n(\ell^N) \neq \frac{v_{\ell^N}}{N}$ implies that for each $n' \in \mathcal{N}$, $\mu_{n'}(\ell^N) \neq \frac{v_{\ell^N}}{N}$, which contradicts Proposition 2. Hence, $\mu_n(\ell^N) = \frac{v_{\ell^N}}{N}$. Thus, $\mu_n(\ell) \leq \mu_n(\ell^N) = \frac{v_{\ell^N}}{N} = \frac{v_\ell}{N}$.

Since the choice of n is arbitrary, the analysis in the previous paragraph shows that for each $n \in \mathcal{N}$, $\mu_n(\ell) \leq \frac{v_\ell}{N}$. By definition of a sharing rule, $\sum_{n \in \mathcal{N}} \mu_n(\ell) = v_\ell$. Since $\sum_{n \in \mathcal{N}} \mu_n(\ell) = v_\ell$ and for each $n \in \mathcal{N}$, $\mu_n(\ell) \leq \frac{v_\ell}{N}$, $\mu(\ell) = \frac{v_\ell}{N}$. $\square$

A.2 Local egalitarianism with transfers — an n-agent version of Example 5.

Imagine that N agents are divided into two groups according to their exogenous types. Consider the redistribution function that divides equally within each group the sum of the values of its group members. For instance, a partnership firm runs two businesses and adopts the equal sharing rule respectively for its partners involved in each business (Burrows and Black [13], Baskenille-Morley and Beechey [7]).

Formally, let $\mathcal{S} \subsetneq \mathcal{N}$ be such that $\mathcal{S} \neq \emptyset$. Let $M := |\mathcal{S}|$. Then $0 < M < N$. Define $r : \mathbb{R}^N_+ \to \mathbb{R}^N_+$ by setting for each $x \in \mathbb{R}^N_+$,

$$r_n(x) = \begin{cases} \frac{1}{M} \sum\limits_{n' \in \mathcal{S}} x_{n'} & \text{if } n \in \mathcal{S}, \\ \frac{1}{N-M} \sum\limits_{n' \in \mathcal{N} \setminus \mathcal{S}} x_{n'} & \text{if } n \in \mathcal{N} \setminus \mathcal{S}. \end{cases}$$

Note that for each $x \in r(\mathbb{R}^N_+)$ and each pair n, n' that are both in $\mathcal{S}$ or $\mathcal{N} \setminus \mathcal{S}$, $x_n = x_{n'}$. When $N = 2$, r is simply the identity mapping that assigns to each agent his individual value.

For each $x \in \mathbb{R}^N_+$, let $\bar{x} := \frac{1}{N} \sum\limits_{n \in \mathcal{N}} x_n$. Define a binary relation $\succsim$ on $r(\mathbb{R}^N_+)$ by setting for each pair $x, y \in r(\mathbb{R}^N_+)$, $x \succsim y$ if there is $\lambda \in [0, 1]$ such that

$$x \geq \lambda y + (1 - \lambda)\bar{y}\mathbf{1}.$$

Note that for each pair $x, y \in r(\mathbb{R}^N_+)$, when $x \geq y$, $x \succsim y$; when $\sum\limits_{n \in \mathcal{N}} x_n = \sum\limits_{n \in \mathcal{N}} y_n$, $x \succsim y$ if and only if x is a mixture between y and the equal-sharing allocation $\bar{y}\mathbf{1}$.

Lemma 3. *For each pair $x, y \in r(\mathbb{R}^N_+)$, $x \succsim y$ if and only if $\sum\limits_{n \in \mathcal{N}} x_n \geq \sum\limits_{n \in \mathcal{N}} y_n$ and for each $n \in \mathcal{N}$, $x_n \geq \min\{y_n, \bar{y}\}$.*

Proof. The "only if" direction is readily verified, so the proof is omitted. To show the "if" direction, let $x, y \in r(\mathbb{R}^N_+)$ be such that $\sum\limits_{n \in \mathcal{N}} x_n \geq \sum\limits_{n \in \mathcal{N}} y_n$ and for each $n \in \mathcal{N}$, $x_n \geq \min\{y_n, \bar{y}\}$. Assume without loss of generality that $1 \in \mathcal{S}$ and $2 \in \mathcal{N} \setminus \mathcal{S}$. Since $x, y \in r(\mathbb{R}^N_+)$, by definition of r, for each $n \in \mathcal{S}$, $x_n = x_1$ and $y_n = y_1$, and for each $n \in \mathcal{N} \setminus \mathcal{S}$, $x_n = x_2$ and $y_n = y_2$. Note that either $y_1 \leq \bar{y} \leq y_2$ or $y_2 < \bar{y} < y_1$.

We shall prove $x \succsim y$ only in the case $y_1 \leq \bar{y} \leq y_2$, and analogous arguments can be applied to the other case. Since $y_1 \leq \bar{y} \leq y_2$, $x_1 \geq \min\{y_1, \bar{y}\} = y_1$ and $x_2 \geq \min\{y_2, \bar{y}\} = \bar{y}$. If $x_2 \geq y_2$, then $x \geq y$, and thus by definition of $\succsim$, $x \succsim y$. Suppose that $x_2 < y_2$. Let $\lambda := \frac{(N-M)(x_2 - \bar{y})}{M(\bar{y} - y_1)}$. Since $\bar{y} \leq x_2 < y_2$ and $y_1 = \frac{1}{M}[N\bar{y} - (N - M)y_2]$, we have $y_1 < \bar{y}$, and thus

