

Robust Equilibria in Tournaments with Externalities

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May 2018

Abstract

Agents form coalitions with other agents. The coalition that forms with the largest power wins the tournament. A partition of the agents is a no threat equilibrium (NTE) if whenever a group of agents gains by forming their own coalition, there exist another group of agents that gains by forming their own coalition and harms at least one agent who initially deviated from the partition. We characterize the class of preferences over group of agents that guarantees the existence of a NTE partition. This class of preferences includes more limited versions where the CORE, the traditional equilibria, does not exist.

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1 Introduction

Coalitions are often built around multiple issues and at any scale of society, from neighborhood issues to international conflict. However, two aspects constantly influence the formation of coalitions. On one hand, through coalitions, weaker parties in a conflict can increase their power. Coalition building is the primary mechanism through which weaker parties can develop their power base and thereby better defend their interests. Therefore, there is a need to develop a model of coalition formation that incorporates *power*. On the other hand, coalitions form around a central issue. People tend to associate with others of similar goals and characteristics. Altruism toward race, language, religion or ancestral homeland are usual determinants on how coalitions form. In political games, a winning coalition of leftist parties might impose a negative externality on rightist parties. On the other hand, two parties belonging to the same political spectrum might positively benefit if the other wins the election. Similarly, an agent of religion A might prefer to associate with other agent of the same religion, whereas he might dislike to associate with agents of religion B. Therefore, there is a need to analyze how preferences toward others with similar characteristics affect the behavior of coalitions, that is how parties are being influenced by *externalities* toward other agents.

In order to do so, we introduce a model (henceforth called a *tournament*) in which agents form coalitions with other agents and have externalities. The coalition that forms with the largest power wins the tournament. Agents belonging to a winning coalition receive the benefits whereas losing agents do not get any benefit. Unlike previous work on the literature of coalition formation with externalities, we take a naive approach on power and externalities, since both power and externalities may be difficult to measure. In our model, the power of a coalition is represented by an arbitrary function that is non-decreasing with respect to the inclusion relation. The preferences of an agent are represented by arbitrary preferences over the coalitions in which the agents belong to. Our work focuses on finding conditions on the structure of coalitions that can form that guarantee the existence of an equilibrium that is robust to the information of power and preferences.

We study two equilibrium notions. First, we study the traditional and most widely accepted equilibrium notion, the *CORE*, where no coalition of agents has the incentive to deviate assuming everyone else remain unchanged after the deviation. Another limitation of the core is that it was developed in settings without externalities. In tournament settings where deviations affect others, this leads for affected parties to reorganize in order to be part of the winning coalition. In order to address these issues, we introduce the no-threat equilibrium (NTE). To understand the NTE, notice that the formation of a coalition can shift the balance of power in a conflict situation and alter the future course of the conflict. People who pool their resources and work together are generally more powerful and more able to advance their interests, than those who do not. Coalition members may be able to resist certain threats or even begin to make counter threats. Generally, low-power groups are much more successful in defending their interests against the dominant group if they work together as a coalition. This is certainly more effective than fighting among themselves and/or fighting the dominant group alone. Under a NTE equilibrium, agents avoid threats, that is NTE is an equilibrium where after the winning coalition is formed, there is no group of agents who can

pose a threat.

In order to illustrate our model and results, consider the case of three agents, $\{1, 2, 3\}$, and preferences such that agent 1 prefers to associate with agent 2 instead of agent 3; agent 2 prefers to associate with agent 3 instead of agent 1; and agent 3 prefers to associate with agent 1 instead of agent 2. For any vector of powers such that no agents is a dictator. Indeed, if $\{1, 2\}$ is the winning coalition, then it can deviate to $\{2, 3\}$; if $\{2, 3\}$ is the winning coalition then it can deviate to $\{3, 1\}$; and if $\{3, 1\}$ is the winning coalition then it can deviate to $\{1, 2\}$. In the simple case of preferences with externalities, neither the CORE and NTE equilibrium might exist. However, limitation on the preferences, typically coming from physical constraints might select them. Such limitation might be generated, for instance, by a matching in which agents might only be able to associate with couples (e.g., pairs of a man and a woman), or there might be physical constraints, where only consecutive agents might associate to others.

