Incentive-Compatible Simple Mechanisms *

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Abstract

We consider mechanisms for allocating a fixed amount of divisible resources among multiple agents when they have quasilinear preferences and can only report messages in a finite-dimensional space. We show that in contrast with infinite-dimensional message spaces, efficiency is not compatible with implementation in dominant strategies. However, for the weaker notion of implementation, such as in the Nash equilibrium, we find that a class of 'VCG-like' mechanisms is the only efficient selection in one-dimensional message spaces. The trifecta in mechanism design, namely efficiency, fairness and simplicity of implementation, is achieved via a mechanism that we introduce and characterize in this paper.

Keywords Resource-sharing; Cost-sharing; Implementation; Envy-Free; VCG Mechanisms.

JEL Classification D44; D79

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1 Introduction

The overarching goal of mechanism design over the last forty years has been to find efficient, fair and incentive-compatible mechanisms in the presence of strategic agents with private information. Unfortunately, however, the three properties are often in conflict. An earlier work in this regard, i.e. Gibbard and Satterthwaite's theorem, states a fundamental impossibility in mechanism design, in that the dictatorial decision is the only incentive-compatible mechanism for general preference domains (Gibbard, 1973; Satterthwaite, 1975). To evade the pessimistic implication of this impossibility, economists have focused on studying more specific preference domains, finding, for example, that in the domain of single-peaked preferences, there exists a dominant-strategy incentive-compatible mechanism that satisfies Pareto efficiency and envy freeness (Sprumont, 1991). The other successful case of mechanism design comes by examining quasi-linear preferences. Holmstrom (1979), for instance, characterises Vickrey-Clarke-Groves (VCG) mechanisms by full-efficiency, i.e. maximised social welfare, and dominant strategy incentive compatibility. However, the extensive literature indicates that even when restricted to quasi-linear preferences, achieving full-efficiency, incentive compatibility and fairness is impossible in direct mechanisms.

In this paper, we show that the trifecta of mechanism design is achievable in the quasi-linear preference domain when allocating a divisible commodity. The success results from employing a simplified message space rather than full preference revelation. In a divisible commodity allocation like auctioning emission permits, electricity or Treasury bonds, buyers submit multiple price and quantity pairs as demand schedules. Each buyer can then stake a claim to purchasing any quantity (including a fraction of permits) between zero and the amount available at any desirable price. In this case, all direct mechanisms generate large communication costs, as they require agents to report infinite dimensional vectors, which applies also to VCG mechanisms; moreover, finding an efficient allocation under the VCG mechanisms is computationally intractable (Rothkopf, Pekec, and Harstad, 1998; Nisan and Ronen, 2007). As such, employing simplified message spaces will improve the practicality of executing these and similar mechanisms.

Our main findings are threefold. First, when restricted to a finite-dimensional message space, efficient allocations cannot be achieved in a dominant strategy (Theorem 1). This result contrasts with the case of infinite-dimensional strategy spaces, where VCG mechanisms achieve efficiency in dominant strategies. In addition, dominant strategy implementation is a demanding property that is difficult to satisfy with another desirable property in finite-dimensional message spaces. Our positive results are illustrated in the rest of the paper, when adopting a weaker notion of implementation, that is, incentive compatibility in the Nash equilibrium.

Second, a mechanism restricted to much simpler message spaces can achieve full efficiency in the Nash equilibrium. In fact, we provide a group of one-dimensional message mechanisms that are characterised by full efficiency in the Nash equilibrium (Theorem 2). We call this group *VCG-like mechanisms*, because their payment structure mimics that of VCG mechanisms, and so they inherit the budget imbalance property of VCG mechanisms as well (Proposition 1). With the Inada condition satisfied for at least two agents, where their marginal utility at 0 is arbitrarily large, every Nash equilibrium of the VCG-like mechanisms is efficient when the mechanism has multiple equilibria (Theorem 3).

Third, we present a one-dimensional message mechanism that achieves the trifecta in the Nash equilibrium. On the simple message domain, we can in fact guarantee a strong fairness notion of envy freeness. The *no envy (envy-free)* axiom is a central standard of fairness in mechanism design theory, since each participant is maximally satisfied with his or her resource share and payment, compared to what others receive and pay (Foley, 1967; Thomson, 2011). The *simple envy-free* (SEF) mechanism in this paper is the only one-dimensional message mechanism that achieves efficient and envy-free allocations with a simple payment structure (Theorem 4). Additionally, the SEF mechanism satisfies other basic fairness properties, such as voluntary participation and ranking (Proposition 2).

This paper is organized as follows. Section 1.1 discusses the related literature, while section 2 describes the model of finite-dimensional mechanisms and presents our first finding in relation to the impossibility of efficient dominant strategy implementation in finite-dimensional spaces. Section 3 studies efficient mechanisms in one-dimensional message spaces and characterizes the VCG-like mechanisms by efficiency and incentive compatibility in the Nash equilibrium. The Inada condition guarantees the uniqueness of efficient equilibria of these mechanisms. Section 4 characterizes the SEF mechanism and discusses its other fairness properties as well as the size of budget imbalances in great detail. Section 5 concludes, suggesting natural extensions of our results for future research. The proofs follow the conclusion.

1.1 Related Literature

Auctioning divisible resources has been discussed theoretically in Back and Zender (1993, 2001), Wang and Zender (2002) and Ausubel et al. (2014), in order to compare uniform price and discriminatory auctions according to the seller's expected revenue. However, the purpose of auctioning a limited amount of public resources is often to attain efficiency and fairness rather than revenue maximization. In fact, any multi-priced mechanisms, including our one-dimensional message mechanisms, should satisfy compelling fairness properties such as the envy-free axiom; otherwise, multi-priced mechanisms often generate heated debate and even legal disputes, as buyers consider different average prices unfair.[1]

Unfortunately, no envy axiom is in conflict with full efficiency in most resource allocation problems for agents with general quasilinear preferences. Maskin (1999) and Fleurbaey and Maniquet (1997) show that for preferences satisfying monotonic closedness, the no-envy axiom is satisfied if an allocation rule is Nash-implementable, in addition to satisfying the equitable treatment of equals. Their results are inapplicable in our setting, because quasi-linear preferences are not monotonically closed. Moulin (2010) mentions that efficient cost-sharing demand mechanisms for divisible commodities cannot reach envy freeness.

For the case of indivisible goods, some fair VCG mechanisms have been found by imposing additional requirements on top of quasi-linear preferences (Papai, 2003; Yengin, 2012). For the problem of allocating heterogeneous indivisible objects, Papai (2003) identifies a class of envy-free VCG mechanisms when utilities are superadditive. For the same problem, Yengin (2012) characterises a class of VCG mechanisms satisfying envy-free and egalitarian-equivalence axioms on restricted domains

[1] For example, Amazon's dynamic pricing was blocked by consumers. Whenever a public utility charges different average prices, the public complains about unfair wealth transfer.

over quasi-linear preferences. For the problem relating to allocating indivisible commodities with unit demand, envy freeness implies efficiency (Svensson, 1983; Alkan et al., 1991). This result, namely in terms of envy freeness implying efficiency, however, does not hold in our setting with a divisible commodity with multi-unit demand. Zhang (2005) and Feldman et al. (2005) studied a modified version of the no-envy axiom when each buyer has a fixed budget. Their mechanisms only achieve the weaker version of envy freeness in an inefficient Nash equilibrium. To the best of our knowledge, our SEF mechanism is the first one-dimensional message solution that satisfies envy freeness in an efficient Nash equilibrium.

Regarding the issue of computationally intractable mechanisms, the literature in mechanism design has taken two approaches to date. The first is to employ sequential preference revelation (Conitzer and Sandholm, 2004), and although this approach could increase administrative costs or be infeasible in time-constrained environments, we do not concern ourselves with the sequential structure of the game. The second approach works by limiting the messages that participants can report.[2] Both approaches rely on the revelation principle, in that any performance that can be achieved in equilibrium by a mechanism can be achieved by an incentive-compatible direct mechanism, in which participants make a one-time report listing his or her values for all possible outcomes, following which the outcome is computed from those reports, and the participants' incentives lead them to report truthfully. The two approaches economise on information by seeking reports only about values in relevant ranges for direct mechanisms (Milgrom, 2011; Parkes, 2005).