$\lambda \in [0, \infty)$. Since $x_2 < y_2$, $My_1 + (N-M)x_2 \leq My_1 + (N-M)y_2 = \sum_{n \in \mathcal{N}} y_n = N\bar{y} = M\bar{y} + (N-M)\bar{y}$, and thus $(N-M)(x_2 - \bar{y}) \leq M(\bar{y} - y_1)$ so that $\lambda \leq 1$. Observe that

$$\begin{aligned}
\lambda y_2 + (1-\lambda)\bar{y} =& \frac{(N-M)(x_2-\bar{y})}{M(\bar{y}-y_1)} y_2 + [1 - \frac{(N-M)(x_2-\bar{y})}{M(\bar{y}-y_1)}]\bar{y} \\
=& \frac{(N-M)y_2 - (N-M)\bar{y}}{M(\bar{y}-y_1)} x_2 + \bar{y} - \frac{(N-M)y_2 - (N-M)\bar{y}}{M(\bar{y}-y_1)}\bar{y} \\
=& \frac{N\bar{y} - My_1 - (N-M)\bar{y}}{M(\bar{y}-y_1)} x_2 + \bar{y} - \frac{N\bar{y} - My_1 - (N-M)\bar{y}}{M(\bar{y}-y_1)}\bar{y} \\
=& x_2.
\end{aligned}$$

Since $\lambda y_2 + (1-\lambda)\bar{y} = x_2$ and $\sum_{n \in \mathcal{N}} y_n \leq \sum_{n \in \mathcal{N}} x_n$,

$$\begin{aligned}
& M[\lambda y_1 + (1-\lambda)\bar{y}] + (N-M)x_2 \\
=& M[\lambda y_1 + (1-\lambda)\bar{y}] + (N-M)[\lambda y_2 + (1-\lambda)\bar{y}] \\
=& \lambda[My_1 + (N-M)y_2] + (1-\lambda)\bar{y} \\
=& \sum_{n \in \mathcal{N}} y_n \leq \sum_{n \in \mathcal{N}} x_n \\
=& Mx_1 + (N-M)x_2,
\end{aligned}$$

and thus $\lambda y_1 + (1-\lambda)\bar{y} \leq x_1$. Since $\lambda y_1 + (1-\lambda)\bar{y} \leq x_1$ and $\lambda y_2 + (1-\lambda)\bar{y} = x_2$, $\lambda y + (1-\lambda)\bar{y}\mathbf{1} \leq x$. Hence, by definition of $\succsim$, $x \succsim y$. $\square$

Proposition 3. *The binary relation $\succsim$ is a partial order, and $r(\mathbb{R}^N_+)$ equipped with $\succsim$ is a join-semilattice.*

Proof. First, we show that $\succsim$ is a partial order. By definition of $\succsim$, $\succsim$ is reflexive. To show that $\succsim$ is antisymmetric, let $x, y \in r(\mathbb{R}^N_+)$ be such that $x \succsim y$ and $y \succsim x$. Then by Lemma 3, $\sum_{n \in \mathcal{N}} x_n = \sum_{n \in \mathcal{N}} y_n$, and for each $n \in \mathcal{N}$, $x_n \geq \min\{y_n, \bar{y}\}$ and $y_n \geq \min\{x_n, \bar{x}\}$. If $y = \bar{y}\mathbf{1}$, then for each $n \in \mathcal{N}$, $x_n \geq \bar{y} = \bar{x}$, and thus $x = \bar{x}\mathbf{1} = \bar{y}\mathbf{1} = y$. Suppose that $y \neq \bar{y}\mathbf{1}$. Then we can pick $m \in \mathcal{N}$ such that $y_m < \bar{y}$. Since $x_m \geq \min\{y_m, \bar{y}\}$ and $y_m < \bar{y}$, $x_m \geq y_m$. Since $y_m \geq \min\{x_m, \bar{x}\}$ and $y_m < \bar{y} = \bar{x}$, $y_m \geq x_m$. Since $x_m \geq y_m$ and $y_m \geq x_m$, $x_m = y_m$. Suppose that $m \in \mathcal{S}$. Since $x_m = y_m$, by definition of r, for each $n \in \mathcal{S}$, $x_n = x_m = y_m = y_n$, and since $\bar{x} = \bar{y}$, for each $n \in \mathcal{N} \setminus \mathcal{S}$, $x_n = \frac{1}{N-M}(N\bar{x} - Mx_m) = \frac{1}{N-M}(N\bar{y} - My_m) = y_n$. Hence, $x = y$. Similarly, if $m \in \mathcal{N} \setminus \mathcal{S}$, it can be shown that $x = y$. Lastly, to show that $\succsim$ is transitive, let $x, y, z \in r(\mathbb{R}^N_+)$ be such that $x \succsim y$ and $y \succsim z$. By Lemma 3, $\sum_{n \in \mathcal{N}} x_n \geq \sum_{n \in \mathcal{N}} y_n \geq \sum_{n \in \mathcal{N}} z_n$, and for each $n \in \mathcal{N}$, $x_n \geq \min\{y_n, \bar{y}\}$ and $y_n \geq \min\{z_n, \bar{z}\}$. Hence, $\sum_{n\in\mathcal{N}} x_n \geq \sum_{n\in\mathcal{N}} z_n$ and for each $n \in \mathcal{N}$, $x_n \geq \min\{\min\{z_n, \bar{z}\}, \bar{y}\} = \min\{z_n, \bar{z}\}$. Thus, by Lemma 3, $x \succsim z$. Overall, since $\succsim$ is reflexive, antisymmetric, and transitive, $\succsim$ is a partial order.

