The no-threat equilibrium in tournament games with externalities
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Abstract

Agents form coalitions with other agents. The coalition that forms with the largest power wins the tournament.

A partition of the agents is a no threat equilibrium (NTE) if whenever a group of agents gains by forming their own coalition, there exist another group of agents that gains by forming their own coalition and harms at least one agent who initially deviated from the partition.

We characterize the class of preferences over group of agents that guarantees the existence of a NTE partition. This class includes group of connected agents in networks without cycles, among others.

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1 Introduction

Coalitions are often built around multiple issues and at any scale of society, from neighborhood issues to international conflict. However, two aspects constantly influence the formation of coalitions. On one hand, through coalitions, weaker parties in a conflict can increase their power. Coalition building is the primary mechanism through which weaker parties can develop their power base and thereby better defend their interests. Therefore, there is a need to develop a model of coalition formation that incorporates *power*.

On the other hand, coalitions form around a central issue. People tend to associate with others of similar goals and characteristics. Altruism toward race, language, religion or ancestral homeland are usual determinants on how coalitions form. In political games, a winning coalition of leftist parties might impose a negative externality on rightist parties. On the other hand, two parties belonging to the same political spectrum might positively benefit if the other wins the election. Similarly, a Muslim agent might prefer to associate with other Muslims whereas he might dislike to associate with Jews. Therefore, there is a need to analyze how preferences toward others with similar characteristics affect the behavior of coalitions, that is how parties are being influenced by *externalities* toward other agents.

In the first paper of this project, Juarez[2011] proposes three equilibrium notions on how coalitions form in a static setting. The traditional and most widely accepted notion is the CORE, but unfortunately it is often empty. The MPSP equilibrium is also a great predictor, especially when there is cost of forming and searching for other coalitions. However, a more comprehensive predictor is the no-threat equilibrium (NTE).

To understand the NTE, notice that the formation of a coalition can shift the balance of power in a conflict situation and alter the future course of the conflict. People who pool their resources and work together are generally more powerful and more able to advance their interests, than those who do not. Coalition members may be able to resist certain threats or even begin to make counter threats. Generally, low-power groups are much more successful in defending their interests against the dominant group if they work together as a coalition. This is certainly more effective than fighting among themselves and/or fighting the dominant group alone. Under a NTE equilibrium, agents avoid threats, that is NTE is an equilibrium where after the winning coalition is formed, there is no group of agents who can pose a threat.

In the simple case of preferences with externalities, the NTE equilibrium might not exists. For instance when there are three agents, $\{1, 2, 3\}$, and preferences are such that agent 1 prefers to associate with agent 2 instead of agent 3; agent 2 prefers to associate with agent 3 instead of agent 1; and agent 3 prefers to associate with agent 1 instead of agent 2, we can see that for any vector of powers such that no agents is a dictator the NTE does not exists. Indeed, if $\{1, 2\}$ is the winning coalition, then it can deviate to $\{2, 3\}$; if $\{2, 3\}$ is the winning coalition then it can deviate to $\{3, 1\}$; and if if $\{3, 1\}$ is the winning coalition then it can deviate to $\{1, 2\}$.

Overview of the result

The goal of the paper is to find the restrictions of preferences of the agents that always guarantee the existence of a NTE. Theorem 1 provides the necessary and sufficient condition for this existence. In particular, such restriction on the set of feasible coalition are such that for any set of feasible coalition, non-empty pairwise intersection implies a common intersection. Moreover, under minimally rich conditions on the set of feasible coalitions, this set is is a subset of the group of connected coalitions in a network without cycles.

2 The model

The model is composed of three charateristics: power, preferences, and coalition formation equilibrium. We discuss them below.

2.1 Power

Definition 1 *A power function as a mapping* $\pi : 2^N \to \mathbb{R}_+$ *such that if* $S \subseteq T$ *then* $\pi(T) \geq \pi(S)$.

Example 1 (Linear power) *Suppose every agent* i *has a linear power* π_i*, and the power of a coalition* S *is* $\pi(S) = \sum_{i \in S} \pi_i$.