Our approach goes beyond the revelation principle by studying indirect mechanisms, which only require agents to report a one-dimensional strategy (or scalar strategy), and the message each agent reports is not necessarily a value of received allocation. Scalar strategy mechanisms have drawn attention in the literature on network capacity allocation mechanisms reducing computational intensity. Initially, Kelly (1997) and Kelly et al. (1998) proposed efficient network bandwidth allocation algorithms for non-strategic agents, and only recently have scalar strategy mechanisms been studied for strategic agents. Maheswaran and Basar (2005), Johari (2004), Johari and Tsitsiklis (2004) and Yang and Hajek (2004) examined the efficiency loss of

[2]The second approach guarantees incentive compatibility in an ϵ-Nash equilibrium upon a condition of outcome closure property (Milgrom, 2010, 2011).

uniform price models in the Nash equilibrium, while Johari and Tsitsiklis (2009), Maheswaran and Basar (2006) and Yang and Hajek (2006, 2007) showed that efficiency is attainable in the Nash equilibrium under multi-price mechanisms. Unlike these papers, we characterise a group of scalar strategy mechanisms that achieve full efficiency and incentive compatibility in the Nash equilibrium. Furthermore, we examine the fairness properties of scalar strategy mechanisms.

2 Finite-dimensional mechanisms

A fixed amount R of a divisible resource is available to be divided among a group of agents, $N = \{1, \cdots, n\}$ with $n \geq 2$. Let x_i be the resource share of agent $i \in N$ and $x = (x_1, \cdots, x_n)$. An allocation x is *feasible* if it belongs to the set $\mathcal{X} = \{x : \sum_{i \in N} x_i = R,\ x_i \geq 0 \ \text{ for all } i \in N\}$. Agent i's utility function u_i is strictly increasing, concave and continuously differentiable on $[0, R]$. Let $u_i(0) = 0$ for each $i \in N$ and $u = (u_1, \cdots, u_n)$. Denoted by $\mathcal{U}$, the set of utility functions satisfies the aforementioned properties. The utility functions of the agents are unknown to the mechanism designer.

A *mechanism* consists of a triple (Θ, x, t), where Θ is the set of allowable strategies of the form $\theta = (\theta_1, \cdots, \theta_n)$ for each $\theta_i \in \Theta_i$, $x(\theta) \in \mathcal{X}$ is an allocation vector and $t(\theta) \in \mathcal{R}^n$ is the payment vector for the agents. With a money transfer $t_i(\theta) \in \mathcal{R}$, agent i's *net utility* $p_i(\theta)$ takes the quasilinear form $p_i(\theta_i, \theta_{-i}) = u_i(x_i(\theta)) - t_i(\theta)$, whereby $\theta_{-i} =(\theta_1, \cdots, \theta_{i-1},\theta_{i+1}, \cdots, \theta_n)$ is a vector of strategies (or messages) submitted by others. For simplicity, we assume that the functions x and t are twice-continuously differentiable in θ. The mechanism is *m-dimensional* if the strategy space $\Theta_i = \mathcal{R}^m_+$ for all $i \in N$. A mechanism that is not m-dimensional for any m is said to be *infinite-dimensional.*

Given $u \in \mathcal{U}^n$, if a resource allocation $y \in \mathcal{X}$ maximizes $\sum_{i \in N} u_i(y_i)$, then the allocation y is said to be *efficient* at u. A natural goal in mechanism design is to implement an efficient allocation. Note that when applying an infinite-dimensional mechanism to a resource allocation problem, finding efficient allocations is computationally intractable (Nisan and Ronen, 2007).

For the first result of the paper, we require a strong notion of implementability, such that an efficient allocation should be implemented in a dominant strategy equilibrium. A mechanism is *efficient in dominant strategies* if for every $u \in \mathcal{U}^n$ there exists $\theta^* \in \Theta$ such that $x(\theta^*)$ is efficient at u and θ^* is in equilibrium in dominant strategies, i.e. $p_i(\theta_i^*, \tilde{\theta}_{-i}) \geq p_i(\tilde{\theta}_i, \tilde{\theta}_{-i})$ for all $\tilde{\theta}_i$ and $\tilde{\theta}_{-i}$. In a dominant strategy equilibrium, agents have no incentive to deviate, regardless of the reports made by other agents.

One important infinite-dimensional class is the Vickrey-Clarke-Groves (VCG) mechanisms, where the space of the message reports equals $\Theta_i = \mathcal{U}$. Given the reported utility functions $\hat{u} \in \mathcal{U}^n$, VCG mechanisms select a resource allocation x such that $x(\hat{u}) \in \text{argmax}_{x \in \mathcal{X}} \sum_{i \in N} \hat{u}_i(x_i)$ and choose a payment scheme for each $i \in N$, such that $t_i(\hat{u}) = -\sum_{j \neq i} \hat{u}_j(x_j(\hat{u})) + h_i(\hat{u}_{-i})$, where $h_i(\cdot)$ is an arbitrary function of $\hat{u}_{-i}$. The VCG mechanisms select an efficient allocation by aligning each agent's self-interest with achieving efficiency. The function h_i for agent i depends only on the messages $\hat{u}_{-i}$ sent by other agents. Thus, agent i maximizes $u_i(x_i(\hat{u}_i, \hat{u}_{-i})) + \sum_{j \neq i} \hat{u}_j(x_j(\hat{u}_i, \hat{u}_{-i}))$ by choosing $\hat{u}_i = u_i$ and thus is efficient in dominant strategies. Among direct revelation mechanisms, the VCG mechanisms are the only efficient mechanisms prevalent in dominant strategies (Holmstrom, 1979). In contrast, when restricting to finite-dimensional strategy spaces, we have the impossibility of achieving efficiency in dominant strategies.

Theorem 1. *There is no m-dimensional mechanism that is efficient in dominant strategies.*

Our result in Theorem 1 extends to any countable dimensional strategy space and therefore implies that we can achieve efficiency in dominant strategies, albeit only in infinite-dimensional strategy spaces.

While most of the literature has focused on infinite-dimensional mechanisms, we maintain our assumption of finite-dimensional strategies and relax our notion of implementability to that of the Nash equilibrium: a mechanism is *efficient* if for every $u \in \mathcal{U}^n$ there exists $\theta^* \in \Theta$, such that $x(\theta^*)$ is efficient at u and θ^* is a Nash equilibrium, i.e. $p(\theta_i^*, \theta_{-i}^*) \geq p(\tilde{\theta}_i, \theta_{-i}^*)$ for all $\tilde{\theta}_i \in \Theta_i$.

In the next section, we show that efficient mechanisms exist even in one-dimensional strategies, the smallest dimension where they may exist.[3]

3 Efficient mechanisms in one-dimensional spaces

In this section, we characterize a large class of efficient mechanisms for the message space, $\Theta = \mathcal{R}_+^n$, that is, when each agent has a one-dimensional strategy space $\mathcal{R}_+$.

