Optimality of the Uniform Rule*

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Abstract

A fixed amount of a divisible resource is distributed among agents whose preferences are single-peaked. Under the condition of no free disposal, the uniform rule has been characterized by a number of interesting properties. Taking a cardinal approach with concave utility functions, this paper provides another characterization of the uniform rule. We show that the uniform rule is the only consistent rule that maximizes the worst-case relative surplus for any group of agents among strategy-proof and ordinally efficient mechanisms.

Keywords: Single-peaked preferences; strategy-proofness; worst-case analysis; efficiency; uniform rule; consistency; divisible good.

JEL Classification: D63; D70; D71

1 Introduction

Consider the problem of allocating an infinitely divisible good to agents with single-peaked preferences: each agent has a most preferred level of consumption, and moving away from that level makes him or her worse off. The most famous solution for this problem is the uniform rule (Benassy, 1982). The uniform rule has been studied from various perspectives: strategy-proofness by Sprumont (1991) and Ching (1994), consistency by Thomson (1994b) and Dagan (1996), monotonicity by Thomson (1994a,1995,1997) and Sönmez (1994), no-envy by Ching (1992), and convex no-envy by Chun (2000). The literature only uses ordinal information about preferences and characterizes the uniform rule based on ordinal efficiency. Unlike this literature, we study the case when preference intensities are part of the model. We restrict our attention to preference intensities that are concave.

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When cardinal utilities matter, an optimality criterion used to rank allocation rules in the literature is the **worst relative surplus (WS)**, that is, the smallest relative surplus over all utility profiles under an allocation rule.[1] This worst-case criterion avoids misspecification of utilities and it corresponds to a planner's extreme risk aversion (Johari and Tsitsiklis, 2004, You, 2015). Here, our optimality criterion strengthens WS as it considers the performance of a rule at every subgroup of agents.

Our results show that the uniform rule is uniquely optimal and consistent among strategy-proof and ordinally efficient rules. It guarantees the greatest surplus at any coalition of agents when the good is either overdemanded or underdemanded. The uniform rule achieves WS of $\frac{1}{|S|}$ when the good is overdemanded, and WS of $\frac{|S|-1}{|S|}$ when underdemanded, for a given coalition $S \subseteq N$.

The paper is organized as follows. Section 2 explains the model and introduces optimality. Section 3 proves that the uniform rule is the only optimal and consistent mechanism and computes its WS for any coalition.

2 Model

Let $C > 0$ be a perfectly divisible resource, to be divided among agents. There is no free disposal. A planner distributes the resource to a group of agents $N = \{1, \ldots, n\}$ where $n \geq 2$. For each $i \in N$, agent i's resource share will be denoted by y_i. A **feasible allocation** of C among the agents in $S \subset N$ is a vector y in the set $Y(S, C) = \{y \in [0, C]^S | \sum_{i \in S} y_i = C\}$. The value of agent i's share is presented by a utility function u_i that is continuous and concave on $[0, C]$. The resource is an economic good; thus $u_i(y_i) \geq 0$ for all $i \in N$ and all $y_i \in [0, C]$. In addition, each agent i's utility is **single-peaked**: there exists a peak, $x_i^*(u_i) \in [0, C]$, such that for all y_i,$z_i \in [0, C]$, if either $x_i^*(u_i) < y_i < z_i$ or $z_i < y_i < x_i^*(u_i)$, then $u_i(x_i^*) > u_i(y_i) > u_i(z_i)$. We denote by $\mathcal{U}(S, C)$ the set of utility functions for the agents in S with amount of resource C satisfying the aforementioned properties. Let $u_{-i} = (u_j)_{j \in S \setminus i}$ and $u_S = (u_i)_{i \in S}$. For $u \in \mathcal{U}(S, C)$, let $x_S^*(u) = \sum_{i \in S} x_i^*(u_i)$ be the sum of the peaks. If $x_S^*(u) > C$ or $x_S^*(u) < C$, then a resource is **overdemanded** or **underdemanded**, respectively. Let $\bar{\mathcal{U}}(S, C)$ and $\underline{\mathcal{U}}(S, C)$ be the set of utility functions that are overdemanded and underdemanded for the agents in S and amount of resource C, respectively.