Example 2 (Cobb-Douglass) *Suppose that every agent* i *has an endowment* ω^i *and the power of coalition* S *is given by* $\pi(S) = \prod_k (\sum_{j \in S} \omega_k^j)^{\alpha_j}$.

2.2 Feasible sets

We define a feasible set as the set of coalitions that can form.

Definition 2 *The set of feasible coalitions* $\mathbb{F}$ *is a collection of subsets* $\mathbb{F} \subseteq 2^N$ *such that* $\{i\} \in \mathbb{F}$ *for all* $i \in N$.

We interpret $\mathbb{F}$ as the set of coalition that can form. We assume that all coalitions containing a single agent can always form, therefore $\mathbb{F}$ contains all the singletons.

Example 3

- *If* $\mathbb{F} = 2^N$ *then group of agents can form.*
- *If* $\mathbb{F} = \{\{1\}, \{2\}, \ldots, \{n-1\}, \{n\}\}$ *then no group of two or more agents can form.*
- *If* $\mathbb{F}$ *be the sets of coalition that contains one or two elements.*

Network analysis would be helpful to find the type of coalitions that are feasible. Given a network, we will show how to generate a set of feasible coalitions. A central idea is our notion of "connectedness".

Definition 3 *Consider a network* H *of* N *agents. We say a coalition* S *is* **connected** *if the subnetwork restricted to the agents in* S *has a single component.*

Denote by $C(H)$ the set of all connected coalitions. As we will see below, the set $\mathbb{F}$ of feasible coalitions would be related to the set $C(H)$ for some networks.

We can interpret H as the network of friendships. That is, two agents are friends if there exists a link in H that connects them. Alternatively, H might be interpreted as a location of roads or political affiliation of agents.

Example 4

- *If M is a line, then $\mathbb{F}$ contains all the segments that are subsets of M. We can interpret M as the set of agents an affiliation from leftist to rightist. Feasible coalitions are those that contain only agents of consecutive characteristics.*

2.3 Tournament

A player i has strict preference $\succ_i$ over feasible coalitions that contain himself. That is, the domain of preference of agent i is $\mathbb{S}_i = \{S | i \in S, S \in \mathbb{F}\}$.

A tournament in this setting is a game such that agents form coalitions and the coalition with the largest power wins the tournament.

Given N players, let the set of partitions of N be denoted by κ.

Definition 4 • *A tournament is a function $G : \kappa \to 2^N$ such that for all $\Pi \in \kappa$:*

a) $G(\Pi) \in \Pi$
b) $\pi(G(\Pi)) \geq \pi(S)$, $\forall S \in \Pi$

• *The outcome of a tournament for agent i at partition Π is:*

$$G_i(\Pi) = \begin{cases} G(\Pi) \text{ if } i \in G(\Pi) \\ \emptyset \text{ if } i \notin G(\Pi) \end{cases}$$

That is, the function G gives the winning coalition for every partition Π. This coalition is the coalition with the largest power from Π.

2.3.1 No threat equilibrium

Definition 5 *The set of winning coalitions is:*

$$W = \{S | \pi(S) > \pi(T), \forall T \subseteq N \setminus S, T \in \mathbb{F}\}$$

Equilibrium winning coalitions need not necessarily be stable to deviations of a group of players. Thus, we employ the stability notion called the No-Threat Equilibrium (NTE)

Definition 6 *A partition Π is an NTE if whenever there is a T such that $G_i(\Pi \setminus T, T) \succ_i G_i(\Pi), \forall i \in T$, there exists a $V \subseteq N \setminus T$ such that $G_i(\Pi \setminus (T \cup V), T, V) \succ_i G_i(\Pi \setminus T, T), \forall i \in V$ and $G_i(\Pi) \succ_i G_i(\Pi \setminus (T \cup V), T, V)$ for some $i \in T$.*

Example 5 *We now show examples of networks that give rise to a set of feasible coalitions where NTE might or might not exist. Consider the network in Figure which gives the set of connected coalitions*

$C(H) = \{\{1\},\{2\},\{3\},\{1,2\},\{2,3\},\{1,2,3\}\}$. Suppose that the feasible set is equal to $C(H)$ and let the power set be: $\pi = (.34, .31, .35)$.