A *VCG-like mechanism* requires each agent i to report a scalar strategy θ_i which selects a *surrogate utility* function $\bar{u}(\cdot, \theta_i)$ from a given single-parameter family of functions, $\bar{\mathcal{U}} = \{\bar{u}(\cdot, \theta_i) | \; \theta_i \in \mathcal{R}_+\}$. We assume that $\bar{u}(x_i, \theta_i)$ for all $i \in N$ is strictly concave and strictly increasing for $x_i \in \mathcal{R}_{++}$, and twice-continuously differentiable for $x_i, \theta_i \in \mathcal{R}_{++}$. If $\theta_i = 0$, then $\bar{u}(x_i, \theta_i) = 0$. For simplicity, we denote by $\bar{u}'(x_i, \theta_i) = \frac{d\bar{u}(x_i,\theta_i)}{dx_i}$ and assume that $\bar{u}'(0, 0) \leq 0$. In addition, for every $\gamma, x_i \in \mathcal{R}_{++}$, there exists $\theta_i \in \mathcal{R}_{++}$ such that $\bar{u}'(x_i, \theta_i) = \gamma$. These assumptions together imply richness of $\bar{\mathcal{U}}$, such that all the functions in $\bar{\mathcal{U}}$ cover the entire space of $\mathcal{R}_{++}^2$, whose single element is (x_i, γ). Each agent i can express his or her marginal utility at any amount of resource by selecting θ_i.

Once the mechanism collects θ, its resource allocation $x \in \mathcal{X}$ maximizes the sum of surrogate utilities, $\sum_{i \in N} \bar{u}(x_i, \theta_i)$, for the given θ. The VCG-like mechanism sets its payment scheme analogously to the payment scheme of the VCG mechanisms, such that each agent's payment depends on both the sum of surrogate utilities of other agents and an arbitrary function of strategies submitted by other agents.

Formally, for θ collected, VCG-like mechanisms choose the resource allocation x, such that

$$x(\theta) \in \operatorname*{argmax}_{x \in \mathcal{X}} \sum_{i \in N} \bar{u}(x_i, \theta_i) \tag{1}$$

and the payment scheme t such that $t_i(\theta) = -\sum_{j \neq i} \bar{u}(x_j(\theta), \theta_j) + h_i(\theta_{-i})$ for all $i \in N$, where $h_i(\cdot)$ is an arbitrary function of θ_{-i}.

The VCG-like mechanisms are computationally tractable. Computation of the

[3] That is, $\Theta = \mathcal{R}_+^n$ is a space with the smallest cardinality where efficiency can be achieved, assuming Cantor's continuum hypothesis, which we do not dispute.

efficient allocation in (1) is quite simple, unlike in VCG mechanisms. Note that the first-order conditions of (1) give $\bar{u}'(x_i, \theta_i) = \bar{u}'(x_j, \theta_j)$ for all $i, j \in N$, and $\theta_i, \theta_j > 0$. Since $\bar{u}$ is strictly concave, $x(\theta)$ is unique for a given θ. These first-order conditions are necessary and sufficient for the computation of (1). Once a class of surrogate functions is determined, the mechanism only needs to solve a set of linear equations of marginal utilities. For example, with a family of surrogate functions $\bar{\mathcal{U}} = \{\theta_i \ln x_i | \ \theta_i \in \mathcal{R}_{++}\}$ for two agents, the mechanism determines the allocation by solving $\theta_1/x_1 = \theta_2/x_2$ and $x_1 + x_2 = R$ for reported θ_1 and θ_2.

The VCG-like mechanisms achieve efficient allocations in the Nash equilibrium. Furthermore, only VCG-like mechanisms have this property for the strategy space $\Theta = \mathcal{R}^1_+$.

Theorem 2. *A one-dimensional mechanism is efficient if and only if it is a VCG-like mechanism.*

We note that Johari and Tsitsiklis (2009) have already proved the "if" part of this result on a model under different assumptions about the surrogate functions and their proof is adapted to our model. The converse of this result ("only if"), which is the main contribution of this Theorem 2, has not been addressed in the past. Our assumptions on the surrogate functions, especially such as twice-continuous differentiability, are critical for Proposition 1 and Theorem 4.

Using surrogate functions and properly designed payment schemes, VCG-like mechanisms extract information on true marginal utilities at an efficient allocation and therefore achieve efficient equilibria. In an interior solution, each agent i's optimal strategy θ_i, given θ_{-i}, is determined by the first-order condition: $u_i'(x_i) = \frac{\partial t_i(\theta)}{\partial \theta_i} / \frac{\partial x_i(\theta)}{\partial \theta_i}$. The term $\frac{\partial t_i(\theta)}{\partial \theta_i} / \frac{\partial x_i(\theta)}{\partial \theta_i}$ is a *marginal price*, the ratio of the additional amount of money agent i has to pay for the additional units of the good agent i receives when he increases his strategy θ_i. Agent i chooses θ_i such that his marginal utility equals his marginal price. VCG-like mechanisms set the same marginal price for all agents so that $\frac{\partial t_i(\theta)}{\partial \theta_i} / \frac{\partial x_i(\theta)}{\partial \theta_i} = g(\theta)$ where $g(\theta) = \bar{u}'(x_i, \theta_i)$ for all $i \in N$. Therefore, agent i selects an equilibrium strategy θ_i, thus satisfying $u_i'(x_i) = \bar{u}'(x_i, \theta_i)$. For the efficient allocation x^*, each agent i chooses θ_i^*, thereby satisfying $u_i'(x_i^*) = \bar{u}'(x_i^*, \theta_i^*)$.

A VCG-like mechanism can have multiple Nash equilibria and inefficient equilibria, as exemplified in Example 1 and Example 2, respectively.

Example 1 (Multiplicity of efficient equilibria)**.** Consider a case with $\bar{u}(x,\theta) = \theta \ln x$ and two agents. If each agent's utility function has a constant slope over part of the domain, there can be multiple equilibria. Let u_1 and u_2 have the same constant slope over $[x_1, \frac{R}{2}]$ and $[\frac{R}{2}, x_2]$, respectively, where $x_1 + x_2 = R$. Then, $u_1'(x_1) = u_2'(x_2)$, and x is an efficient allocation. There are a pair of equilibrium strategies θ_1, θ_2 which satisfy $x_1 = \frac{\theta_1}{\theta_1+\theta_2}R$ and $x_2 = \frac{\theta_2}{\theta_1+\theta_2}R$. Likewise, if $q_1 \in [x_1, \frac{R}{2}]$ and $q_2 \in [\frac{R}{2}, x_2]$ with $q_1+q_2 = R$, we again have $u_1'(q_1) = u_2'(q_2)$, and there is a pair of equilibrium strategies θ_1' and θ_2', which satisfies $q_1 = \frac{\theta_1'}{\theta_1'+\theta_2'}R$ and $q_2 = \frac{\theta_2'}{\theta_1'+\theta_2'}R$. We can find infinitely more equilibria in this example.

Example 2 (Existence of inefficient equilibria)**.** Let $n = 2$ and $R = 1$, and then let a surrogate function be $\bar{u}(x_i, \theta_i) = -\frac{\theta_i^2}{x_i}$ and a residual payment scheme be $h(\theta_{-i}) = -2\theta_j^2$. Then, agent i's net utility is $p_i(\theta_i, \theta_j) = u_i\big(\frac{\theta_i}{\theta_i+\theta_j}\big) - \theta_i\theta_j + \theta_j^2$. Suppose $u_1(x_1) = ax_1$ and $u_2(x_2) = bx_2$ for $0 < a < b$. At an efficient allocation, agent 1 receives nothing and agent 2 should receive 1. However, there are multiple equilibria where agent 1 receives everything and agent 2 gets nothing. Assume that agent 1 reports $\theta_1 = \epsilon > 0$ and agent 2 reports θ_2 =0. Agent 1 does not have any incentive to change his strategy, since he receives all of the resource but pays nothing. Agent 2 does not have any incentive to change his strategy if his net utility decreases by submitting a positive number. This is the case when $\frac{\partial p_2(\epsilon,0)}{\partial \theta_2} \leq 0$, that is, $b \leq \epsilon^2$. Therefore, $(\epsilon, 0)$ is a Nash equilibrium if $\epsilon \geq \sqrt{b}$, albeit the allocation is inefficient.