Second, we show that $r(\mathbb{R}^N_+)$ equipped with $\succsim$ is a join-semilattice. Let $x, y \in r(\mathbb{R}^N_+)$. Suppose without loss of generality that $\sum_{n\in\mathcal{N}} x_n \geq \sum_{n\in\mathcal{N}} y_n$, and that $1 \in \mathcal{S}$ and $2 \in \mathcal{N} \setminus \mathcal{S}$. Since $x, y \in r(\mathbb{R}^N_+)$, by definition of r, for each $n \in \mathcal{S}$, $x_n = x_1$ and $y_n = y_1$, and for each $n \in \mathcal{N} \setminus \mathcal{S}$, $x_n = x_2$ and $y_n = y_2$. Let *med* denote the operator that takes the median of a finite list of real numbers. Let $z^* \in \mathbb{R}^N$ be such that

$$z^*_n = \begin{cases} med\{\min\{y_1, \bar{y}\}, x_1, \frac{1}{M}[N\bar{x} - (N-M)\min\{y_2, \bar{y}\}]\} & \text{if } n \in \mathcal{S}, \\ \frac{1}{N-M}(N\bar{x} - Mz^*_1) & \text{if } n \in \mathcal{N} \setminus \mathcal{S}. \end{cases} \tag{7}$$

Since $\sum_{n\in\mathcal{N}} x_n \geq \sum_{n\in\mathcal{N}} y_n$, $\min\{y_2, \bar{y}\} \leq \bar{y} \leq \bar{x}$, and thus

$$\frac{1}{M}[N\bar{x} - (N-M)\min\{y_2, \bar{y}\}] \geq \bar{x} \geq \bar{y} \geq \min\{y_1, \bar{y}\}. \tag{8}$$

By (8), for each $n \in \mathcal{S}$, $\frac{1}{M}[N\bar{x} - (N-M)\min\{y_2, \bar{y}\}] \geq z^*_n \geq \min\{y_1, \bar{y}\} \geq 0$. Since $1 \in \mathcal{S}$, $\frac{1}{M}[N\bar{x} - (N-M)\min\{y_2, \bar{y}\}] \geq z^*_1$, and thus for each $n \in \mathcal{N} \setminus \mathcal{S}$, $z^*_n = \frac{1}{N-M}(N\bar{x} - Mz^*_1) \geq \frac{1}{N-M}(N-M)\min\{y_2, \bar{y}\} = \min\{y_2, \bar{y}\} \geq 0$. Since for each $n \in \mathcal{N}$, $z^*_n \geq 0$ and $r(z^*) = z^*$, $z^* \in r(\mathbb{R}^N_+)$. We want to show that $z^* = \max_{\succsim}\{x, y\}$. That is, $z^* \succsim x$, $z^* \succsim y$, and for each $z \in r(\mathbb{R}^N_+)$ such that $z \succsim x$ and $z \succsim y$, $z \succsim z^*$.

Since $\sum_{n\in\mathcal{N}} z^*_n = N\bar{x} = \sum_{n\in\mathcal{N}} x_n \geq \sum_{n\in\mathcal{N}} y_n$, and since for each $n \in \mathcal{S}$, $z^*_n \geq \min\{y_1, \bar{y}\} = \min\{y_n, \bar{y}\}$, and for each $n \in \mathcal{N} \setminus \mathcal{S}$, $z^*_n \geq \min\{y_2, \bar{y}\} = \min\{y_n, \bar{y}\}$, by Lemma 3, $z^* \succsim y$. Fix $z \in r(\mathbb{R}^N_+)$ be such that $z \succsim x$ and $z \succsim y$. By (8), either $\frac{1}{M}[N\bar{x}-(N-M)\min\{y_2, \bar{y}\}] = \min\{y_1, \bar{y}\}$ or $\frac{1}{M}[N\bar{x} - (N-M)\min\{y_2, \bar{y}\}] > \min\{y_1, \bar{y}\}$. We shall show that $z \succsim z^* \succsim x$ in each of the two cases.

Case 1: $\frac{1}{M}[N\bar{x} - (N-M)\min\{y_2, \bar{y}\}] = \min\{y_1, \bar{y}\}$. Then by (8),

$$\frac{1}{M}[N\bar{x} - (N-M)\min\{y_2, \bar{y}\}] = \bar{x} = \bar{y} = \min\{y_1, \bar{y}\},$$

which implies that $N\bar{y} = N\bar{x} = M\min\{y_1, \bar{y}\} + (N-M)\min\{y_2, \bar{y}\}$. Hence, $y_1 = y_2 = \bar{y} = \bar{x}$, and thus $y = \bar{y}\mathbf{1} = \bar{x}\mathbf{1} = z^*$. Since $z^* = \bar{x}\mathbf{1}$, by definition of $\succsim$, $z^* \succsim x$. Since $z \succsim y$ and $y = z^*$, $z \succsim z^*$.

Case 2: $\frac{1}{M}[N\bar{x}-(N-M)\min\{y_2, \bar{y}\}] > \min\{y_1, \bar{y}\}$. Since $1 \in S$, there are three possibilities:

$z_1^* = \min\{y_1, \bar{y}\}$, $z_1^* = x_1$, or $z_1^* = \frac{1}{M}[N\bar{x} - (N-M)\min\{y_2, \bar{y}\}]$. First, suppose first that $z_1^* = x_1$. Then for each $n \in \mathcal{S}$, $z_n^* = z_1^* = x_1 = x_n$, and for each $n \in \mathcal{N} \setminus \mathcal{S}$, $z_n^* = \frac{1}{N-M}(N\bar{x} - Mz_1^*) = \frac{1}{N-M}(N\bar{x} - Mx_1^*) = x_n$. Thus, $z^* = x$. Since $z^* = x$ and $\succsim$ is reflexive, $z^* \succsim x$. Since $z^* = x$ and $z \succsim x$, $z \succsim z^*$.