Assume that preferences are as follows:

$\{1\} \succ_1 \{1,2\} \succ_1 \{1,2,3\}$
$\{2\} \succ_2 \{1,2\} \succ_2 \{2,3\} \succ_2 \{1,2,3\}$
$\{3\} \succ_3 \{2,3\} \succ_3 \{1,2,3\}$

In this case, {1,2} is the NTE. If player 1 deviates to form, say, a singleton coalition $\{1\}$, then another feasible coalition, in this case $\{2,3\}$ can form and win over $\{1\}$.

Now consider the network in Figure that generates the feasible set:

$\mathbb{F} = \{\{1\},\{2\},\{3\},\{1,2\},\{2,3\},\{1,3\},\{1,2,3\}\}$

and let the power set be $\pi = (.34, .31, .35)$

Let preferences be as follows:

$\{1,2\} \succ_1 \{1,3\}$
$\{2,3\} \succ_2 \{1,2\}$
$\{1,3\} \succ_3 \{2,3\}$

In this case the NTE may not exist, since this network generates a cycle. Thus player 2 would deviate from $\{1,2\}$ and form $\{2,3\}$. However, player 3 would like to deviate and form $\{1,3\}$ but player 1 would like to form $\{1,2\}$, and so on.

3 Main Theorem

The main theorem in this paper states the set of restrictions on the preferences that guarantee the existence of NTE. Moreover, if preferences are rich enough, the set of restrictions will be coming from a set of connected coalition in a network without cycles.

Definition 7 *A collection of feasible sets $\mathbb{F}$ is* ***pairwise stable*** *if for any sets $\{S_1, S_2, \ldots, S_k\} \subset \mathbb{F}$ such that $S_i \cap S_j \neq \emptyset$ for all i, j, implies that $\bigcap_i S_i \neq \emptyset$*

Theorem 1

- *Given a set of feasible coalition $\mathbb{F}$, the NTE equilibrium exists for all preferences defined on $\mathbb{F}$ and all power functions if and only if $\mathbb{F}$ is pairwaise stable.*
- *Given a set of feasible coalition $\mathbb{F}$ that is minimally rich, the NTE equilibrium exists for all preferences defined on $\mathbb{F}$ and all power functions if and only if there exists a network without cycles H such that $\mathbb{F} \subset C(H)$.*

4 Fighting for a divisible prize

Let Y be a divisible resource, say money. Let N be a set of agents $N = \{1, 2, ..., n\}$, and each agent has an additive preference on his share of money and other agents' shares. Each agent has a power described as $\pi_1, \pi_2, ..., \pi_n$ with $\pi_i \geq 0$ and $\sum_{i=1}^{n} \pi_i = 1$. Also, without loss of generality, we assume that $\sum_{i \in S} \pi_i \neq \sum_{j \in T} \pi_j$ for all $S \neq T$, so we do not have any ties.

A *partition* is a collection of disjoint subsets $S_1, S_2, .., S_k$ of N where $\bigcup_{i=1}^{k} S_i = N$. Each subset S_j in the partition is called a *coalition*. The power of a coalition S is given by $\pi(S) = \sum_{i \in S} \pi_i$. The winning coalition for a partition $\Pi = (S_1, ..., S_k)$ is the subset S_j with $\pi(S_j)$ maximum.

In a tournament, agents form coalitions in order to win the game, or more generally, in order to increase their net utilities (see below).

Let ξ be a function that specifies the allocations of the resource across the winning agents. That is, for any agent $i \in S \subseteq N$, $\xi_i(S)$ is the allocation of the money to agent i with $\sum_{i=1}^{n} \xi_i(S) = Y$ when coalition S is winning. We assume that ξ is *cross-monotonic* on the size of the coalition, that is $\xi_i(S) > \xi_i(T)$ for $i \in S \subset T$.