For each $i \in N$, u_i is private information. A direct mechanism F associates to each report $u \in \mathcal{U}(S, C)$ a feasible allocation $F(S, u, C) \in Y(S, C)$ for every $S \subseteq N$ and $C > 0$. When there is no confusion, $F(S, u, C)$ will be denoted by $F(u, C)$ or $F(u)$. A mechanism F is **consistent** if for all $C > 0$, all $u \in \mathcal{U}(N, C)$ and all $S \subseteq N$, $F_i(N, u, C) = F_i(S, u_S, \sum_{i \in S} F_i(N, u, C))$ for all $i \in S$. A mechanism F is **strategy-proof** if for all $S \subset N$, $i \in S$, all $u \in \mathcal{U}(S, C)$, and all $u_i' \in \mathcal{U}(S, C)$, $u_i(F(u_i, u_{-i})) \geq u_i(F(u_i', u_{-i}))$. A mechanism F is **same-sided** if the following property holds: for all $S \subseteq N$, $i \in S$, $u \in \mathcal{U}(S, C)$, if $x_S^* \leq C$,

[1]The worst-case analysis is adopted in Aggarwal et al. (2005), Goldberg et al. (2001, 2006), Hartline and McGrew (2005), Jahari and Tsitsiklis (2004), Juarez and Kumar (2012), Moulin (2009), Moulin and Shenker (2001) and Fischer and Klimm (2015). The worst-case analysis that adopts an absolute instead of relative measure has been explored in Juarez (2008), Juarez (2015), and Moulin and Shenker (1999).

then $x_i^*(u_i) \leq F_i(u)$, and if $x_S^* \geq C$, then $x_i^*(u_i) \geq F_i(u)$. Same-sidedness is equivalent to ordinal efficiency for strategy-proof mechanisms.

The **uniform rule** F^U is a well-known mechanism defined as follows: for all $i \in S$, if $x_S^* \geq C$, then $F_i^U(u) = \min\{x_i^*(u_i), \mu\}$ where μ solves the equation $\sum_{i \in S} \min\{x_i^*(u_i), \mu\} = C$. If $x_S^* \leq C$, then $F_i^U(u) = \max\{x_i^*(u_i), \nu\}$ where ν solves $\sum_{i \in S} \max\{x_i^*(u_i), \nu\} = C$. The uniform rule meets all the above properties.

We measure the performance of a mechanism by computing the worst relative surplus at every coalition. Given a utility profile $u \in \mathcal{U}(S, C)$, the **relative surplus** at an allocation is the ratio of the economic surplus to efficient surplus. If $y \in Y(S, C)$ satisfies $y \in \arg\max_{z \in Y(S,C)} \sum_{i \in S} u_i(z_i)$, the allocation is **efficient**. Each allocation $y \in Y(S, C)$ generates **economic surplus** of $\sigma^y(u) = \sum_{i \in S} u_i(y_i)$. When y is efficient, the economic surplus is said to be **efficient surplus**. We denote by $Eff(u, C)$ the efficient surplus given the utility profile u and budget C.

Definition 1 (Optimality of a mechanism) *i. The worst relative surplus of the mechanism F at a coalition $S \subseteq N$ and resource C for the overdemanded case is*

$$WS(F, S, C) = \inf_{u \in \bar{\mathcal{U}}(S,C)} \frac{\sum_{i \in S} u_i(F_i(S, u, C))}{Eff(u, C)}$$

ii. A strategy-proof and same-sided mechanism F^* is optimal for the overdemanded case if for any other strategy-proof and same-sided mechanism F, we have that $WS(F, S, C) \leq WS(F^*, S, C)$ for any $S \subseteq N$ and $C > 0$.

iii. The worst relative surplus of the mechanism G at a coalition $S \subseteq N$ and resource C for the underdemanded case is

$$WS(G, S, C) = \inf_{u \in \underline{\mathcal{U}}(S,C)} \frac{\sum_{i \in S} u_i(G_i(S, u, C))}{Eff(u, C)}$$

iv. A strategy-proof and same-sided mechanism G^* is optimal for the underdemanded case if for any other strategy-proof and same-sided mechanism G, we have that $WS(G, S, C) \leq WS(G^*, S, C)$ for any $S \subseteq N$ and $C > 0$.

3 The Result

We will show that the uniform rule is uniquely optimal and consistent among strategy-proof and same-sided mechanisms. First we present a characterization of strategy-proof and same-sided mechanisms. The following proposition is due to Barberà, Jackson and Neme (1997).