Our main results provides necessary and sufficient conditions for the existence of a CORE and NTE. Indeed, our main result shows that NTE always exists if and only if agents can be connected in a network without cycles, and the set of feasible coalitions is a subset of such coalitions (Theorem 1). The generality of the results allows for general application, for instance for the selection of a facility under single peaked preferences, or the selection of a winning couple to be included in the domain.

Our results also show that the CORE is highly restrictive and the class of feasible coalitions where the CORE can actually be defined if and only if non-disjoint coalitions are actually included on each other (Proposition 2). Thus, providing severe restrictions on the partitions of the agents.

1.1 Related Literature

There is a large literature studying equilibria in coalition formation with externalities, many of them resembling the CORE. For instance, Bogomolnaia and Jackson [5] study different stability notions and characterizes necessary and sufficient conditions for their existence. This has been extended to a variety of settings, for instance, Papai [16], Ehlers [9], Bloch and Dutta [4], Chatterjee et al. [6], Pycia [18], Romero-Medina [21], Banerjee et al. [2]. Several of these include equilibria that work in dynamic settings as well, for instance, Greenberg [10], Chwe [7], Bloch [3], Xue [22], Arnold and Schwalbe [1], Diamantoudi and Xue [8], Ray and Vohra [20], Iñarra et al. [11], Ray and Vohra [19]. Unfortunately, such literature does not study the role that power plays in the formation of coalitions.

A recent literature does focus on the issue of power. For instance, Piccione and Razin [17] study how the relationships of power among agents actually determine the overall ranking of the society. They characterize coalitions under which their identity (the power of agents within those coalition) determines the social order and thus the structure of society. Similarly, Jordan [15] characterizes the core and the stable set in a class of coalitional games called “pillage games”. In this model, wealth is allocated among the finite agents in the game. A reallocation of wealth among the agents is only made possible by using force. A power function, which is monotonically increasing in membership

and the members' wealth, regulates the ability of the agents to use force. The coalition is then able to appropriate the wealth of other less powerful coalitions. Although newer coalition formation models are dynamic in nature, most of them are limited to a static distribution of power among the agents. We fill this gap by exploring the possibility that agents are able to deviate from a currently formed coalition. Jandoc and Juarez[12] studies a model where agents are endowed with power and characterizes the formation of coalitions when power accumulates. Such formation is highly dependent on the power accumulation rule. Jandoc and Juarez[14] study dynamic coalition formation when agents disagree on the sharing rule. It characterizes the structure of power the guarantees the existence of stable coalitions, even if they disagree on how to share the resource. Jandoc and Juarez[13] tests the self-enforcing coalition formation equilibrium and shows that over time it becomes a good predictor on the way coalitions form

Despite the abundance of coalition formation models, little work has been done studying the robustness of equilibria, especially when the power of the agents or their preferences may change. Our work is the first to introduce and characterize the structure on the coalitions that can form that guarantees the existence of equilibria regardless of whether power and preferences of the agents change.

2 The model

We study coalition formation models for a fixed group of agents $N = \{1, \dots, n\}$. Agents are endowed with power, which is an arbitrary functions among coalitions.

Definition 1. *A power function as a mapping $\pi : 2^N \rightarrow \mathbb{R}_+$, such that:*

- $\pi(\emptyset) = 0$.
- If $S \subset T$ then $\pi(S) < \pi(T)$.
- If $S \cap T = \emptyset$ then $\pi(S) \neq \pi(T)$ when $\pi(S) \neq 0$ or $\pi(T) \neq 0$.

This function represents a ranking of power over the set of coalitions. $\pi(S) > \pi(T)$ means that coalition S is more powerful than coalition T . Moreover, two disjoint coalitions can not have the same positive power. This break ties in competitive situations between different groups of players.

Some coalitions are feasible to form while others are not. This might represent, for instance, a physical constraint.

Definition 2. *The set of feasible coalitions \mathbb{F} is a collection of subsets $\mathbb{F} \subseteq 2^N$ such that if $S, T \in \mathbb{F}$ then there is a partition of $S \setminus T$ composed of sets in \mathbb{F} .*

We interpret \mathbb{F} as the set of coalition that can be formed. If coalition S is formed, but players in $S \cap T$ decide to form coalition T , then the remaining players in $S \setminus T$ can be rearranged into feasible coalitions.

Example 1. • If $\mathbb{F} = 2^N$ then every group of agents is a feasible coalition.

- If $\mathbb{F} = \{\{1\}, \{2\}, \dots, \{n-1\}, \{n\}\}$ then no group of two or more agents can be formed.