We are going to consider by far the two rules for dividing money to agents in the winning coalition, which are equal-sharing and proportional sharing. Further extensions consider convex combinations of them, and additional variations.

Let S be the winning coalition.

1. Equal sharing is given by

$$\xi_i(S) = \begin{cases} \dfrac{Y}{|S|} & \text{if } i \in S, \\ 0 & \text{otherwise.} \end{cases}$$

   So under equal sharing, all agents in the winning coalitions share the same amount of the resource.

2. Proportional sharing is given by

$$\xi_i(S) = \begin{cases} \dfrac{\pi_i}{\pi(S)} Y & \text{if } i \in S, \\ 0 & \text{otherwise.} \end{cases}$$

   where $\pi(S) = \sum_{j \in S} \pi_j$ for notational convenience. So under proportional sharing, each agent's share depends on his power and the total power in the winning coalition.

Definition 8 Externality *is a situation in which each agent cares not only about himself, but also possibly cares about the other agents. Those relationships are represented by an $n \times n$ matrix for n agents, with entries M_{ij} representing the externality that agent j imposes on agent i.*

Definition 9 *For a partition Π, with winning coalition S^* (with largest power) the* net utility *to agent $i \in S^*$ is $v_i(\Pi) = \sum_{j \in S^*} M_{ij} \xi_j(S^*)$. If $k \notin S^*$, then $v_k(\Pi) = 0$. The object of the game is to maximize your net utility. That is, agent i tries to find the partition that maximizes $v_i(\Pi)$. Notice in the case without externalities the agents try to maximize $\xi_i(\Pi)$.*

4.1 The NTE equilibrium

Definition 10 *Under the* No-Threat Equilibrium (NTE) *if a group of agents find it profitable deviate from a coalition, then there is another group of agents who can react to that deviation in a way that harms the agents who originally deviated. Thus,* Π *is NTE (or* Π *is* NTE-stable*) if whenever* $S \subseteq N$ *is such that* $v_i(\Pi - S, S) > v_i(\Pi)$ *for all* $i \in S$*, then there exists* $T \subseteq N - S$ *such that* $v_i(\Pi - (S \cup T), T, S) > v_i(\Pi - S, S)$ *for all* $i \in T$ *and* $v_i(\Pi - (S \cup T), T, S) < v_i(\Pi)$ *for some* $i \in S$*. This means that at least one member of* S *will in the long run not profit by deviating to form the coalition* S*.*

With an NTE-stable partition, it may be possible for a group to deviate and gain a temporary advantage by forming a new coalition - but if they do so, then yet another coalition can react to punish them. NTE is related to the $\beta-$core discussed in the literature of cooperative games, but has not been used before in the literature of coalition formation.

4.2 Stability in tournament without externalities

As suggested by a referee, we provide below the comparison of equilibriums in the case without externalities.

Definition 11

- *A* minimally winning coalition *is a winning coalition* $S^* \subseteq N$ *satisfying* $\pi(S^*) > 1/2 > \pi(S^* - \{j\})$ *for all* $j \in S^*$*.*
- *A* minimally winning coalition of minimal size (MWCMZ) *is a minimally winning coalition* $S^* \subseteq N$ *satisfying* $|S^*| \leq |S|$ *for all* $S \subseteq N$ *with* S *minimally winning.*
- *The* minimally winning coalition of minimal weight (MWCMW) *is minimally winning coalition* $T^* \subseteq N$ *such that* $\pi(T^*) \leq \pi(T)$ *for all* $T \subseteq N$ *minimally winning.*

Example 6 *Suppose an externality-free environment with five agents and vector of power* $\pi = (.41, .34, .12, .10, .8)$*.*