Second, suppose that $z_1^* = \min\{y_1, \bar{y}\}$. Since $\frac{1}{M}[N\bar{x} - (N-M)\min\{y_2, \bar{y}\}] > \min\{y_1, \bar{y}\} = z_1^*$, by definition of z_1, $z_1^* \geq x_1$. Since $z_1^* = \min\{y_1, \bar{y}\}$, by (8), $\bar{x} \geq z_1^*$. Since $\bar{x} \geq z_1^* \geq x_1$, for each $n \in \mathcal{S}$, $z_n^* = z_1^* \geq \min\{x_1, \bar{x}\} = \min\{x_n, \bar{x}\}$ and for each $n \in \mathcal{N} \setminus \mathcal{S}$, $z_n^* = \frac{1}{N-M}(N\bar{x} - Mz_1^*) \geq \frac{1}{N-M}(N\bar{x} - M\bar{x}) = \bar{x} \geq \min\{x_n, \bar{x}\}$. Since $\sum_{n\in\mathcal{N}} z_n^* = \sum_{n\in\mathcal{N}} x_n$ and for each $n \in \mathcal{N}$, $z_n^* \geq \min\{x_n \bar{x}\}$, by Lemma 3, $z^* \succsim x$. Moreover, since $\sum_{n\in\mathcal{N}} z_n^* = \sum_{n\in\mathcal{N}} x_n$ and $\bar{x} \geq z_1^* \geq x_1$,

$$\min\{z_1^*, \bar{z}^*\} = \min\{z_1^*, \bar{x}\} = z_1^* = \min\{y_1, \bar{y}\}, \tag{9}$$

and

$$\min\{z_2^*, \bar{z}^*\} = \min\{\frac{1}{N-M}(N\bar{x} - Mz_1^*), \bar{x}\} \leq \min\{\frac{1}{N-M}(N\bar{x} - Mx_1), \bar{x}\} = \min\{x_2, \bar{x}\}. \tag{10}$$

Since $z \succsim x$ and $z \succsim y$, by Lemma 3, $\sum_{n\in\mathcal{N}} z_n \geq \sum_{n\in\mathcal{N}} x_n = \sum_{n\in\mathcal{N}} z_n^*$, $z_1 \geq \min\{y_1, \bar{y}\}$, and $z_2 \geq \min\{x_2, \bar{x}\}$. Since $z_1 \geq \min\{y_1, \bar{y}\}$, by (9), for each $n \in \mathcal{S}$, $z_n = z_1 \geq \min\{z_1^*, \bar{z}^*\} = \min\{z_n^*, \bar{z}^*\}$. Since $z_2 \geq \min\{x_2, \bar{x}\}$, by (10), for each $n \in \mathcal{N} \setminus \mathcal{S}$, $z_n = z_2 \geq \min\{z_2^*, \bar{z}^*\} = \min\{z_n^*, \bar{z}^*\}$. Since $\sum_{n\in\mathcal{N}} z_n \geq \sum_{n\in\mathcal{N}} z_n^*$ and for each $n \in \mathcal{N}$, $z_n \geq \min\{z_n^*, \bar{z}^*\}$, by Lemma 3, $z \succsim z^*$.

Lastly, suppose that $z_1^* = \frac{1}{M}[N\bar{x} - (N-M)\min\{y_2, \bar{y}\}]$. Since $z_1^* = \frac{1}{M}[N\bar{x} - (N-M)\min\{y_2, \bar{y}\}] > \min\{y_1, \bar{y}\}$, by definition of z_1^*, $x_1 \geq z_1^*$. Since $z_1^* = \frac{1}{M}[N\bar{x} - (N-M)\min\{y_2, \bar{y}\}]$, by (8), $z_1^* \geq \bar{x}$. Since $x_1 \geq z_1^* \geq \bar{x}$, for each $n \in \mathcal{S}$, $z_n^* = z_1^* \geq \min\{x_1, \bar{x}\} = \min\{x_n, \bar{x}\}$ and for each $n \in \mathcal{N} \setminus \mathcal{S}$, $z_n^* = \frac{1}{N-M}(\bar{x} - Mz_1^*) \geq \frac{1}{N-M}(N\bar{x} - Mx_1) = x_n \geq \min\{x_n, \bar{x}\}$. Since $\sum_{n\in\mathcal{N}} z_n^* = \sum_{n\in\mathcal{N}} x_n$ and for each $n \in \mathcal{N}$, $z_n^* \geq \min\{x_n \bar{x}\}$, by Lemma 3, $z^* \succsim x$. Moreover, since $\sum_{n\in\mathcal{N}} z_n^* = \sum_{n\in\mathcal{N}} x_n$, $x_1 \geq z_1^* \geq \bar{x} \geq \bar{y}$, and $z_1^* = \frac{1}{M}[N\bar{x} - (N-M)\min\{y_2, \bar{y}\}]$,

$$\min\{z_1^*, \bar{z}^*\} = \min\{z_1^*, \bar{x}\} = \bar{x} = \min\{x_1, \bar{x}\}, \tag{11}$$

and

$$\min\{z_2^*, \bar{z}^*\} = \min\{\frac{1}{N-M}(N\bar{x} - Mz_1^*), \bar{x}\} = \min\{\min\{y_2, \bar{y}\}, \bar{x}\} = \min\{y_2, \bar{y}\}. \tag{12}$$