- \mathbb{F} be the set of coalitions containing one or two agents.
- Fix a group of agents T , if $\mathbb{F} \subset \{S \mid T \subseteq S\}$ then any coalition need the group T to be formed.

Network analysis would be helpful to find the type of coalitions that are feasible. Given a network, we will show how to generate a set of feasible coalitions. A central idea is our notion of “connectedness”.

Definition 3. Consider a network H of N agents. We say a coalition S is **connected** if the subnetwork restricted to the agents in S has a single component.

Denote by $C(H)$ the set of all connected coalitions. As we will see below, the set \mathbb{F} of feasible coalitions would be related to the set $C(H)$ for some networks.

We can interpret H as the network of friendships. That is, two agents are friends if there exists a link in H that connects them. Alternatively, H might be interpreted as a location of roads or political affiliation of agents.

Example 2. • If M is a line, then \mathbb{F} contains all the segments that are subsets of M . We can interpret M as the set of agents an affiliation from leftist to rightist. Feasible coalitions are those that contain only agents of consecutive characteristics.

2.1 Tournament

A player i has strict preference \succ_i over feasible coalitions that contain himself and the empty set. That is, the domain of preference of agent i is $\mathbb{S}_i = \{S \in \mathbb{F} \mid i \in S\} \cup \{\emptyset\}$ such that $S \succ_i \emptyset$ for all $S \in \mathbb{S}_i$. A tournament in this setting is a game such that agents form coalitions and the coalition with the largest power wins the tournament.

Given N players, let the set of partitions of N be denoted by Π .

Definition 4. A tournament is a function $G : \Pi \rightarrow 2^N$ such that for all $P \in \Pi$:

- $G(P) \in P$.
- $\pi(G(P)) \geq \pi(S), \forall S \in P$.

The outcome of a tournament for agent i at partition P is:

$$G_i(P) = \begin{cases} G(P) & \text{if } i \in G(P) \\ \emptyset & \text{if } i \notin G(P) \end{cases}$$

That is, the function G gives the winning coalition for every partition P . This coalition is the coalition with the largest power from P .



Figure 1: Network of three agents in a line

2.1.1 No threat equilibrium

Definition 5. *The set of winning coalitions is:*

$$W = \{S \in \mathbb{F} \mid \pi(S) > \pi(T), \forall T \subseteq N \setminus S, T \in \mathbb{F}\}.$$

Definition 6. *Let $P \in \Pi$ be a partition and $T \subseteq \mathbb{F}$ be a feasible coalition. The set*

$$[P \setminus T] = \{\bar{P} \in \Pi(N \setminus T) \mid \text{if } S \in P \text{ and } S \setminus T \in \mathbb{F} \text{ then } S \setminus T \in \bar{P}\}$$

represents a class of maximal partitions when P and T agree.

Definition 7. *We say that a coalition $T \in \mathbb{F}$ defeats a partition $P \in \Pi$ if there is $\bar{P} \in [P \setminus T]$ such that*

$$G_i(\bar{P}, T) \succ_i G_i(P), \quad \forall i \in T,$$

where (\bar{P}, T) denotes the partition $\bar{P} \cup \{T\}$ of N .

In the previous definition, note that the coalition T must be a winning coalition in (\bar{P}, T) is necessary to defeat the partition P . Thus, it will cause no confusion if we say that the winning coalition in a partition defeats the winning coalition in another partition.

Equilibrium winning coalitions need not necessarily be stable to deviations of a group of players. Thus, we employ the stability notion called the No-Threat Equilibrium (NTE).

Definition 8. *Let G be a tournament. A partition P^* is an NTE if whenever there is a coalition $T \in \mathbb{F}$ and $\bar{P} \in [P^* \setminus T]$ such that (\bar{P}, T) defeats P^* , then there is a coalition $V \subseteq N \setminus T$ and $\tilde{P} \in [(\bar{P}, T) \setminus V]$ such that (\tilde{P}, V) defeats (\bar{P}, T) .*

We now show examples of networks that give rise to a set of feasible coalitions where NTE might or might not exist.