This tournament has an empty core. Indeed, it is not difficult to see that no matter what winning coalition forms, there is always another coalition that can block. For instance, if the coalition $\{12, 345\}$ *forms, then agents* 12 *are winning and will split the resource between them. Nevertheless, this partition is not stable, since agent 1 can block by deviating:* $\{1, 2, 345\}$*. On the other hand,* $\{12, 345\}$ *is a NTE equilibrium, since if agent 1 decides to split to form his own partition* $\{1, 2, 345\}$*, then agent 2 can threat agent 1 by merging with 2345 and form partition* $\{1, 2345\}$*, under which agent 1 is worse off.*

Theorem 2

- *Under Equal-Sharing, the NTE is non-empty and contains the minimally winning coalition of minimal size.*
- *Under Proportional-Sharing, the NTE is non-empty and contains the minimally winning coalition of minimal weight.*

4.3 Existence of NTE under single peaked preferences

Consider the matrix:

$$\tilde{M} = \begin{bmatrix} 1 & .9 & 0 \\ 0 & 1 & .9 \\ .9 & 0 & 1 \end{bmatrix}$$

$\pi = (.45, .3, .25)$ and equal-sharing rule.
then MPSP equals $(23, 1)$. However NTE is empty.
$(23, 1)$ blocked by $(31, 2)$.
$(31, 2)$ blocked $(12, 3)$.
$(12, 3)$ blocked by $(31, 2)$.

Theorem 3 *For a matrix with positive externalities and agent ordered in a line with single peaked preferences among themselves, a NTE equilibrium always exists under equal-sharing.*

On the other hand, for the proportional sharing rule, the NTE is always empty.

Proofs

4.4 Proof of Theorem 1

Proof. ($\Leftarrow$) Assume $S_1, S_2, \ldots, S_K \in W(H)$.

Step 1. We show that $\bigcap_{k=1}^{K} S_k \neq \emptyset$

We prove by induction that $\bigcap_{k=1}^{K} S_k \neq \emptyset$. The base of induction is $n = 3$. Clearly, we can arrange the players in a line (see Figure) because this is the only possible network for three players if there are no cycles. If $\pi_i > 0.5$ then player i emerges as the winner. On the other hand, suppose we have no player i such that $\pi_i > 0.5$. In this scenario we have two cases. First, if $\{1, 2\}, \{2, 3\} \in \mathbb{F}$ then the coalition that maximizes player 2's payoff is the NTE. Second, if $\{1, 2\}, \in \mathbb{F}$ but $\{2, 3\} \notin \mathbb{F}$ and $\pi_3 < \pi_2$ or $\pi_3 < \pi_1$ then the NTE is the singleton coalition S such that $\arg\max_S\{\pi_1, \pi_1\}$. If, however, $\pi_3 > \pi_2, \pi_1$ then $\{1, 2\}$ is the NTE (see Example 2). Thus the NTE exists for three players.

Let our induction hypothesis be that this is true for any network for n agents. We will then prove this for $n+1$ players. Let $S_1, S_2, \ldots, S_K$ be elements in an $n+1$ network structure such that $S_i \cap S_j \neq \emptyset$.

• If $S_1, S_2, \ldots, S_K$ only contains n players then we are done by the induction hypothesis.

• Assume that all $n+1$ players are in $S_1, S_2, \ldots, S_K$. Take an agent a with degree one in the network and take an agent b that is connected to a. If $a \in \bigcap_{k=1}^{K} S_k$ we are done.

• If $a \notin \bigcap_{k=1}^{K} S_k$, we know that $a \notin S_k$ for some k. We show that if $a \in S_i$ it must be that $b \in S_i$. To see this, notice that if $a \notin S_k$ then for some player $x \neq a$ it must be that $x \in S_i \cap S_k$. Take the path $[x, a]$. Since

$a \in S_i$ and $x \in S_i$, by connectedness it must be that players on the path $[x, a]$ should be a subset of S_i, that is, $[x, a] \subseteq S_i$. Since $b \in [x, a]$ then $b \in S_i$.