Proposition 1 *A mechanism F is strategy-proof and same-sided if and only if for each $i \in N$ there exists $a_i : \mathcal{U}(N \setminus i, C) \to [0, C]$ and $b_i : \mathcal{U}(N \setminus i, C) \to [0, C]$, such that $a_i(u_{-i}) \leq b_i(u_{-i})$ and $\sum_{i \in N} \min[x_i^*(u_i), b_i(u_{-i})] = C$ for all u such that $x_N^*(u) > C$, $\sum_{i \in N} \max[x_i^*(u_i), a_i(u_{-i})] = C$ for all u such that $x_N^*(u) \leq C$, and $F_i(u) = \min[x_i^*(u_i), b_i(u_{-i})]$ if $x_N^*(u) > C$, $F_i(u) = \max[x_i^*(u_i), a_i(u_{-i})]$ if $x_N^*(u) \leq C$.*

Proposition 1 states that each agent will receive his or her most preferred share or personalized cap (floor) for the case of overdemanded (underdemanded, respectively) resource if and only if the mechanism is strategy-proof and same-sided. Using this result, we show that the uniform rule is uniquely optimal.

Theorem 1 *i. The uniform rule F^U is the **uniquely** optimal and consistent mechanism for the overdemanded case. Moreover, $WS(F^U, S, C) = \frac{1}{|S|}$ for any $S \subseteq N$ and any $C > 0$.*

*ii. The uniform rule F^U is the **uniquely** optimal and consistent mechanism for the underdemanded case. Moreover, $WS(F^U, S, C) = \frac{|S|-1}{|S|}$ for any $S \subseteq N$ and any $C > 0$.*

Proof.

Proof for the overdemanded case:

Step A1. $WS(F^U, S, C) = \frac{1}{s}$ for any coalition $S \subseteq N$ such that $|S| = s$ and any $C > 0$. First, we show that $WS(F^U, S, D) \geq \frac{1}{s}$ for $D > 0$. Consider $v_S \in \bar{\mathcal{U}}(S, D)$ and suppose that E_i is the peak of v_i for each $i \in S$. Since v_i is non-decreasing and concave in the interval $[0, E_i]$, we have $v_i(\min\{\frac{D}{s}, E_i\}) \geq \frac{v_i(\min\{D, E_i\})}{s}$. This implies

$$\frac{\sum_{i\in S} v_i(\min\{\frac{D}{s}, E_i\})}{\sum_{i\in S} v_i(\min\{D, E_i\})} \geq \frac{1}{s}$$

Furthermore, since $F_i^U(v_S, D) \geq \min\{\frac{D}{s}, E_i\}$ for each $i \in S$, we have

$$\frac{\sum_{i\in S} v_i(F_i^U(v_S, D))}{\sum_{i\in S} v_i(\min\{D, E_i\})} \geq \frac{\sum_{i\in S} v_i(\min\{\frac{D}{s}, E_i\})}{\sum_{i\in S} v_i(\min\{D, E_i\})} \geq \frac{1}{s}$$

Finally, from $Eff(v_S, D) \leq \sum_{i\in S} v_i(\min\{D, E_i\})$, we have

$$\frac{\sum_{i\in S} v_i(F_i^U(v_S, D))}{Eff(v_S, D)} \geq \frac{\sum_{i\in S} v_i(F_i^U(v_S, D))}{\sum_{i\in S} v_i(\min\{D, E_i\})} \geq \frac{1}{s}$$

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Hence, $WS(F^U, S, D) \geq \frac{1}{s}$.

Next, we show that $WS(F^U, S, D) \leq \frac{1}{s}$. Consider the set of linear utility functions $u_i(x) = \alpha_i x$ for $x \leq C$, where $\alpha_i = \delta^i \alpha_1$ for some arbitrary $0 < \delta < 1$, $\alpha_1 > 0$ and $i \geq 2$. This implies that $\alpha_1 > \alpha_2 > \cdots > \alpha_n > 0$. Since $x_i^*(u_i) \geq C$ for all $i \in N$ with these utility functions, the uniform rule makes an allocation $F_i^U = \frac{C}{n}$ for each $i \in N$. Consider an arbitrary coalition $S \subseteq N$, where $S = \{i_1, i_2, \ldots, i_s\}$ and $i_1 < i_2 < \cdots < i_s$. The uniform rule $F^U(N, u, C)$ generates a surplus of $\sum_{k\in S} \alpha_k \frac{C}{n}$ for S with $v_S = u_S$. Efficiency is giving agent i_1 total resource for S: $Eff(u_S, D) = \alpha_{i_1} D$ for $D = s\frac{C}{n}$. Thus, we have