Example 2 demonstrates that as long as every other agent $j \in N$, $j \neq i$ has finite $u_j'(0)$, agent i has an opportunity to take the entirety of the resource, resulting in an inefficient equilibrium. Thus, to prevent this from happening, for each agent $i \in N$ there should be at least one other agent $j \neq i$ with $u_j'(0) = +\infty$. That is, the presence of at least two actively participating agents with infinite marginal utility at zero allocation will guarantee efficient allocations. The following assumption ensures that all Nash equilibria of a VCG-like mechanism are efficient: $u_i'(0) = \infty$ for at least two agents. We will call this assumption the Inada condition in this paper.[4]

[4]This condition is one of the Inada conditions that are commonly used in macroeconomics for production functions that guarantee the stability of an economic growth path.

Theorem 3 (Uniqueness of Efficient Equilibria)**.** *Consider a problem where at least two agents meet the Inada condition. Then, every Nash equilibrium of a VCG-like mechanism is efficient.*

A mechanism (Θ, x, t) is *budget-balanced* if $\sum_{i \in N} t_i(\theta) = 0$ holds for any pair of equilibrium strategies θ. Any budget-balanced mechanism makes an amount of transfers to the agents exactly equal to the amount it collects from others, and thus it has no deficit or surplus in an equilibrium. Budget balance is often a desirable property for economies that are collectively owned by agents. Budget balance, however, is difficult to satisfy, and it is well known that traditional VCG mechanisms fail to be budget-balanced (Green and Laffont, 1979; Moulin, 2009, 2010; Moulin and Shenker, 2001; You, 2015). Our following result extends such an impossibility to VCG-like mechanisms.

Proposition 1. *Every VCG-like mechanism fails to be budget-balanced.*

4 SEF: A fair, efficient and simple mechanism

Having explored the issue of efficiency in simple mechanisms, we now look at fairness. Our main results in this section show that the trifecta of fairness, efficiency and simplicity of implementation is possible.

While fairness may have different meanings, we adopt the traditional interpretation provided by Foley (1967) of envy freeness under which no agent should prefer any other agent's allocation to his or her own. Envy freeness has played a central role in the distributive justice literature. Formally, a mechanism is *envy-free*, or satisfies *no envy*, if for each $u \in \mathcal{U}^n$ and each Nash-equilibrium θ, $u_i(x_i(\theta)) - t_i(\theta) \geq u_i(x_j(\theta)) - t_j(\theta)$ holds for all pairs $i, j \in N$. Agent i does not envy agent j if his or her net utility at equilibrium is at least as high as his or her net utility in the case in which he or she receives agent j's share and payment instead. Our next example shows that VCG-like mechanisms might not be envy-free, and further restrictions might be needed to characterise our desired trifecta.

Example 3 (VCG-like mechanisms may not be envy-free)**.** Consider an efficient VCG-like mechanism with $\bar{u}(x_i, \theta_i) = \theta_i \ln x_i$ for all $i \in N$. In this case, $x_i = \frac{\theta_i}{\theta_N} R$

and $\frac{\partial p_i}{\partial \theta_i} = \frac{R\theta_{N\setminus i}}{\theta_N^2}\left[u_i'(x_i) - \frac{\theta_N}{R}\right]$. With the Inada condition, we have $\theta_{N\setminus i} \neq 0$ for all $i \in N$. The first-order condition of the equilibrium is $u_i'(x_i) = \frac{\theta_N}{R}$ if $x_i > 0$ and $u_i'(x_i) \leq \frac{\theta_N}{R}$ if $x_i = 0$. From Lemma 1, no envy or efficiency hold if and only if $h_i(\theta_{-i}) - h_j(\theta_{-j}) = (\theta_i - \theta_j) - \theta_i \ln(\theta_i R) + \theta_j \ln(\theta_j R) + (\theta_i - \theta_j) \ln \theta_N$. The left-hand side of the equation is additively separable with respect to θ_i and θ_j, so that its cross derivative should be zero. However, the right-hand side's cross derivative is $(\theta_j - \theta_i)/\theta_N^2 \neq 0$. Therefore, with $\bar{u}(x_i, \theta_i) = \theta_i \ln x_i$, the efficient VCG-like mechanism is not envy-free. Likewise, with $\bar{u}(x_i, \theta_i) = \theta_i \sqrt{x_i}$, the efficient VCG-like mechanism generates envy among agents.

The *simple envy-free (SEF) mechanism* is an envy-free VCG-like mechanism with a simple payment scheme. Resource allocation is proportional to strategies: $x_i = \frac{\theta_i}{\theta_N} R$ for each $i \in N$. Agent i pays $t_i = \theta_i \theta_{N\setminus i} - S_{-i}$, where $S = \sum_{i \in N} \theta_i^2$, and $S_{-i} = S - \theta_i^2$. Payment made by some agents can be negative, thereby allowing a subsidy from the mechanism, in which case agent i's net utility is $p_i(\theta_i, \theta_{-i}) = u_i\left(\frac{\theta_i}{\theta_N} R\right) - \theta_i \theta_{N\setminus i} + S_{-i}$. We can construct the SEF mechanism by setting a surrogate function at $\bar{u}(x_i, \theta_i) = -\frac{\theta_i^2}{x_i} R$ and the residual payment scheme at $h(\theta_{-i}) = -\theta_{N\setminus i}^2 - S_{-i}$ within the VCG-like mechanisms.

We now examine whether the SEF mechanism satisfies desirable fairness properties and measure the size of budget imbalance. In addition to the no envy concept, we introduce two additional fairness concepts. First, a mechanism satisfies *ranking* if $x_i \leq x_j$ implies $t_i \leq t_j$ for any $i, j \in N$, $i \neq j$ at every equilibrium. If an agent receives a bigger share of the resource than the other agents, he or she has to pay a greater amount than them. Second, a mechanism satisfies *voluntary participation* if each agent $i \in N$ has net utility $p_i(\theta)$ which is non-negative at equilibrium θ. Individuals are not forced to participate in the mechanism if they would be made worse off by participating. We assume that $x_i = t_i = 0$ if agent i does not participate, i.e. he or she does not submit any bid. Proposition 2a shows that the SEF mechanism is not only envy-free, but also satisfies the two aforementioned fairness properties. Proposition 2b shows that the SEF mechanism can generate a budget deficit.

Proposition 2. *a. The SEF mechanism (i) satisfies ranking, (ii) achieves voluntary participation and (iii) guarantees no envy.*

b. The SEF mechanism yields a budget deficit. However, when every agent submits the same strategy, namely $\theta = (\alpha, \cdots, \alpha)$, the mechanism is budget-balanced.

The SEF mechanism's budget deficit ranges from 0 to $R\lambda(n-1)$, where λ is the marginal utility in the efficient equilibrium. We can interpret λ as the market clearing price for price-taking buyers.

A two-part tariff payment scheme is a common practice for pricing a divisible good. A *simple two-part tariff* for each agent i consists of a variable price in θ_i and a fixed price independent of θ_i: $t_i(\theta) = \sum_{j \neq i} \alpha(\theta_j)\theta_i + \sum_{j \neq i} k(\theta_j)$. Here, $\alpha(\theta_j)$ and $k(\theta_j)$ are polynomials of (non-negative) degrees m and γ, respectively, in θ_j for each $j \in N$, where either m or γ is 1. We show that the SEF mechanism is characterized by efficiency, no envy and a simple two-part tariff.

Theorem 4. *The SEF mechanism is the only efficient, envy-free and simple two-part tariff scalar mechanism (up to affine transformations).*

We note that this result is tight. Removing a simple two-part tariff, a large class of scalar mechanisms that are efficient and envy-free is discussed in Lemma 1 (see Proofs section). Efficient and simple two-part tariff mechanisms generate envy when $\alpha(\theta_i)$ is polynomial of a degree larger than one. Finally, the egalitarian allocation, $x_i(\theta) = \frac{R}{n}$ and $t_i(\theta) = 0$ for all $i \in N$, is not efficient, but it is envy-free and satisfies a simple two-part tariff, generated by $\alpha(\theta_j) = k(\theta_j) = 0$.