Example 3. *Consider the network in Figure 1 which gives the set of connected coalitions*

$$C(H) = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Suppose that the feasible set is equal to $C(H)$ and let the power function such that

$$\pi(\{2\}) < \pi(\{1\}) < \pi(\{3\}) < \pi(\{1, 2\}) < \pi(\{2, 3\}) < \pi(\{1, 2, 3\})$$

Assume that preferences are as follows:

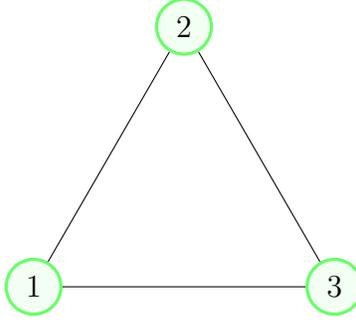


Figure 2: Network with three agents in a cycle.

$$\begin{aligned} \{1\} \succ_1 \{1, 2\} \succ_1 \{1, 2, 3\} \\ \{2\} \succ_2 \{1, 2\} \succ_2 \{2, 3\} \succ_2 \{1, 2, 3\} \\ \{3\} \succ_3 \{2, 3\} \succ_3 \{1, 2, 3\} \end{aligned}$$

In this case, $\{1, 2\}$ is the NTE. If player 1 deviates to form, say, a singleton coalition $\{1\}$, then another feasible coalition, in this case $\{2, 3\}$ can be formed and win over $\{1\}$.

Example 4. Now consider the network in Figure 2 that generates the set of connected coalitions:

$$C(H) = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}.$$

Suppose that $\mathbb{F} = C(H)$ and let π the power function such that

$$\pi(\{2\}) < \pi(\{1\}) < \pi(\{3\}) < \pi(\{1, 2\}) < \pi(\{2, 3\}) < \pi(\{1, 3\}) < \pi(\{1, 2, 3\})$$

$$\begin{aligned} \{1, 2\} \succ_1 \{1, 3\} \succ_1 \{1, 2, 3\} \succ_1 \{1\} \\ \{2, 3\} \succ_2 \{1, 2\} \succ_2 \{1, 2, 3\} \succ_2 \{2\} \\ \{1, 3\} \succ_3 \{2, 3\} \succ_3 \{1, 2, 3\} \succ_3 \{3\} \end{aligned}$$

In this case the NTE does not exist, since this network generates a cycle. Thus player 2 would deviate from $\{1, 2\}$ and $\{2, 3\}$ can be formed, coalition $\{2, 3\}$ defeats $\{1, 2\}$ but $\{1\}$ can not defeat $\{1, 2\}$. However, player 3 would like to deviate and $\{1, 3\}$ can be formed, coalition $\{1, 3\}$ defeats $\{2, 3\}$ but $\{2\}$ can not defeat $\{1, 3\}$. But player 1 would like to form $\{1, 2\}$, coalition $\{1, 2\}$ defeats $\{1, 3\}$ but $\{3\}$ can not defeat $\{1, 2\}$, and so on.

3 Main Theorem

The main theorem in this paper states the set of restrictions on \mathbb{F} that guarantee the existence of NTE for all preferences and all power functions. Moreover, if \mathbb{F} are rich enough, the set of feasible coalitions will be coming from a set of connected coalition in a network without cycles.

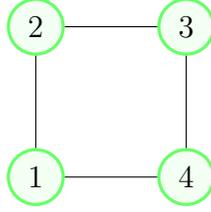


Figure 3: Network with four agents in a square.

By definition of set of winning coalitions, we can show that $S, T \in W$ then $S \cap T \neq \emptyset$ ¹. Thus, the following definition establishes a necessary condition for a set has the opportunity to be winning condition.

Definition 9. We say that a set $\{S_1, S_2, \dots, S_k\} \subseteq \mathbb{F}$ is composed of **potentially winning coalitions** if $S_i \cap S_j \neq \emptyset$ for all i, j .

The following stronger condition will guarantee us the existence of at least one NTE.

Definition 10. A collection of feasible coalitions \mathbb{F} is **pairwise stable** if every family $\{S_1, S_2, \dots, S_k\} \subset \mathbb{F}$ composed of potentially winning coalitions satisfies that $\bigcap_i S_i \neq \emptyset$

Theorem 1. Given a set of feasible coalition \mathbb{F} , a NTE exists for all preferences defined on \mathbb{F} and all power functions if and only if \mathbb{F} is pairwise stable.

Proposition 1. Given a set \mathbb{F} that is composed of potentially winning coalitions, a NTE exists for all preferences defined on \mathbb{F} and all power functions if and only if there exists a network without cycles H such that $\mathbb{F} \subset C(H)$.