$$WS(F^U, S, D) \leq \lim_{\delta\to 0} \frac{\sum_{k\in S} \alpha_k \frac{C}{n}}{\alpha_{i_1} D} = \lim_{\delta\to 0} \frac{\frac{C}{n} \sum_{k\in S} \alpha_k}{\alpha_{i_1}(s\frac{C}{n})} = \frac{1}{s} \lim_{\delta\to 0} \frac{\sum_{k\in S} \alpha_k}{\alpha_{i_1}}$$

$$\leq \frac{1}{s}\lim_{\delta\to 0}\frac{\alpha_{i_1}+\delta\alpha_{i_1}+\cdots+\delta^{s-1}\alpha_{i_1}}{\alpha_{i_1}}=\frac{1}{s}$$

Step A2. Suppose that a strategy-proof and same-sided mechanism F is optimal for the agents in N and resource C. Then for every utility profile u, we have that $b_i(u_{-i}) \geq \frac{C}{n}$ for any $i \in N$.

We prove step A2 by contradiction. Suppose there exists a utility profile u and agent i such that $b_i(u_{-i}) < \frac{C}{n}$. Let agent i's utility function be $u_i^\alpha(x) = \alpha x$ for $x \leq C$ and $\alpha > 0$. For a large enough α, the efficient allocation is giving C to agent i. Thus, $Eff((u_i^\alpha, u_{-i}), C) = \alpha C$ for large α. On the other hand, $\sigma^F((u_i^\alpha, u_{-i}), C) = \alpha b_i(u_{-i}) + \sum_{k\in N\setminus i} u_k(F_k(u)) \leq \alpha b_i(u_{-i}) + \sum_{k\in N\setminus i} u_k(x_k^*(u_k))$. Thus,

$$WS(F,N,C) \leq \lim_{\alpha\to\infty}\frac{\alpha b_i(u_{-i}) + \sum_{k\in N\setminus i} u_k(x_k^*(u_k))}{\alpha C} = \frac{b_i(u_{-i})}{C} < \frac{1}{n} = WS(F^U, N, C).$$

This contradicts that F is optimal.

In particular, Proposition 1 and Step A2 imply the following two conditions: (a) if $x_i^*(u_i) < \frac{C}{n}$, then $y_i^F(u,C) = x_i^*(u_i)$; (b) if $y_i^F(u,C) < x_i^*(u_i)$, then $y_i^F(u,C) \geq \frac{C}{n}$.

For the next two steps, we fix a strategy-proof and same-sided mechanism F that is optimal for any coalition S.

Step A3. For any profile $u \in \bar{\mathcal{U}}(N,C)$, the mechanism F either assigns the agents their peak or a constant amount.

Consider the set $S^* = \{i \in N | y_i^F(u,C) < x_i^*(u_i)\}$ composed of the agents who receive allocations different from their peaks. Consistency implies $y_i^F(u_{S^*}, D) = y_i^F(u,C)$ for any $i \in S^*$ where $D = \sum_{i\in S^*} y_i^F(u,C)$. Since F is optimal at coalition S^* and condition (b) holds for coalition S^*, we have $y_i^F(u_{S^*}, D) \geq \frac{\sum_{i\in S^*} y_i^F(u_{S^*},D)}{|S^*|}$ for any $i \in S^*$. Hence $y_i^F(u,C) = y_i^F(u_{S^*}, D) = \frac{\sum_{i\in S^*} y_i^F(u_{S^*},D)}{|S^*|}$ for any $i \in S^*$.

Step A4. For any utility profile u, assume, without loss of generality, that $x_1^*(u_1) \leq x_2^*(u_2) \leq \cdots \leq x_n^*(u_n)$. There exists $k \leq n$ such that $S^* = \{k, k+1, \ldots, n\}$. That is, the set of agents who do not receive their peaks is a set of consecutive agents with the highest peaks.