5 Conclusion

We have studied the class of incentive-compatible mechanisms in terms of the problem inherent in allocating a divisible resource among agents, within the boundaries of simplicity. Our SEF mechanism reconciles the three desirable properties in the mechanism design literature that have more than often been impossible to achieve simultaneously: fairness, efficiency and simplicity of implementation.

The SEF mechanism does not meet more robust notions of implementation, such as implementation in dominant strategies, but neither does any mechanism defined on a domain of finite-dimensional strategies. Therefore, a natural extension of our work is to study more robust equilibrium selections where these three properties can be achieved.

Finally, our work has focused on the division of a fixed amount of divisible resource. Natural extensions can include more general resource or cost-sharing problems, where the amount to divide is not fixed and the strategy spaces are more complex. Our reductionist approach may lead to positive results even on these problems.

Proofs

Proof of Theorem 1

We show the non-existence for the simplest case of $n = 2$. The extension to any number of agents follows trivially by fixing the utility of the other agents at 0.

Note that a scalar mechanism implements an efficient equilibrium if and only if, for a given utility profile $u \in U$, each equilibrium θ with an equilibrium allocation $x(\theta)$ satisfies the first-order condition for efficient equilibria, $u_i'(x_i(\theta)) = g(\theta)$ for all $i \in N$, where g is a continuous and positive function. The function g determines the properties of the scalar strategy mechanism.

Suppose that a scalar strategy mechanism $M(g)$ implements an efficient dominant strategy equilibrium. Given the mechanism $M(g)$, for every pair of utility profiles (u_1, u_2), there is a corresponding pair of dominant strategy equilibrium $(\tilde{\theta}_1, \tilde{\theta}_2)$. This means that $\tilde{\theta}_1$ should be a best response of agent 1 to every $\theta_2 \in \mathcal{R}_+$, i.e. we have

$$u_1'(x_1(\tilde{\theta}_1, \theta_2)) = g(\tilde{\theta}_1, \theta_2) \text{ for every } \theta_2 \in \mathcal{R}_+. \tag{2}$$

Likewise, $\tilde{\theta}_2$ should be a best response of agent 2 to every $\theta_1 \in \mathcal{R}_+$, in which case we have

$$u_2'(x_2(\theta_1, \tilde{\theta}_2)) = g(\theta_1, \tilde{\theta}_2) \text{ for every } \theta_1 \in \mathcal{R}_+. \tag{3}$$

To use simple notation, we denote u_1' by f and u_2' by h, in which case f and h

are functions of θ_1 and θ_2, i.e. $f = f(\theta_1, \theta_2)$ and $h = h(\theta_1, \theta_2)$. Since a pair of utility profiles $u \in \mathcal{U}^2$ can be chosen arbitrarily, there are a set F and a set H such that

$$\begin{aligned} F &= \{f(x,y): \ f \text{ is continuous and positive}\} \\ H &= \{h(x,y): \ h \text{ is continuous and positive}\}. \end{aligned}$$

Denoting $x = \theta_1$, $y = \theta_2$, $x^* = \tilde{\theta_1}$ and $y^* = \tilde{\theta_2}$, equations (2) and (3) are rewritten as follows:

$$f(x^*, y) = g(x^*, y) \text{ for every } y \in Y \tag{4}$$

and

$$h(x, y^*) = g(x, y^*) \text{ for every } x \in X \tag{5}$$

where $X = Y = [0, +\infty)$ and $X \times Y$ are the domains of functions f and h. In addition, we should have

$$f(x^*, y^*) = h(x^*, y^*). \tag{6}$$

The equations (4)-(6) should hold for any pair of f and h from F and H, respectively. Note that x^* and y^* depend on the choice of f and h, but g is fixed by the mechanism.

Let us choose three pairs of (f_1, h_1), (f_2, h_2) and (f_3, h_3) from F and H. There are corresponding dominant strategy equilibrium pairs: (x_1, y_1), (x_2, y_2) and (x_3, y_3), respectively. According to equation (6), we have

$$\begin{aligned} f_1(x_1, y_1) = h_1(x_1, y_1) = g(x_1, y_1), \\ f_2(x_2, y_2) = h_2(x_2, y_2) = g(x_2, y_2), \\ f_3(x_3, y_3) = h_3(x_3, y_3) = g(x_3, y_3). \end{aligned}$$

Notice that the function $g(x, y)$ should have the same values at each point (x_1, y_2),

(x_1, y_3), (x_2, y_1), (x_2, y_3), (x_3, y_1), and (x_3, y_2),, so we have

$$\begin{aligned}
f_1(x_1, y_2) &= h_2(x_1, y_2) = g(x_1, y_2),\\
f_1(x_1, y_3) &= h_3(x_1, y_3) = g(x_1, y_3),\\
h_1(x_2, y_1) &= f_2(x_2, y_1) = g(x_2, y_1),\\
f_2(x_2, y_3) &= h_3(x_2, y_3) = g(x_2, y_3),\\
h_1(x_3, y_1) &= f_3(x_3, y_1) = g(x_3, y_1),\\
f_3(x_3, y_2) &= h_2(x_3, y_2) = g(x_3, y_2).
\end{aligned}$$

Taking x_1, x_2, x_3, y_1, y_2, and y_3 as unknown variables to solve, we have six unknown variables with nine equations. Considering that f and h are arbitrarily selected, a function g cannot exist in this situation. ■

Proof of Theorem 2

We prove the "if" part.

Step A1. We show that θ is a Nash equilibrium if and only if for all $i \in N$,

$$x(\theta) \in \operatorname*{argmax}_{x \in \mathcal{X}} \left[u_i(x_i) + \sum_{j \neq i} \bar{u}(x_j, \theta_j) \right]. \tag{7}$$

The optimal value of (7) is an upper bound to agent i's net utility, without $h_i(\theta_{-i})$. Given θ, if (7) holds for all agents, then their net utilities are maximised so that θ is a Nash equilibrium. For the sake of contradiction, suppose that given a Nash equilibrium θ, (7) is not satisfied for some agent i. The problem (7) has an optimal solution x^*, since $\mathcal{X}$ is compact and $x^* \neq x(\theta)$. Next, x^* satisfies the first-order conditions that include only the first derivatives $u_i'(x_i^*)$ and $\bar{u}'(x_j^*, \theta_j)$ for all $j \neq i$. Since $\bar{u}'(0, 0) \leq 0$, agent i can find $\theta_i' > 0$ satisfying $u_i'(x_i^*) = \bar{u}'(x_i^*, \theta_i')$. Thus, the optimal solution x^* also satisfies the first-order conditions of problem (1). Solution $x(\theta_i', \theta_{-i})$ is the unique solution to problem (1) for given strategy vector (θ_i', θ_{-i}).

Thus, we have $x^* = x(\theta_i', \theta_{-i})$, which implies

$$\begin{aligned} u_i(x_i(\theta)) + \sum_{j\neq i} \bar{u}(x_j(\theta), \theta_j) - h_i(\theta_{-i}) &< u_i(x_i^*) + \sum_{j\neq i} \bar{u}(x_j^*, \theta_j) - h_i(\theta_{-i}) \\ &= u_i(x_i(\theta_i', \theta_{-i})) + \sum_{j\neq i} \bar{u}(x_j(\theta_i', \theta_{-i}), \theta_j) - h_i(\theta_{-i}) \end{aligned}$$

and contradicts the notion that θ is a Nash equilibrium.

Step A2. We prove that every VCG-like mechanism has an efficient Nash equilibrium. For a vector of efficient allocations x^*, each agent i, $i \in N$, can choose $\theta_i > 0$ such that $u_i'(x_i^*) = \bar{u}'(x_i^*, \theta_i)$ for $x_i^* > 0$ or selects $\theta_i = 0$ for $x_i^* = 0$. Since (1) has a unique solution for θ, we have $x^* = x(\theta)$. From $u_i'(x_i^*) = \bar{u}'(x_i^*, \theta_i)$, problems (1) and (7) only involve the same first derivatives, thus suggesting that efficient allocation $x^*(= x(\theta))$ is also a solution of (7). Therefore, we conclude that θ is a Nash equilibrium.