In following example we show a case in which \mathbb{F} is not composed of *potentially winning coalitions*, but there is a network H with a cycle such that $\mathbb{F} \subseteq C(H)$ and a NTE exists for all preferences defined on \mathbb{F} and all power functions.

Example 5. Let $\mathbb{F} = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\}$ be the set of feasible coalitions. Now consider the network in Figure 3.

Each side of the square is a feasible coalition. But \mathbb{F} is not composed of potentially winning coalitions. In fact, if a side belongs to a family composed of potentially winning coalitions then the opposite side does not belongs to that family.

For example, the side $\{1, 2\}$ only belongs to two families composed of potentially winning coalitions:

$$F_1 = \{\{1, 2\}, \{2, 3\}\} \quad \text{and} \quad F_2 = \{\{1, 2\}, \{4, 1\}\}$$

Both families F_1 and F_2 are pairwise stable.

¹Proof of Theorem 1 in Step 1 of only if part

This is analogous for the families composed of potentially winning coalitions in which the other sides belong. Thus, every family that is composed of potentially winning coalitions is pairwise stable. Therefore, in this case a NTE exists for all preferences defined on \mathbb{F} and all power functions.

3.1 The CORE

Definition 11. A partition $P \in \Pi$ is **stable** if no coalition defeats it. The **core** is the set of all stable partitions.

Note that if a partition P is stable then P is a NTE, but the converse does not hold. Moreover, the following example shows a situation in which there is a NTE equilibrium but the core is empty.

Example 6. Power matchings. Let $N = \{A, B, 1, 2\}$ the set of players and

$$\mathbb{F} = \{\{A\}, \{B\}, \{1\}, \{2\}, \{A, 1\}, \{A, 2\}, \{B, 1\}, \{B, 2\}\}$$

be the set of feasible coalitions. The power of every singleton is null and the ranking of power for couples is

$$\{A, 2\} > \{A, 1\} > \{B, 1\} > \{B, 2\}$$

The preferences are defined as follow:

$$\{A, 1\} \succ_A \{A, 2\} \succ_A \{A\} \{B, 1\} \succ_B \{B, 2\} \succ_B \{B\} \{B, 1\} \succ_1 \{A, 1\} \succ_1 \{1\} \{B, 2\} \succ_2 \{A, 2\} \succ_2 \{2\}$$

Thus, each agent prefers to be in any couple than alone.

This tournament has an empty core. Indeed, it is not difficult to see that no matter what partition be formed, there is always a coalition that can defeat it.

For instance, if the partition $\{A1, B2\}$ be formed, then coalition $A1$ wins the tournament. Nevertheless, this partition is not stable, since agents B and 1 can defeat by deviating: $\{A, B1, 2\}$.

On the other hand, $\{A1, B2\}$ is a NTE equilibrium, since if agents B and 1 decide to form the partition $\{A, B1, 2\}$, then coalition $A2$ can defeat $B1$ in the partition $\{A2, B1\}$.

Proposition 2. Given a set of feasible coalition \mathbb{F} , the core is not empty for all preferences defined on \mathbb{F} and all power functions if and only if for all pair of coalitions $A, B \in \mathbb{F}$ such that $A \cap B \neq \emptyset$ then $A \subset B$ or $B \subset A$.

Proofs

3.2 Proof of Theorem 1

Proof. (\Leftarrow) **Step 1:** Any collection of winning coalitions is pairwise stable.

First, observe if S and T are winning coalition then $S \cap T \neq \emptyset$. Because, if $S \cap T = \emptyset$, then $T \subseteq N \setminus S$. But, by definition if $S \in W$ and $T \subseteq N \setminus S$ then $T \notin W$.

Thus, if $S_1, S_2, \dots, S_K \in W$ be winning coalitions, then $S_i \cap S_j \neq \emptyset$ for all $i, j \in \{1, 2, \dots, K\}$. Therefore, by hypotheses we have $\bigcap_{k=1}^K S_k \neq \emptyset$.

Step 2: Let $V = \bigcap_{k=1}^K S_k$ and let $i \in V$. Let $S^* \in W$ such that $S^* \succ_i S$, $\forall S \in W \setminus \{S^*\}$. Let there be a partition $P = (S^*, [N \setminus S^*])$ where $[N \setminus S^*]$ is a partition of $N \setminus S^*$ composed of feasible coalitions. Assume that T deviates at partition P .