Suppose for the sake of contradiction that there exists $i < j$ such that $y_i^F(u,C) < x_i^*(u_i)$ and $y_j^F(u,C) = x_j^*(u_j)$. Let $D = y_i^F(u,C) + y_j^F(u,C)$. By consistency, optimality on the set $\{i,j\}$ and condition (b), we have $y_i^F(u,C) = y_i^F(\{i,j\}, u_{\{i,j\}}, D) \geq \frac{y_i^F(\{i,j\},u_{\{i,j\}},D)+y_j^F(\{i,j\},u_{\{i,j\}},D)}{2}$ $= \frac{y_i^F(u,C)+y_j^F(u,C)}{2}$. This contradicts to $y_j^F(u,C) = x_j^*(u_j) \geq x_i^*(u_i) > y_i^F(u,C)$.

Finally, there is a unique rule that satisfies steps A3 and A4. This rule is the uniform rule.

<u>Proof for the underdemanded case:</u>

The following steps parallel Step A1-A4 above.

Step B1. $WS(F^U, S, C) = \frac{s-1}{s}$ for any coalition $S \subseteq N$ such that $|S| = s$ and any $C > 0$. First, we show that $WS(F^U, S, D) \geq \frac{s-1}{s}$ for $D > 0$. Consider $v_S \in \underline{\mathcal{U}}(S,D)$ and suppose that E_i is the peak of v_i for each $i \in S$. Since v_i is non-increasing, non-negative and concave in the interval $[E_i, D]$, we have $v_i(\max\{\frac{D}{s}, E_i\}) \geq \frac{v_i(E_i)(s-1)}{s}$. This implies

$$\frac{\sum_{i\in S} v_i(\max\{\frac{D}{s}, E_i\})}{\sum_{i\in S} v_i(E_i)} \geq \frac{s-1}{s}$$

Furthermore, since $E_i \leq F_i^U(v_S, D) \leq \max\{\frac{D}{s}, E_i\}$ for each $i \in S$, we have

$$\frac{\sum_{i\in S} v_i(F_i^U(v_S, D))}{\sum_{i\in S} v_i(E_i)} \geq \frac{\sum_{i\in S} v_i(\max\{\frac{D}{s}, E_i\})}{\sum_{i\in S} v_i(E_i)} \geq \frac{s-1}{s}$$

Finally, from $Eff(v_S, D) \leq \sum_{i\in S} v_i(E_i)$, we have

$$\frac{\sum_{i\in S} v_i(F_i^U(v_S, D))}{\sum_{i\in S} Eff(v_S, D)} \geq \frac{\sum_{i\in S} v_i(F_i^U(v_S, D))}{\sum_{i\in S} v_i(E_i)} \geq \frac{s-1}{s}$$

.

Hence, $WS(F^U, S, D) \geq \frac{s-1}{s}$.

Next, we show that $WS(F^U, S, D) \leq \frac{s-1}{s}$. Consider an arbitrary coalition S with resource D to divide, where $S = \{i_1, i_2, \ldots, i_s\}$ and $i_1 < i_2 < \cdots < i_s$. Let $v_i = \alpha_i(D - x)$ for $x \leq D$ for all $i \in S$ where $\alpha_i > \alpha_j > 0$ if $i < j$. The efficient allocation gives D to agent i_s and nothing to all other agents: $Eff(v_S, D) = \sum_{j=1}^{s-1} \alpha_{i_j} D$. However, the uniform rule $F^U(S, v_S, D)$ generates a surplus of $\sum_{k\in S} \alpha_k(D - \frac{D}{s})$. Thus

$$WS(F^U, S, D) \leq \min_{\alpha:\alpha_{i_1} > \alpha_{i_2} > \cdots > \alpha_{i_s}} \frac{\sum_{k\in S} \alpha_k(D - \frac{D}{s})}{\sum_{j=1}^{s-1} \alpha_{i_j}(D)} = \min_{\alpha:\alpha_{i_1} > \alpha_{i_2} > \cdots > \alpha_{i_s}} \frac{s-1}{s} \frac{\sum_{k\in S} \alpha_k}{\sum_{j=1}^{s-1} \alpha_{i_j}} = \frac{s-1}{s}$$

Step B2. Suppose that the strategy-proof and same-sided mechanism F is optimal for the agents in N and resource C. Then for every utility profile u, we have that $a_i(u_{-i}) \leq \frac{C}{n}$ for any i.