Since $z \succsim x$ and $z \succsim y$, by Lemma 3, $\sum_{n\in\mathcal{N}} z_n \geq \sum_{n\in\mathcal{N}} x_n = \sum_{n\in\mathcal{N}} z_n^*$, $z_1 \geq \min\{x_1, \bar{x}\}$, and $z_2 \geq$

$\min\{y_2, \bar{y}\}$. Since $z_1 \geq \min\{x_1, \bar{x}\}$, by (11), for each $n \in \mathcal{S}$, $z_n = z_1 \geq \min\{z_1^*, \bar{z}^*\} = \min\{z_n^*, \bar{z}^*\}$. Since $z_2 \geq \min\{y_2, \bar{y}\}$, by (12), for each $n \in \mathcal{N} \setminus \mathcal{S}$, $z_n = z_2 \geq \min\{z_2^*, \bar{z}^*\} = \min\{z_n^*, \bar{z}^*\}$. Since $\sum_{n \in \mathcal{N}} z_n \geq \sum_{n \in \mathcal{N}} z_n^*$ and for each $n \in \mathcal{N}$, $z_n \geq \min\{z_n^*, \bar{z}^*\}$, by Lemma 3, $z \succsim z^*$. □

Proposition 4. *The pair $(r, \succsim)$ is a monotone and continuous redistribution scheme. Moreover, for each finite list of elements of $r(\mathbb{R}_+^N)$, $x^1, \cdots, x^K$, if $x^* = \max_{\succsim}\{x^1, \cdots, x^K\}$, then $\sum_{n \in \mathcal{N}} x_n^* = \max\{\sum_{n \in \mathcal{N}} x_n^1, \cdots, \sum_{n \in \mathcal{N}} x_n^K\}$.*

Proof. By definition of r, for each $x \in \mathbb{R}_+^N$, $\sum_{n \in \mathcal{N}} r_n(x) = \sum_{n \in \mathcal{N}} x_n$ and $r(r(x)) = r(x)$. By Proposition 3, $\succsim$ is a partial order over $r(\mathbb{R}_+^N)$ equipped with which $r(\mathbb{R}_+^N)$ is a join-semilattice. Thus, $(r, \succsim)$ is a redistribution scheme.

To show that $(r, \succsim)$ is monotone, let $x, y \in \mathbb{R}_+^N$ be such that $x \geq y$. Since $x \geq y$, for each $n \in \mathcal{S}$, $r_n(x) = \frac{1}{M} \sum_{n' \in \mathcal{S}} x_{n'} \geq \frac{1}{M} \sum_{n' \in \mathcal{S}} y_{n'} = r_n(y)$ and for each $n \in \mathcal{N} \setminus \mathcal{S}$, $r_n(x) = \frac{1}{N-M} \sum_{n' \in \mathcal{N} \setminus \mathcal{S}} x_{n'} \geq \frac{1}{N-M} \sum_{n' \in \mathcal{N} \setminus \mathcal{S}} y_{n'} = r_n(y)$. Thus, $r(x) \geq r(y)$. Since $r(x) \geq r(y)$, by definition of $\succsim$, $x \succsim y$.

Next, we show that $(r, \succsim)$ is continuous. By definition of r, r is continuous. To show that the join operator $\max_{\succsim}\{\cdot, \cdot\} : r(\mathbb{R}_+^N)^2 \to r(\mathbb{R}_+^N)$ is continuous, let $x, y \in r(\mathbb{R}_+^N)$, $z := \max_{\succsim}\{x, y\}$, and for each $k \in \mathbb{N}$, $x^k, y^k \in r(\mathbb{R}_+^N)$, $z^k := \max_{\succsim}\{x^k, y^k\}$ be such that $\{(x^k, y^k)\}_{k=1}^{\infty}$ converges to (x, y). We shall show that $\lim_{k \to \infty} z^k = z$. Suppose without loss of generality that $\sum_{n \in \mathcal{N}} x_n \geq \sum_{n \in \mathcal{N}} y_n$, and that $1 \in \mathcal{S}$ and $2 \in \mathcal{N} \setminus \mathcal{S}$. We divide the proof into two cases: $\sum_{n \in \mathcal{N}} x_n > \sum_{n \in \mathcal{N}} y_n$ and $\sum_{n \in \mathcal{N}} x_n = \sum_{n \in \mathcal{N}} y_n$.

Consider the case $\sum_{n \in \mathcal{N}} x_n > \sum_{n \in \mathcal{N}} y_n$. Since $\sum_{n \in \mathcal{N}} x_n > \sum_{n \in \mathcal{N}} y_n$ and $\{(x^k, y^k)\}_{k=1}^{\infty}$ converges to (x, y), for each sufficiently large $k \in \mathbb{N}$, $\sum_{n \in \mathcal{N}} x_n^k > \sum_{n \in \mathcal{N}} y_n^k$. Since $\sum_{n \in \mathcal{N}} x_n > \sum_{n \in \mathcal{N}} y_n$ and $z = \max_{\succsim}\{x, y\}$, by proof of Proposition 3 (see (7)), z is given by

$$z_n = \begin{cases} med\{\min\{y_1, \bar{y}\}, x_1, \frac{1}{M}[N\bar{x} - (N-M)\min\{y_2, \bar{y}\}]\} & \text{if } n \in \mathcal{S}, \\ \frac{1}{N-M}(N\bar{x} - Mz_1) & \text{if } n \in \mathcal{N} \setminus \mathcal{S}. \end{cases} \tag{13}$$

Similarly, for each sufficiently large $k \in \mathbb{N}$, z^k is given by

$$z_n^k = \begin{cases} med\{\min\{y_1^k, \bar{y}^k\}, x_1^k, \frac{1}{M}[N\bar{x}^k - (N-M)\min\{y_2^k, \bar{y}^k\}]\} & \text{if } n \in \mathcal{S}, \\ \frac{1}{N-M}(N\bar{x}^k - Mz_1^k) & \text{if } n \in \mathcal{N} \setminus \mathcal{S}. \end{cases} \tag{14}$$

Since $\{(x^k, y^k)\}_{k=1}^{\infty}$ converges to (x, y), by (13) and (14), $\lim_{k \to \infty} z^k = z$.