We prove the "only if" part.

Step B1. Consider an efficient one-dimensional mechanism $(\Theta,\ x,\ t)$, where $\Theta = \mathcal{R}_+^1$ and $x(\theta) \in \mathcal{X}$. We construct a space of surrogate utility function for this mechanism. Note that since $x(\theta)$ is an allocation of R units, then we can find functions $f_i : \mathcal{R}_+ \to \mathcal{R}_+$ such that $f_i(0) = 0$ and $x_i(\theta) = \frac{f_i(\theta_i)}{\sum_{j=1}^n f_j(\theta_j)} R$ for all $\theta \in \mathcal{R}_+^n$. Thus, the function $\bar{u}(x_i, \theta_i) = f_i(\theta_i) \ln x_i$ meets the assumption of a surrogate utility function. Furthermore, since $\bar{u}'(x_i, \theta_i) = \bar{u}'(x_j, \theta_j)$, the function $x(\theta)$ is the unique solution of (1) at any given θ.

Step B2. Pick an arbitrary utility profile u and the surrogate function $\bar{u}$ constructed in Step B1. The analytic implicit function theorem and the twice-continuous differentiability assumptions on the functions $\bar{u}$ imply that $x_i(\theta)$ is twice continuously differentiable. Thus, if θ is a Nash equilibrium, θ satisfies the following first-order condition with a continuously differentiable $t_i(\theta)$:

$$\frac{\partial t_i(\theta_i, \theta_{-i})}{\partial \theta_i} = \frac{\partial u_i(x_i(\theta_i, \theta_{-i}))}{\partial x_i} \frac{\partial x_i(\theta_i, \theta_{-i})}{\partial \theta_i}. \tag{8}$$

Step B3. Let x^* be an efficient allocation. The richness assumption of $\bar{u}$ implies that for each $i \in N$, there exists θ_i^* satisfying $u_i'(x_i^*) = \bar{u}'(x_i^*, \theta_i^*)$. Since the efficient solution x^* satisfies the first-order condition $u_i'(x_i^*) = u_j'(x_j^*)$ for all $i, j \in N$, we have

$$\bar{u}'(x_i^*, \theta_i^*) = u_i'(x_i^*) = u_j'(x_j^*) = \bar{u}'(x_j^*, \theta_j^*), \tag{9}$$

for all $i, j \in N$, thus suggesting that x^* is the unique solution to problem (1) for the submitted θ^*. Also, $x^*(\theta^*)$ is a solution of (7), whose first-order conditions only include $u_i'(x_i^*)$ and $\bar{u}'(x_j^*, \theta_j)$ for all $j \neq i$. The result in Step A1 implies that θ^* is an efficient Nash equilibrium.

Step B4. For an efficient Nash equilibrium θ and an associated efficient allocation x^*, we can rewrite (8) as the following:

$$t_i(\theta_i, \theta_{-i}) = t_i(0, \theta_{-i}) + \int_0^{\theta_i} \frac{\partial u_i(x_i^*(s, \theta_{-i}))}{\partial x_i} \frac{\partial x_i(s, \theta_{-i})}{\partial s} ds. \tag{10}$$

From the differentiability of $x(\theta)$ and resource constraint $R = \sum_{i \in N} x_i(\theta)$, we have $\frac{\partial x_i}{\partial \theta_i} = -\sum_{j \neq i} \frac{\partial x_j}{\partial \theta_i}$. Using the equation (9) and $\frac{\partial x_i}{\partial \theta_i} = -\sum_{j \neq i} \frac{\partial x_j}{\partial \theta_i}$, we can rewrite equation (10) as

$$\begin{aligned} t_i(\theta_i, \theta_{-i}) &= t_i(0, \theta_{-i}) + \int_0^{\theta_i} -\sum_{j \neq i} \bar{u}'(x_j^*(s, \theta_{-i}), \theta_j) \frac{\partial x_j(s, \theta_{-i})}{\partial s} ds \\ &= t_i(0, \theta_{-i}) - \sum_{j \neq i} \int_{x_j^*(0, \theta_{-i})}^{x_j^*(\theta_i, \theta_{-i})} \bar{u}'(x_j, \theta_j) dx_j \\ &= t_i(0, \theta_{-i}) - \sum_{j \neq i} \bar{u}(x_j^*(\theta_i, \theta_{-i}), \theta_j) + \sum_{j \neq i} \bar{u}(x_j^*(0, \theta_{-i}), \theta_j). \end{aligned}$$

This is true if and only if $t_i(\theta)$ satisfies:

$$t_i(\theta_i, \theta_{-i}) = -\sum_{j \neq i} \bar{u}(x_j^*(\theta), \theta_j) + h_i(\theta_{-i}).$$

Therefore, we conclude that the payment scheme of the scalar strategy mechanism follows a VCG-like mechanism. ■

Proof of Theorem 3

Without loss of generality, suppose that $u_1'(0) = u_2'(0) = \infty$. In this case, agents 1 and 2 will receive positive amounts of resources. From Step A1 in the proof of Theorem 2, θ is a Nash equilibrium if and only if, for all $i \in N$, we have (7). The optimality conditions of (7) imply that for $i = 1, 2$ and for every $j \neq i$, $u_i'(x_i) = \bar{u}'(x_j, \theta_j)$ if $x_j > 0$, or $u_i'(x_i) \geq \bar{u}'(x_j, \theta_j)$ if $x_j = 0$, which includes $u_1'(x_1) = \bar{u}'(x_2, \theta_2)$ and $u_2'(x_2) = \bar{u}'(x_1, \theta_1)$. Since $x(\theta)$ is the solution to problem (1), we also have $\bar{u}'(x_1, \theta_1) = \bar{u}'(x_2, \theta_2)$, and thus we have $u_1'(x_1) = \bar{u}'(x_1, \theta_1)$. The optimality conditions of (7) indicate that for an arbitrary agent $i \in N$, we have $u_i'(x_i) = \bar{u}'(x_1, \theta_1) = u_1'(x_1)$ if $x_i > 0$, or $u_i'(x_i) \leq \bar{u}'(x_1, \theta_1) = u_1'(x_1)$ if $x_i = 0$. These are the first-order conditions of efficient allocations. Therefore, with $u_i'(0) = \infty$ for at least two agents, every Nash equilibrium is efficient. ■

Proof of Proposition 1

An agent i's net utility in a VCG-like mechanism is $p_i(\theta) = u_i(x_i) + \sum_{j \neq i} \bar{u}(\theta_j, x_j) - h_i(\theta_{-i})$, so his payment is $t_i(\theta) = -\sum_{j \neq i} \bar{u}(\theta_j, x_j) + h_i(\theta_{-i})$ at an equilibrium θ and a corresponding allocation x. The mechanism's budget is

$$\sum_{i \in N} t_i = -\sum_{i \in N} \sum_{j \neq i} \bar{u}(\theta_j, x_j) + \sum_{i \in N} h_i(\theta_{-i}).$$

Budget balance means that $\sum_{i \in N} h_i(\theta_{-i}) = \sum_{i \in N} \sum_{j \neq i} \bar{u}(\theta_j, x_j)$ holds for any pair of equilibrium strategies θ.

Suppose that there exists a $\bar{u}$ which yields a budget balance. Let $n = 2$. Without loss of generality, we can assume that $x_1 > 0,\ x_2 > 0$ at equilibrium allocations, and so the following budget balance shows that $h_1(\theta_2) + h_2(\theta_1) = \bar{u}(\theta_1, x_1) + \bar{u}(\theta_2, x_2)$ for any equilibrium strategies θ and corresponding allocations x. Equilibrium strategies θ vary in line with the ways in which utility profiles u vary. Thus, the right-hand side of this equation should be additively separable in θ_1 and θ_2. Therefore, a budget balance implies $\frac{\partial^2[\bar{u}(\theta_1, x_1) + \bar{u}(\theta_2, x_2)]}{\partial\theta_1 \partial\theta_2} = 0$. The partial derivative of $\bar{u}$ in x_i, $\bar{u}'(\theta_i, x_i)$ is denoted by $\bar{u}_{(0,1)}(\theta_i, x_i)$ for $i = 1, 2$. Recall that the equilibrium condition is written as $\bar{u}_{(0,1)}(\theta_1, x_1) = \bar{u}_{(0,1)}(\theta_2, x_2) = g(\theta)$.