We prove step 2 by contradiction. Suppose there exists a utility profile u and agent i such that $a_i(u_{-i}) > \frac{C}{n}$. Let agent i's utility function be $u_i^\alpha(x) = \alpha - \frac{\alpha}{c}x$ for $x \leq C$ and $\alpha > 0$. For a large enough α, the efficient allocation requires agent i to receive nothing and the other agents to split the resource C. Thus, $Eff((u_i^\alpha, u_{-i}), C) = \alpha + \sum_{j\neq i} u^j(E^j) \geq \alpha$ for large α, where E^j is the efficient allocation of the resource to agent j. On the other hand, $\sigma^F((u_i^\alpha, u_{-i}), C) \leq \alpha - \frac{\alpha}{C} a_i(u_{-i}) + \sum_{k\neq i} u_k(x_k^*(u_k))$ from Proposition 1. Thus, we have

$$WS(F, N, C) \leq \lim_{\alpha\to\infty} \frac{\alpha - \frac{\alpha}{C} a_i(u_{-i}) + \sum_{k\neq i} u_k(x_k^*(u_k))}{\alpha} = 1 - \frac{a_i(u_{-i})}{C} < \frac{n-1}{n}.$$

Therefore, F is not optimal, which is a contradiction.

In particular, Proposition 1 and Step B2 imply the following conditions: (a) if $x_i^*(u_i) > \frac{C}{n}$, then $y_i^F(u, C) = x_i^*(u_i)$; (b) if $y_i^F(u, C) > x_i^*(u_i)$, then $y_i^F(u, C) \leq \frac{C}{n}$.

For the next two steps, we fix a strategy-proof and same-sided mechanism F that is optimal.

Step B3. For any profile $u \in \mathcal{U}(N, C)$, the mechanism F either assigns the agents their peak or a constant amount.

Consider the set $S^* = \{i \in N | y_i^F(u, C) > x_i^*(u_i)\}$ composed of the agents who receive an allocation different from their peak. Let $D = \sum_{i \in S^*} y_i^F(u, C)$. By applying optimality on S^*, consistency and condition (b), we have $y_i^F(u, C) = y_i^F(u_{S^*}, D) \leq \frac{\sum_{i \in S^*} y_i^F(u_{S^*}, D)}{|S^*|}$ for any $i \in S^*$. Hence $y_i^F(u, C) = \frac{\sum_{i \in S^*} y_i^F(u, C)}{|S^*|}$ for any $i \in S^*$.

Step B4. For any utility profile u, assume, without loss of generality, that $x_1^*(u_1) \leq x_2^*(u_2) \leq \cdots \leq x_n^*(u_n)$. There exists $k \leq n$ such that $S^* = \{1, \ldots, k\}$. That is, the set of agents who do not receive their peak is a set of consecutive agents with the lowest peaks.

Suppose that there exists $i > j$ such that $y_i^F(u, C) > x_i^*(u_i)$ and $y_j^F(u, C) = x_j^*(u_j)$. Let $D = y_i^F(u, C) + y_j^F(u, C)$. By consistency, optimality on the set $\{i, j\}$ and condition (b), we have $y_i^F(u, C) = y_i^F(\{i, j\}, u_{\{i,j\}}, D) \leq \frac{y_i^F(\{i,j\}, u_{\{i,j\}}, D) + y_j^F(\{i,j\}, u_{\{i,j\}}, D)}{2} = \frac{y_i^F(u,C) + y_j^F(u,C)}{2}$. This contradicts to $y_j^F(u, C) = x_j^*(u_j) \leq x_i^*(u_i) < y_i^F(u, C)$.