Consider the other case $\sum_{n\in\mathcal{N}} x_n = \sum_{n\in\mathcal{N}} y_n$. Then we have both $\sum_{n\in\mathcal{N}} x_n \geq \sum_{n\in\mathcal{N}} y_n$ and $\sum_{n\in\mathcal{N}} y_n \geq \sum_{n\in\mathcal{N}} x_n$. Since $\sum_{n\in\mathcal{N}} x_n \geq \sum_{n\in\mathcal{N}} y_n$, $\sum_{n\in\mathcal{N}} y_n \geq \sum_{n\in\mathcal{N}} x_n$, and $z = \max_{\succsim}\{x, y\}$, by proof of Proposition 3 (see (7)), z is given by either (13) or

$$z_n = \begin{cases} med\{\min\{x_1, \bar{x}\}, y_1, \frac{1}{M}[N\bar{y} - (N-M)\min\{x_2, \bar{x}\}]\} & \text{if } n \in \mathcal{S}, \\ \frac{1}{N-M}(N\bar{y} - Mz_1) & \text{if } n \in \mathcal{N} \setminus \mathcal{S}. \end{cases} \tag{15}$$

Similarly, for each $k \in \mathbb{N}$ such that $\sum_{n\in\mathcal{N}} x_n^k \geq \sum_{n\in\mathcal{N}} y_n^k$, z^k is given by (14), and for each $k \in \mathbb{N}$ such that $\sum_{n\in\mathcal{N}} y_n^k > \sum_{n\in\mathcal{N}} x_n^k$, z^k is given by

$$z_n^k = \begin{cases} med\{\min\{x_1^k, \bar{x}^k\}, y_1^k, \frac{1}{M}[N\bar{y}^k - (N-M)\min\{x_2^k, \bar{x}^k\}]\} & \text{if } n \in \mathcal{S}, \\ \frac{1}{N-M}(N\bar{y}^k - Mz_1^k) & \text{if } n \in \mathcal{N} \setminus \mathcal{S}. \end{cases} \tag{16}$$

For each subsequence $\{(x^{k_j}, y^{k_j})\}_{j=1}^{\infty}$ of $\{(x^k, y^k)\}_{k=1}^{\infty}$ such that for each $j \in \mathbb{N}$, $\sum_{n\in\mathcal{N}} x_n^{k_j} \geq \sum_{n\in\mathcal{N}} y_n^{k_j}$, since $\{(x^{k_j}, y^{k_j})\}_{j=1}^{\infty}$ converges to (x, y), by (13) and (14), $\lim_{j\to\infty} z^{k_j} = z$. For each subsequence $\{(x^{k_j}, y^{k_j})\}_{j=1}^{\infty}$ of $\{(x^k, y^k)\}_{k=1}^{\infty}$ such that for each $j \in \mathbb{N}$, $\sum_{n\in\mathcal{N}} y_n^{k_j} > \sum_{n\in\mathcal{N}} x_n^{k_j}$, since $\{(x^{k_j}, y^{k_j})\}_{j=1}^{\infty}$ converges to (x, y), by (15) and (16), $\lim_{j\to\infty} z^{k_j} = z$. Hence, $\lim_{k\to\infty} z^k = z$.

Lastly, let $x^1, \cdots, x^K$ be a finite list of elements of $r(\mathbb{R}_+^N)$ and $x^* := \max_{\succsim}\{x^1, \cdots, x^K\}$. Note that for each pair $x, y \in r(\mathbb{R}_+^N)$, if $z^* = \max\{x, y\}$, then by proof of Proposition 3 (see (7)), $\sum_{n\in\mathcal{N}} z_n^* = \max\{\sum_{n\in\mathcal{N}} x_n, \sum_{n\in\mathcal{N}} y_n\}$. Thus, since x^* can be obtained by repeatedly applying the join operator to pairs of elements of $r(\mathbb{R}_+^N)$, $\sum_{n\in\mathcal{N}} x_n^* = \max\{\sum_{n\in\mathcal{N}} x_n^1, \cdots, \sum_{n\in\mathcal{N}} x_n^K\}$. □