Now we have

$$\begin{aligned}\frac{\partial(\bar{u}(\theta_1,x_1)+\bar{u}(\theta_2,x_2))}{\partial\theta_1} &= \bar{u}_{(1,0)}(\theta_1,x_1)+\bar{u}_{(0,1)}(\theta_1,x_1)\frac{\partial x_1}{\partial\theta_1}+\bar{u}_{(0,1)}(\theta_2,x_2)\frac{\partial x_2}{\partial\theta_1}\\ &= \bar{u}_{(1,0)}(\theta_1,x_1)+g(\theta)\left(\frac{\partial x_1}{\partial\theta_1}+\frac{\partial x_2}{\partial\theta_1}\right)=\bar{u}_{(1,0)}(\theta_1,x_1).\end{aligned}$$

The last equality holds, since $x_1(\theta)+x_2(\theta)=R$. Likewise, we have $\frac{\partial}{\partial\theta_2}(\bar{u}(\theta_1,x_1)+\bar{u}(\theta_2,x_2))=\bar{u}_{(1,0)}(\theta_2,x_2)$, and because $\frac{\partial^2}{\partial\theta_1\partial\theta_2}=\frac{\partial^2}{\partial\theta_2\partial\theta_1}$, the budget balance holds if and only if $\frac{\partial\bar{u}_{(1,0)}(\theta_1,x_1)}{\partial\theta_2}=\frac{\partial\bar{u}_{(1,0)}(\theta_2,x_2)}{\partial\theta_1}=0$. This is equivalent to $\bar{u}_{(1,1)}(\theta_1,x_1)\frac{\partial x_1}{\partial\theta_2}=\bar{u}_{(1,1)}(\theta_2,x_2)\frac{\partial x_2}{\partial\theta_1}=0$. We proved that x_i is a differentiable function of θ, so $\frac{\partial x_2}{\partial\theta_1}$ and $\frac{\partial x_1}{\partial\theta_2}$ cannot be zero. Thus, the budget balance requests $\bar{u}_{(1,1)}(\theta_1,x_1)=\bar{u}_{(1,1)}(\theta_2,x_2)=0$, thereby suggesting that for $i=1,2$, $\bar{u}(x_i,\theta_i)$ is additively separable in x_i and θ_i, i.e. we should have $\bar{u}(\theta_i,x_i)=f(\theta_i)+k(x_i)$ for some functions f and k. Then, for $i=1,2$, $\bar{u}'(\theta_i,x_i)=\frac{\partial k(x_i)}{\partial x_i}$. This violates an assumption about $\bar{u}$ of VCG-like mechanisms, such that for every $\gamma\in(0,\infty)$ and $x_i>0$, there exists $\theta_i>0$ s.t. $\bar{u}'(x_i,\theta_i)=\gamma$ for $i=1,2$. Therefore, there is no $\bar{u}$ that satisfies the budget balance. ■

Conditions for Envy Freeness

Lemma 1.[Necessary and sufficient conditions for envy freeness and efficiency] *An allocation mechanism with a side payment is envy-free and efficient if and only if it satisfies the following condition: given $u\in\mathcal{U}^n$, at every equilibrium allocation and payment (x,t), for every $i\in N$ and all $j\neq i$, if $x_i\neq x_j$, then $u_i'(x_i)=\frac{t_i-t_j}{x_i-x_j}$ for $x_i>0$, and $u_i'(x_i)\leq\frac{t_i-t_j}{x_i-x_j}$ for $x_i=0$, and if $x_i=x_j$, then $t_i=t_j$.*

Proof of Lemma 1. Given $u\in\mathcal{U}^n$, let the mechanism have an equilibrium allocation and payment (x,t). By definition, no envy holds if and only if $u_i(x_i)-u_i(x_j)\geq t_i-t_j$ and $u_j(x_j)-u_j(x_i)\geq t_j-t_i$. This is equivalent to $\frac{u_i(x_i)-u_i(x_j)}{x_i-x_j}\leq\frac{t_i-t_j}{x_i-x_j}$ and $\frac{u_j(x_j)-u_j(x_i)}{x_j-x_i}\geq\frac{t_j-t_i}{x_j-x_i}$ for $x_i<x_j$ for all $i,j\in N$. The concavity of u_i for $x_i<x_j$ implies $\frac{u_i(x_i)-u_i(x_j)}{x_i-x_j}\leq u_i'(x_i)$, and so agent i is envy-free if and only if $u_i'(x_i)\leq\frac{t_i-t_j}{x_i-x_j}$. Likewise, agent j is envy-free if and only if $u_j'(x_j)\geq\frac{t_j-t_i}{x_j-x_i}$. Combining the two inequalities, we have $u_i'(x_i)\leq\frac{t_i-t_j}{x_i-x_j}\leq u_j'(x_j)$. With the first-order condition of efficiency, i.e. $u_i'(x_i)=u_j'(x_j)$ for $x_i,x_j\in\mathcal{R}_{++}$, the mechanism is efficient and envy-free if and only if $u_i'(x_i)=\frac{t_i-t_j}{x_i-x_j}$ for $x_i>0$, and if $x_i=x_j$, no envy holds, but if and only if $t_i=t_j$. ■

Proof of Proposition 2

(i) Ranking: Suppose that $x_i \leq x_j$, which is equivalent to $\theta_i \leq \theta_j$. Remember that $t_i = \theta_i\theta_{N\setminus i} - S_{-i}$, in which case $t_i - t_j = \theta_i\theta_{N\setminus i} - S_{-i} - \theta_j\theta_{N\setminus j} + S_{-j} = \theta_N(\theta_i - \theta_j)$. Thus, $t_i \leq t_j$ if and only if $\theta_i \leq \theta_j (x_i \leq x_j)$.

(ii) Voluntary Participation: Since $p_i(\theta_i, \theta_{-i})$ is concave in θ_i, it is sufficient to check if $p_i(0, \theta_{-i}) \geq 0$. We see $p_i(0, \theta_{-i}) = S_{-i} > 0$, and so VP holds.

(iii) No-envy: Under the Inada condition, the SEF mechanism is efficient, as $u_i'(x_i) = \frac{\theta_N^2}{R} = u_j'(x_j)$ for all $i, j \in N$. No envy holds if and only if $u_i(x_i) - t_i \geq u_i(x_j) - t_j$ at equilibrium x and t for all $i, j \in N$, which is written as $u_i(x_i) - u_i(x_j) \geq \theta_i\theta_{N\setminus i} - \theta_j\theta_{N\setminus j} - S_{-i} + S_{-j} = (\theta_i - \theta_j)\theta_N = (x_i - x_j)\frac{\theta_N^2}{R} = (x_i - x_j)u_i'(x_i)$. The concavity of u_i implies $u_i(x_i) - u_i(x_j) \geq (x_i - x_j)u_i'(x_i)$, and thus no envy holds. ■

Part b. The mechanism collects $\sum_{i\in N} t_i$ and we have $\sum_{i\in N} t_i = \sum_{i\in N}[\theta_i\theta_{N\setminus i} - S_{-i}] = \sum_{i\in N}\theta_i\theta_N - nS = (\sum_{i\in N}\theta_i)^2 - n\sum_{i\in N}\theta_i^2 \leq 0$. The last inequality holds due to the Cauchy-Schwartz inequality, and so the SEF mechanism yields a budget deficit of $n\sum_{i\in N}\theta_i^2 - (\sum_{i\in N}\theta_i)^2$. Substituting in $\theta_i = x_i\frac{\theta_N}{R}$, the budget deficit is $n\big(\sum_{i\in N} x_i^2\big)\frac{\theta_N^2}{R} - \frac{\theta_N^2}{R}\big(\sum_{i\in N} x_i\big)^2 = n\big(\sum_{i\in N} x_i^2\big)\frac{\theta_N^2}{R^2} - \theta_N^2$. There is a $i \in N$ with $x_i > 0$, and so $u_i'(x_i) = \frac{\theta_N^2}{R}$, following which the budget deficit is $\frac{n}{R}u_i'(x_i)\sum_{i\in N} x_i^2 - Ru_i'(x_i) = \frac{n}{R}\lambda\big[\sum_{i\in N} x_i^2 - \frac{R^2}{n}\big]$.