Case 1: $i \in T$. If $T \in W$ then i gets a lower payoff. On the other hand, if $T \notin W$ then there exists a coalition in $N \setminus T$ that can be formed and defeats the coalition T .

Case 2: $i \notin T$. Then from our assumption, $T \notin W$ since $i \in V$ is part of all winning coalitions. Therefore there exists a coalition in $N \setminus T$ that can be formed and defeats the coalition T .

Therefore, S^* is a NTE. Note that every winning coalition in the top of preferences (restricted to W) of any player is a NTE.

(\Rightarrow)

We will prove the contrapositive version: Given a set of feasible coalition \mathbb{F} , if there is a collection $C = \{S_1, S_2, \dots, S_K\} \subseteq \mathbb{F}$ such that every $S_i \cap S_j \neq \emptyset$ but $\bigcap_{k=1}^K S_k = \emptyset$, then there are preferences defined on \mathbb{F} and a power function such that NTE does not exist.

Consider the power function as follow:

$$\pi_i(S) = \begin{cases} 1 & \text{if } S \in C \\ 0 & \text{if } S \notin C \end{cases}$$

Clearly, S_1, S_2, \dots, S_K are the only winning coalitions for this power function.

Now, we will proof that there are preferences defined on \mathbb{F} such that NTE does not exist.

Let $i \in N$ be any agent. Due to $\bigcap_{k=1}^K S_k = \emptyset$, we have that i belongs in $r < K$ winning coalitions. Assume that i belongs to the coalitions $S_{i_1}, S_{i_2}, \dots, S_{i_r}$ such that $i_1 < i_2 < \dots < i_r$.

Consider the following strict preference over \mathbb{F} , for each agent $i \in N$:

- If $i \notin S_1 \cap S_K$, then

$$T \prec_i S_{i_1} \prec_i S_{i_2} \prec_i \dots \prec_i S_{i_r},$$

for all $T \neq S_{i_k}$ in \mathbb{S}_i , $k = 1, \dots, r$.

- If $i \in S_1 \cap S_K$, then

$$S_{i_r} \prec_i S_{i_1} \prec_i S_{i_2} \prec_i \dots \prec_i S_{i_{r-1}},$$

for all $T \neq S_{i_k}$ in \mathbb{S}_i , $k = 1, \dots, r$.

Observe that these preferences satisfies the following conditions:

$$\begin{aligned} S_1 \prec_j S_2, \quad \forall j \in S_1 \cap S_2 \\ S_2 \prec_j S_3, \quad \forall j \in S_2 \cap S_3 \\ \vdots \\ S_K \prec_j S_1, \quad \forall j \in S_K \cap S_1. \end{aligned}$$

These properties in the preferences cause that the NTE does not exist, since this situation generates a cycle. Thus, for $r = 1, \dots, K - 1$, the group $S_r \cap S_{r+1}$ would deviate from S_r and S_{r+1} can be formed, then coalition S_{r+1} defeats coalition S_r but there is not winning coalitions in $N \setminus S_{r+1}$. Finally, group $S_K \cap S_1$ would like to deviate and S_1 can be formed, then coalition S_1 defeats coalition S_K and there is not winning coalitions in $N \setminus S_1$. But group $S_1 \cap S_2$ would like to form S_2 and this coalition defeats coalition S_1 again, and so on.

□

3.3 Proof of Proposition 1

Proof. (\Leftarrow) Assume that $\mathbb{F} = \{S_1, S_2, \dots, S_K\}$ is composed of potentially winning coalitions and there is a network H such that $\mathbb{F} \subseteq C(H)$.

First, we show that $\bigcap_{k=1}^K S_k \neq \emptyset$

We prove by induction that $\bigcap_{k=1}^K S_k \neq \emptyset$. The base of induction is $n = 3$. Clearly, we can arrange the players in a line H (see Figure 1) because this is the only possible network for three players if there are no cycles. There are essentially two cases in which \mathbb{F} is composed of potentially winning coalitions and $\mathbb{F} \subseteq C(H)$ ²:

- $\mathbb{F} = \{\{i\}, \{i, 2\}\}$ for $i = 1$ or 3 . Thus, player i belongs to every potentially winning coalition.
- $\mathbb{F} = \{\{1, 2\}, \{2, 3\}\}$. Thus, player 2 belongs to every potentially winning coalition. This case is analogous for $\mathbb{F} = \{\{2\}, \{1, 2\}, \{2, 3\}\}$.