Define $\varphi : \mathcal{P} \rightrightarrows \bigcup_{G\in\mathcal{G}} \mathcal{L}^G$ and $\mu : \mathcal{P} \to \mathbb{R}_+^N$ by setting for each $(G, v) \in \mathcal{P}$, $\varphi(G, v) := \{L \in \mathcal{L}^G : v_L = v_G\}$ and $\mu(G, v) := \max_{\succsim}\{r(\sum_{j=1}^{J} v(e_j)) : \{e_j\}_{j=1}^J \in \mathcal{L}^G\}$. Since for each $x \in \mathbb{R}_+^N$, $\sum_{n\in\mathcal{N}} r_n(x) = \sum_{n\in\mathcal{N}} x_n$, for each $(G, v) \in \mathcal{P}$, $\max\{\sum_{n\in\mathcal{N}} r_n(\sum_{j=1}^{J} v(e_j)) : \{e_j\}_{j=1}^J \in \mathcal{L}^G\} = \max\{v_L : L \in \mathcal{L}^G\} = v_G$. Thus, by Proposition 4 and the definitions of φ and μ, for each $(G, v) \in \mathcal{P}$ and each $L \in \varphi(G, v)$, $\sum_{n\in\mathcal{N}} \mu_n(G, v) = v_G = v_L$. Hence, (φ, μ) is a well-defined solution. By Proposition 4, $(r, \succsim)$ is a monotone and continuous redistribution scheme. By definition of μ, μ is $(r, \succsim)$ - rationalizable. Thus, by Theorem 3, (φ, μ) satisfies *split invariance*, *irrelevance of dominated paths*, and *parallel composition*.

References

[1] Ambec, S., and Ehlers, L., 2008. Sharing a river among satiable agents. Games and Economic Behavior, 64(1), pp.35-50.

[2] Ambec, S., and Sprumont, Y., 2002. Sharing a river. Journal of Economic Theory, 107(2), pp.453-462.

[3] Aumann, R., and Maschler, M., 1985. Game theoretic analysis of a bankruptcy problem from the Talmud. Journal of Economic Theory, 36(2), pp.195-213.

[4] Bag, P.K., 1997. Public goods provision: applying JacksonMoulin mechanism for restricted agent characteristics. journal of economic theory, 73(2), pp.460-472.

[5] Banker, R. 1981. Equity considerations in traditional full cost allocation practices: an axiomatic perspective. In Joint Cost Allocations, e.d. University of Oklahoma Press, pp.110-130.

[6] Bartling, B., and von Siemens, F.A., 2010. Equal sharing rules in partnerships. Journal of Institutional and Theoretical Economics, 166(2), pp.299-320.

[7] Baskerville-Morley, R., and Beechey, J. 2003. Profit sharing in the Big 8 firms: Accommodating the rainmakers. Paper presented at the 2nd Annual Auckland Regional Accounting Conference, Auckland.

[8] Bergantiños, G., and Vidal-Puga, J.J., 2007. A fair rule in minimum cost spanning tree problems. Journal of Economic Theory, 137(1), pp.326-352.

[9] van den Brink, R., 1997. An axiomatization of the disjunctive permission value for games with a permission structure. International Journal of Game Theory, 26(1), pp.27-43.

[10] van den Brink, R., and Gilles, R., 1996. Axiomatizations of the conjunctive permission value for games with permission structures. Games and Economic Behavior, 12(3), pp. 113-126.

[11] van den Brink, R., van der Laan, G., and Moes, N., 2012. Fair agreements for sharing international rivers with multiple springs and externalities, Journal of Environmental Economics and Management, 63(3), pp.388-403.

[12] Bose, A., Pal, D., and Sappington, D.E., 2010. Equal pay for unequal work: Limiting sabotage in teams. Journal of Economics and Management Strategy, 19(1), pp.25-53.

[13] Burrows, G., and Black, C., 1998. Profit sharing in Australian Big 6 accounting firms: an exploratory study. Accounting, Organizations and Society, 23(5), pp.517-530.

[14] Chambers, C. 2006. Asymmetric rules for claims problems without homogeneity. Games and Economic Behavior, 54(2), pp.241-260.

[15] Chun, Y., 2000. Agreement, separability, and other axioms for quasi-linear social choice problems. Social Choice and Welfare, 17(3), pp.507-521.

[16] Chun, Y., 2016, Fair Queuing. Springer.

[17] Claus, A., and Kleitman, D., 1973. Cost allocation for a spanning tree. Networks, 3(4), pp.289-304.

[18] Dong, B., Ni, D., and Wang, Y., 2012. Sharing a polluted river network. Environmental and Resource Economics, 53(3), pp.367-387.

[19] Dutta, B., and Kar, A., 2004. Cost monotonicity, consistency and minimum cost spanning tree games. Games and Economic Behavior, 48(2), pp.223-248.

[20] Eilenberg, S., 1941. Ordered topological spaces. American Journal of Mathematics, 63(1), pp.39-45.

[21] Encinosa, W.E., Gaynor, M., and Rebitzer, J.B., 2007. The sociology of groups and the economics of incentives: Theory and evidence on compensation systems. Journal of Economic Behavior and Organization, 62(2), pp.187-214.

[22] Farrell, J., and Scotchmer, S., 1988. Partnerships. The Quarterly Journal of Economics, 103(2), pp.279-297.

[23] Friedman, E., and Moulin, H., 1999. Three methods to share joint costs or surplus. Journal of Economic Theory, 87(2), pp.275-312.

[24] de Frutos, M.A., 1999. Coalitional manipulations in a bankruptcy problem. Review of Economic Design, 4(3), pp.255-272.

[25] Gilles, R.P., Owen, G., and van den Brink, R., (1992), Games with permission structures: the conjunctive approach, International Journal of Game Theory, 20(3), pp.277-293.

[26] Hougaard, J.L., 2009. An introduction to allocation rules. Springer.

[27] Hougaard, J., Moreno-Ternero, J., Tvede, M., and Osterdal, L., 2017. Sharing the proceeds from a hierarchical venture. Games and Economic Behavior, 102, pp.98-110.

[28] Hougaard, J. L., Moulin, H., and Osterdal, L. P., 2010. Decentralized pricing in minimum cost spanning trees. Economic Theory. 44(2), pp.293-306.