Finally, there is a unique rule that satisfies steps B3 and B4. This rule is the uniform rule. ■

This characterization is tight. We do not provide all the details due to space limitations. However, we would like to make two remarks. First, the assumption of same-sideness is necessary to get the result. Without same-sidedness, the equal division rule $F(S, u, C) = \frac{C}{|S|}$ is optimal for both the overdemanded and underdemanded cases. It guarantees the same WS as F^U for any group of agents. Second, the assumption of concavity of the utility function is necessary in order to achieve a non-zero WS. To show this for the overdemanded case, consider a strategy-proof and same-sided mechanism and a utility profile u where some agent i is assigned an amount y_i strictly less than his peak $x_i^*(u_i)$. Consider the utility function $v_i^\alpha(x) = u_i(x)$ for $x \leq y_i$, $v_i^\alpha(x) = \alpha^{x - y_i} u_i(x)$ for $x_i^*(u_i) \geq x \geq y_i$, and $v_i^\alpha(x) = \alpha^{2x_i^*(u_i) - y_i - x} u_i(x)$ for $x \geq x_i^*(u_i)$. Note that v_i^α has the same peak as u_i for $\alpha \geq 1$, but the cardinal value at the peak tends to infinity as α tends to infinity. By strategy-proofness and same-sidedness, the assignment of agent i at (v_i^α, u_{-i}) is equal to y_i for any $\alpha \geq 1$. Thus, the mechanism generates zero WS as α goes to infinity. A similar argument can be done for the underdemanded case.

References

Aggarwal, G., Fiat, A., Goldberg, A., Hartline, J., Immorlica, N., Sudan, M., 2005. Derandomization of auctions. In: Proceedings of the Annual Symposium on Theory of Computing. STOC, 619-625.

Barberà, S., Jackson, M., Neme, A., 1997. Strategy-Proof Allotment Rules. Games and Economic Behavior 18, 1-21.

Benassy, J., 1982. The Economics of Market Disequilibrium. Academic Press, San Diego.

Ching, S., 1992. A simple characterization of the uniform rule. Economics Letters 40, 57-60.

Ching, S., 1994. An alternative characterization of the uniform rule. Social Choice and Welfare 11, 131-136.

Chun, Y., 2000. Distribution properties of the uniform rule in economies with single-peaked preferences. Economic Letters 67, 23-27.

Dagan, N., 1996. A note on Thomson's characterizations of the uniform rule. Journal of Economic Theory 69, 255-261.

Fischer, F., Klimm, M., 2015. Optimal Impartial Selection. SIAM Journal on Computing 44 (5), 1263-1285.

Goldberg, A., Hartline, J., Wright, A., 2001. Competitive auctions and digital goods. In: Proceedings of the Annual ACM-SIAM Symposium on Discrete Algo-rithms. SODA, Washington, DC, pp.735-744.

Goldberg, A., Hartline, J., Karlin, A., Saks, M., Wright, A., 2006. Competitive auctions. Games and Economic Behavior 55, 242-269.

Hartline, J., McGrew, R., 2005. From optimal limited to unlimited supply auctions. In: Proceedings of the ACM Conference on Electronic Commerce. EC, Vancouver, Canada, 175-182.

Johari, R., Tsitsiklis, J., 2004. Efficiency Loss in a Network Resource Allocation Game. Mathematics of Operations Research 29 (3), 407-435.

Juarez, R., 2008. The Worst Absolute Surplus Loss in the Problem of Commons: Random Priority vs. Average Cost. Economic Theory 34 (1), 69-84.

Juarez, R., 2015. Optimal group strategyproof cost sharing. Mimeo University of Hawaii.

Moulin, H., 2009. Almost budget-balanced VCG mechanisms to assign multiple objects. Journal of Economic Theory 144, 96-119.

Moulin, H., Shenker, S., 2001. Strategy-proof sharing of submodular costs: budget balance versus efficiency. Economic Theory 18, 511-533.

Sönmez, T., 1994. Consistency, monotonicity, and the uniform rule. Economics Letters 46, 229-235.

Sprumont, Y., 1991. The division problem with single-peaked preferences: a characterization of the uniform allocation rule. Econometrica 59(2), 509-519.

Thomson, W., 1994a. Resource-monotonic solutions to the problem of fair division when preferences are single-peaked. Social Choice and Welfare 11, 205-223.

Thomson, W., 1994b. Consistent solutions to the problem of fair division when preferences are single-peaked. Journal of Economic Theory 63, 219-245.

Thomson, W., 1995. Population-monotonic solutions to the problem of fair division when preferences are single-peaked. Economic Theory 5, 229-246.

Thomson, W., 1997. The replacement principle in economies with single-peaked preferences. Journal of Economic Theory 76, 145-168.

You, J., 2015. Optimal VCG Mechanisms to Assign Multiple Bads. Games and Economic Behavior 92, 166-190.