Observations:

(a) If $u_i = u_j$ for all $i \neq j$ and $i, j \in N$, then $\theta_i = \theta_j$ for all $i \neq j$ and $x_i = \frac{R}{n}$ for all $i \in N$, in which case it is easy to check that the mechanism has a balanced budget.

(b) Note that the supremum of $\sum_{i\in N} x_i^2$ for $x \in \mathcal{X}$ is achieved at the extreme points of $x \in \mathcal{X}$, so we have a budget deficit of $\frac{n}{R}\lambda\big[\sum_{i\in N} x_i^2 - \frac{R^2}{n}\big] \leq \frac{n}{R}\lambda\big(R^2 - \frac{R^2}{n}\big) = \lambda R(n-1)$. ■

Proof of Theorem 4

First, we show that the SEF mechanism satisfies the properties. It was shown previously that the SEF mechanism is efficient and envy-free under the Inada condition. The payment of the SEF mechanism by agent i is $t_i(\theta) = \theta_i\theta_{N\setminus i} - S_{-i} = \sum_{j\neq i}\theta_j\theta_i + \sum_{j\neq i} -\theta_j^2$, i.e. $\alpha(\theta_j) = \theta_j$ is a polynomial of degree 1 in θ_j, and $k(\theta_j) = -\theta_j^2$ is a polynomial of degree 2 in θ_j.

Now we will show that any efficient, envy-free and simple two-part tariff scalar mechanism is the SEF mechanism up to affine transformations. Fix a scalar mechanism that is efficient, envy-free and a simple two-part tariff, and then denote its marginal price by $g(\theta)$. Agent i's net utility under the simple two-part tariff is

$$p_i(\theta_i, \theta_{-i}) = u_i(x_i(\theta)) - \sum_{j\neq i}\alpha(\theta_j)\theta_i - \sum_{j\neq i}k(\theta_j). \tag{11}$$

According to Lemma 1, the presence of neither envy nor efficiency implies

$$g(\theta) = \frac{t_i - t_j}{x_i - x_j} = \frac{\sum_{l\neq i}\alpha(\theta_l)\theta_i + \sum_{l\neq i}k(\theta_l) - \sum_{l\neq j}\alpha(\theta_l)\theta_j - \sum_{l\neq j}k(\theta_l)}{x_i(\theta) - x_j(\theta)}. \tag{12}$$

Equation (12) illustrates that $x_i(\theta) = \frac{\sum_{j\neq i}\alpha(\theta_j)\theta_i - k(\theta_i) + \tilde{S}(\theta_i,\theta_{-i})}{g(\theta)}$. Recall that for any $i \in N$, $x_i(\theta) = 0$ if $\theta_i = 0$, which means $k(\theta_i) = \tilde{S}(0, \theta_{-i})$ for each $i \in N$, and since k is a polynomial, $\tilde{S}$ is differentiable in θ_i, and it is symmetric. Thus, $\tilde{S}(\theta)$ is additively separable. Hence, we can write

$$x_i(\theta) = \frac{\sum_{j\neq i}\alpha(\theta_j)\theta_i - k(\theta_i)}{g(\theta)}. \tag{13}$$

We then rewrite (11) as

$$p_i(\theta_i, \theta_{-i}) = u_i\left(\frac{\sum_{j\neq i}\alpha(\theta_j)\theta_i - k(\theta_i)}{g(\theta)}\right) - \sum_{j\neq i}\alpha(\theta_j)\theta_i - \sum_{j\neq i}k(\theta_j).$$

The first-order condition of equilibrium gives $0 = \frac{\partial p_i(\theta)}{\partial\theta_i}$, which is equivalent to

$$u_i'(x_i)\frac{\sum_{j\neq i}\alpha(\theta_j)g(\theta) - k'(\theta_i)g(\theta) - \sum_{j\neq i}\alpha(\theta_j)\theta_i\frac{\partial g(\theta)}{\partial\theta_i} + k(\theta_i)\frac{\partial g(\theta)}{\partial\theta_i}}{g(\theta)^2} = \sum_{j\neq i}\alpha(\theta_j)$$

where $k'(\theta_i) = \frac{\partial k(\theta_i)}{\partial \theta_i}$. With $u_i'(x_i) = g(\theta)$ from efficiency, this is equivalent to

$$\frac{k'(\theta_i)}{\sum_{j\neq i} \alpha(\theta_j)\theta_i - k(\theta_i)} = -\frac{\frac{\partial g(\theta)}{\partial \theta_i}}{g(\theta)}. \tag{14}$$

Feasibility $R = \sum_{i\in N} x_i$ and (13) together imply

$$g(\theta) = \{\sum_{i\in N}\sum_{j\neq i} \alpha(\theta_j)\theta_i - \sum_{i\in N} k(\theta_i)\}/R. \tag{15}$$

Differentiating (15) with respect to θ_i, we have

$$\frac{\partial g(\theta)}{\partial \theta_i} = \left\{\sum_{j\neq i} \alpha(\theta_j) + \alpha'(\theta_i)\theta_{N\setminus i} - k'(\theta_i)\right\}/R. \tag{16}$$

Replacing $g(\theta)$ with (15) and $\frac{\partial g(\theta)}{\partial \theta_i}$ with (16) in (14), we have

$$k'(\theta_i)\{\sum_{l\neq i}\sum_{j\neq l} \alpha(\theta_j)\theta_l - \sum_{j\neq i} k(\theta_j)\} = \{k(\theta_i) - \sum_{j\neq i} \alpha(\theta_j)\theta_i\}\{\sum_{j\neq i} \alpha(\theta_j) + \alpha'(\theta_i)\theta_{N\setminus i}\} \tag{17}$$

where $\alpha'(\theta_i) = \frac{\partial \alpha(\theta_i)}{\partial \theta_i}$.

Let $k(\theta_i)$ and $\alpha(\theta_i)$ be a polynomial of degree m and of degree γ in θ_i, respectively. The RHS of (17) is the degree of 2γ in θ_j, and the LHS of (17) is the degree of $\max\{\gamma, m\}$ in θ_j, for any $j \neq i$. Thus, we have $2\gamma = m$ for $m, \gamma \geq 1$. As the mechanism is a simple two-part tariff, we have $\gamma = 1$ and $m = 2$,, namely $k(\theta_i) = k\theta_i^2$ and $\alpha(\theta_i) = \alpha\theta_i$ for the non-zero constants k and α (note that even if we set $k(\theta_i) = k\theta_i^2 + \beta\theta_i + C$ and $\alpha(\theta_i) = \alpha\theta_i + D$ for constants k, β, C, α and D, we have the same result). Plugging $k(\theta_i)$ and $\alpha(\theta_i)$ into (17), we have $\alpha(\alpha + k)(\theta_{N\setminus i})^2 = (\alpha + k)kS_{-i}$, which leads to $k = -\alpha$ and thus $k(\theta_i) = -\alpha\theta_i^2$. This implies $g(\theta) = \alpha\frac{\theta_N^2}{R}$ and $x_i(\theta) = \frac{\theta_i}{\theta_N}R$. ■

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