[29] Hougaard, J., and Tvede, M., 2012. Truth-telling and Nash equilibria in minimum cost spanning tree models. European Journal Operations Research, 222(3), pp.566-570.

[30] Hougaard, J., and Tvede, M., 2015. Minimum cost connection networks: Truth-telling and implementation. Journal of Economic Theory, 157, pp.76-99.

[31] Hougaard, J., Tvede, M., and Osterdal, L., 2013. Cost sharing in chains and other fixed trees. Mimeo, University of Southern Denmark.

[32] Jackson, M. and Moulin, H., 1992. Implementing a public project and distributing its cost. Journal of economic Theory, 57(1), pp.125-140.

[33] Ju, B.G., 2003. Manipulation via merging and splitting in claims problems. Review of Economic Design, 8(2), pp.205-215.

[34] Ju, B.G., 2013. Coalitional manipulations on networks. Journal of Economic Theory, 148(2), pp.627-662.

[35] Ju, B.G., Miyagawa, E., and Sakai, T., 2007. Non-manipulable division rules in claim problems and generalizations. Journal of Economic Theory, 132(1), pp.1-26.

[36] Juarez, R., 2008. The worst absolute surplus loss in the problem of commons: random priority vs. average cost. Economic Theory, 34(1), pp.69-84.

[37] Juarez, R., 2013. Group strategyproof cost sharing: the role of indifferences. Games and Economic Behavior, 82(C), pp.218-239.

[38] Juarez, R., and Kumar, R., 2013. Implementing efficient graphs in connection networks. Economic Theory, 54(2), pp.359-403.

[39] Juarez, R., Nitta, K., Vargas, M. 2017. Profit-sharing and implementation of efficient outcomes. Mimeo University of Hawaii.

[40] Kalai, E., 1977. Proportional solutions to bargaining situations: interpersonal utility comparisons. Econometrica, 45(7), pp.1623-1630.

[41] Kar, A., 2002. Axiomatization of the Shapley value on minimum cost spanning tree games. Games and Economic Behavior, 38(2), pp.265-277.

[42] Kobayashi, H., Ohta, K., and Sekiguchi, T., 2008. Optimal Sharing Rules in Repeated Partnerships. Mimeo, Kyoto University.

[43] Littlechild, S. C., and Owen, G., 1973. A simple expression for the Shapley value in a special case. Management Science, 20(3), pp.370-372.

[44] Moulin, H., 1984. The conditional auction mechanism for sharing a surplus. The Review of Economic Studies, 51(1), pp.157-170.

[45] Moulin, H., 1985. The separability axiom and equal-sharing methods. Journal of Economic Theory, 36(1), pp.120-148.

[46] Moulin, H., 1987. Equal or proportional division of a surplus, and other methods. International Journal of Game Theory, 16(3), pp.161-186.

[47] Moulin, H., 1990. Joint ownership of a convex technology: Comparison of three solutions. The Review of Economic Studies, 57(3), pp.439-452.

[48] Moulin, H., 1991. Axioms of cooperative decision making (No. 15). Cambridge University Press.

[49] Moulin, H., 1994. Serial cost-sharing of excludable public goods. The Review of Economic Studies, 61(2), pp.305-325.

[50] Moulin, H., 2000. Priority rules and other asymmetric rationing methods. Econometrica, 68(3), pp.643-684.

[51] Moulin, H., 2002. Axiomatic cost and surplus sharing. In: In K. J. Arrow, A. K. Sen, K. Suzumura (ed.): Handbook of Social Choice and Welfare, Vol 1, pp.289-357, North Holland.

[52] Moulin, H., and Shenker, S., 1992. Serial cost sharing. Econometrica, 60(5), pp.1009-1037.

[53] Moulin, H., and Sprumont, Y., 2005. On demand responsiveness in additive cost sharing. Journal of Economic Theory, 125(1), pp.1-35.

[54] Moulin, H., and Velez, R. A., 2013. The price of imperfect competition for a spanning network. Games and Economic Behavior, 81, pp.11-26.

[55] Nash, J, 1950. The Bargaining Problem. Econometrica, 18, pp.155-162.

[56] Ni, D., and Wang, Y., 2007. Sharing a polluted river. Games and Economic Behavior, 60(1), pp.176-186.

[57] O'Neill, B., 1982. A problem of rights arbitration from the Talmud. Mathematical Social Sciences, 2(4), pp.345-371.

[58] Shapley, L.S., 1953. A value for n-person games. In Contributions to the Theory of Games II, ed. H. Kuhn and A.W. Tucker. Princeton: Princeton University Press.

[59] Sprumont, Y., 1982. The division problem with single-peaked preferences: A characterization of the uniform allocation rule. Econometrica, 59(2), pp.509-519.

[60] Sprumont, Y., 1998. Ordinal cost sharing. Journal of Economic Theory, 81(1), pp.126-162.

[61] Sprumont, Y., 2005. On the discrete version of the Aumann-Shapley cost-sharing method. Econometrica, 73(5), pp.1693-1712.

[62] Thomson, W., 2003. Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey. Mathematical Social Sciences, 45(3), pp.249-297.

[63] Thomson, W., 2007. On the existence of consistent rules to adjudicate conflicting claims: a constructive geometric approach. Review of Economic Design, 11(3), pp.225-251.

[64] Thomson, W., 2013. Cost allocation and airport problems. Working paper, University of Rochester.

[65] Young, H.P., 1988. Distributive justice in taxation. Journal of Economic Theory, 44(2), pp.321-335.