Let our induction hypothesis be that this is true for any network for n players. We will then prove this for $n + 1$ players. Let $\{S_1, S_2, \dots, S_K\}$ be elements in the set of connected coalitions of $n + 1$ network structure such that $S_i \cap S_j \neq \emptyset$.

- If S_1, S_2, \dots, S_K only contains n players then we are done by the induction hypothesis.
- Assume that all $n + 1$ players are in S_1, S_2, \dots, S_K . Take an agent a with degree one in the network and take an agent b that is connected to a . If $a \in \bigcap_{k=1}^K S_k$ we are done.

²The proof is analogous when $\{1, 2, 3\} \in \mathbb{F}$.

- If $a \notin \bigcap_{k=1}^K S_k$, we know that $a \notin S_k$ for some k . We show that if $a \in S_i$ it must be that $b \in S_i$. To see this, notice that if $a \notin S_k$ then for some player $x \neq a$ it must be that $x \in S_i \cap S_k$. Take the path $[x, a]$. Since $a \in S_i$ and $x \in S_i$, by connectedness it must be that players on the path $[x, a]$ should be a subset of S_i , that is, $[x, a] \subseteq S_i$. Since $b \in [x, a]$ then $b \in S_i$.
- Now take the sets $S_1 \setminus \{a\}, S_2 \setminus \{a\}, \dots, S_K \setminus \{a\}$. We know $(S_i \setminus \{a\}) \cap (S_j \setminus \{a\}) \neq \emptyset$ because $S_i \cap S_j \neq \emptyset$ and if $a \in S_i$ then $b \in S_i$. Since those sets only contain n players and they are subsets of network H without the link $[a, b]$ (which is a network without cycles), then by the induction hypothesis $\bigcap (S_i \setminus \{a\}) \neq \emptyset$ implies $\bigcap (S_i) \neq \emptyset$.

Therefore by Theorem 1, we conclude that a NTE equilibrium exists for all preferences defined on \mathbb{F} and all power functions.

(\Rightarrow) Suppose that $\mathbb{F} = \{S_1, S_2, \dots, S_K\}$ is composed of potentially winning coalitions and a NTE exists for all preferences defined on \mathbb{F} and all power functions.

We prove by induction that there is network H without cycles such that $\mathbb{F} \subseteq C(H)$. The base of induction is $n = 3$. As we show in the only if part, three players in all potential families of potentially winning coalitions can be arranged in one line.

The induction hypothesis be that this is true for any network for n players. We will then prove this for $n + 1$ players. Let $\mathbb{F} = \{S_1, S_2, \dots, S_K\}$ be a family composed of potentially winning coalitions with $n + 1$ players. By Theorem 1, we have $\bigcap_i S_i \neq \emptyset$.

If trivially for all $k = 1, \dots, K$, the coalition $S_k = \bigcap_i S_i$ then the network of $n + 1$ players in a line works.

If there is coalition $S_r \neq \bigcap_i S_i$ then there is a player $a \notin \bigcap_{k=1}^K S_k$. We take the coalitions $S_1 \setminus \{a\}, S_2 \setminus \{a\}, \dots, S_K \setminus \{a\}$ which $\bigcap (S_i \setminus \{a\}) \neq \emptyset$. Then by induction hypothesis there is a network G without cycles on $N \setminus \{a\}$ such that $\{S_1 \setminus \{a\}, S_2 \setminus \{a\}, \dots, S_K \setminus \{a\}\} \subseteq C(G)$.

For some $b \in \bigcap (S_i \setminus \{a\})$, we define the network $H = G \cup [b, a]$ by adding the node a and link $[b, a]$ to network G . Clearly, H has not cycles. Moreover, due to $b \in \bigcap (S_i \setminus \{a\})$ and $[b, a] \in H$, then every $S_k \in \mathbb{F}$ belongs $C(H)$ for $k = 1, \dots, K$.

□

3.4 Proof of Proposition 2

Proof. (\Leftarrow) Let $W = \{S_1, \dots, S_m\}$ the set of winning coalitions. By definition we have that $S_i \cap S_j \neq \emptyset$ for all $i, j = 1, \dots, m$. Thus, by hypothesis $S_i \subset S_j$ or $S_j \subset S_i$. Without loss of generality suppose that $S_1 \subset S_2 \subset \dots \subset S_m$.

Let $i \in S_1$ and let $S^* \in W$ such that $S^* \succ_i S, \forall S \in W \setminus \{S^*\}$. Note that i belongs to every

winning coalition.

There is a partition $P = (S^*, [N \setminus S^*])$ where $[N \setminus S^*]$ is a partition of $N \setminus S^*$ composed of feasible coalitions. Suppose that T deviates at partition P .

Case 1: $i \in T$. If $T \in W$ then i gets a lower payoff. On the other hand, if $T \notin W$, then $T \subset S^*$ and there exists a coalition in $R \subset N \setminus T$ that can be formed and defeats the coalition T . If $R \in [S^* \setminus T]$ then T does not deviate because T loses the tournament. If $R \in N \setminus S^*$ then we take the partition $P = (S^*, [N \setminus S^*])$ such that $R \in [N \setminus S^*]$ and then T does not deviate.

Case 2: $i \notin T$. Then, $T \notin W$ since i is part of all winning coalitions. Therefore there exists a winning coalition in $N \setminus T$ that defeats the coalition T , thus T can not deviate.

Thus, S^* is in the core. Therefore, the core is not an empty set.

(\Rightarrow)

We will prove the contrapositive version: Given a set of feasible coalition \mathbb{F} , if there is two coalitions $A, B \in \mathbb{F}$ such that $A \cap B \neq \emptyset$ but $A \not\subseteq B$ and $B \not\subseteq A$, then there are preferences defined on \mathbb{F} and a power function such that the core is an empty set.

Note that if $A \cap B \neq \emptyset$ but $A \not\subseteq B$ and $B \not\subseteq A$, then $A \setminus B \neq \emptyset$ and $B \setminus A \neq \emptyset$. Moreover, by definition there is a partition $P_{B \setminus A} \subset \mathbb{F}$ of $B \setminus A$.

Let $K \in P_{B \setminus A}$ be a feasible coalition and note that $K \subset B$. Consider a power function π , such that:

$$\pi(A) > \pi(B) > \pi(K) > 0,$$

$\pi(S) = 0$ for all S contained in A and B different to K and $\pi(R) = 0$ for all $R \in N \setminus \{A, B\}$.

Now, suppose that we have a set of preferences such that:

- For all $i \in A$: $A \succ_i S$ for all S such that $A \subset S$.
- For all $i \in B$: $B \succ_i S$ for all S such that $B \subset S$.
- For all $i \in A \cap B$: $B \succ_i A \succ_i T$, for all $T \in \mathbb{F} \setminus \{A, B, K\}$.
- For all $i \in K$: $K \succ_i B \succ_i T$, for all $T \in \mathbb{F} \setminus \{A, B, K\}$.

It is straightforward to check that $W = \{S \mid A \subseteq S \text{ or } B \subseteq S\}$.

We show that the core is empty in this case.

Let S be a winning coalition:

Case 1: $A \subset S$ and $B \not\subseteq S$. Any partition $P = (S, [N \setminus S])$ is defeated by the coalition A , because $A \succ_i S$ for all $i \in A$. Thus, A wins the tournament in partition $(A, [S \setminus A], [N \setminus S])$.

Case 2: $B \subset S$ and $A \not\subseteq S$. This case is analogous to the previous one.

Case 3: $A \cup B \subseteq S$. Any partition $P = (S, [N \setminus S])$ is defeated by the coalition B , because $B \succ_i S$ for all $i \in B$ and $B \succ_j A$ for all $j \in A \cap B$. Thus, B wins the tournament in partition $(B, [S \setminus B], [N \setminus S])$.

Case 4: $S = A$. Any partition $P = (A, [N \setminus A])$ is defeated by the coalition B , because $B \succ_i A$ for all $i \in A \cap B$. Thus B wins the tournament in partition $(B, [A \setminus B], [N \setminus (A \cup B)])$.

Case 5: $S = B$. Any partition $P = (B, [N \setminus B])$ is defeated by the coalition K , because $K \succ_i B$ for all $i \in K \subset B$. Thus, K wins the tournament in partition $(K, [B \setminus K], [N \setminus B])$. Therefore P is not stable. Nevertheless, $K \notin W$ due to $A \subset N \setminus K$ and $\pi(A) > \pi(K)$.

This shows that there are not stable partitions. Therefore, the core is empty. \square

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