• Now take the sets $S_1 \setminus \{a\}, S_2 \setminus \{a\}, \ldots, S_K \setminus \{a\}$. We know $(S_i \setminus \{a\}) \cap (S_j \setminus \{a\}) \neq \emptyset$ because $S_i \cap S_j \neq \emptyset$ and if $a \in S_i$ then $b \in S_i$. Since those sets only contain n players and they are subsets of network H without the link [a,b] (which is a network without cycles), then by the induction hypothesis $\bigcap(S_i \setminus \{a\}) \neq \emptyset$ implies $\bigcap(S_i) \neq \emptyset$.

Step 2. Let $V = \bigcap_{k=1}^{K} S_k$ and let $i \in V$. Let $S^* \in W(H)$ such that $S^* \succ_i S \; \forall \; S \in W(H) \setminus S^*$. Let there be a partition $\Pi = (S^*, [N \setminus S^*])$ where $[N \setminus S^*]$ is the list of players that do not belong to S^*. Assume that T deviates at partition Π.

Case 1: $i \in T$. If $T \in W(H)$ then i gets a lower payoff. On the other hand, if $T \notin W(H)$ then there exists a coalition in $N \setminus T$ that can form and beat T.

Case 2: $i \notin T$. Then from our assumption, $T \notin W(H)$ since $i \in V$ is part of all winning coalitions. Therefore there exists a coalition in $N \setminus T$ that forms and beat T.

($\Rightarrow$) Consider a collection of coalitions $\mathbb{F}$ such that *any* preferences of the players defined in $\mathbb{F}$ have a NTE. We will show the existence of a network N that do not contain cycles such that $\mathbb{F} \subseteq C(N)$. We prove this by induction on the number of agents. Let the base of induction be $n = 3$. If $n = 3$ and NTE exists for all preferences, the preferences do not have cycles. Suppose there is a cycle such that:

$\{1, 2\} \succ_1 \{1, 3\}$
$\{2, 3\} \succ_2 \{1, 2\}$
$\{1, 3\} \succ_3 \{2, 3\}$

we showed in Example 2 that there will be no NTE. Thus, if preferences do not have cycles then we can order the players in a line such as that in Figure.

Our induction hypothesis is that we assume that for any set of coalitions with n players such that NTE exists, we can find a network N without cycles such that $\mathbb{F} \subseteq C(N)$.

We now prove it for $n + 1$. Suppose there are $n + 1$ players.

Case 1: Suppose that there is a player i and j such that if $S \in \mathbb{F}$ and $i \in S$ then $j \in S$. Consider the set $\tilde{\mathbb{F}} = \{S \setminus i \; | S \in F\}$.

Claim 1: $\tilde{\mathbb{F}}$ is a feasible domain for n agents where NTE always exists.

Proof: Assume the contrary, that there exists preferences $(\succ_1, \succ_2, \ldots, \succ_n)$ and there is no NTE. Let us extend these preferences to $\mathbb{F}$ such that for all coalitions $S, T \in \mathbb{F}$:

$S \tilde{\succ}_k T \Leftrightarrow S \setminus i \succ_k T \setminus i$ where $\tilde{\succ}$ denotes the extension of $\succ$ to $\mathbb{F}$.

By assumption there is always an NTE on $\tilde{\succ}$. Let Π be the partition that generates the NTE. Then the partition $\Pi \setminus i$ is also an NTE for $\succ$ which contradicts our assumption that there is no NTE for $\succ$.

By our induction hypothesis, there is a network N defined for all players except i that has no cycles. Consider a network $\tilde{N}$ which coincides with N except we add a link from i to j.

Claim 2: $\tilde{N}$ does not have cycles and $\mathbb{F} \subseteq C(\tilde{N})$

Proof: $\tilde{N}$ having no cycles is obvious looking at Figure. Now if $S \in \tilde{F} \subset C(N)$ then $S \cup i \in C(\tilde{N})$. Therefore $\mathbb{F} \subseteq C(\tilde{N})$

Case 2: Suppose that there is no player i, j such that $i \in S \in \mathbb{F}$ and $j \in S$. This means that if $i, j \in S$ then there exists an $\tilde{S}$ such that $i \in \tilde{S}$ and $j \notin \tilde{S}$ and a $\tilde{T}$ such that $i \notin \tilde{T}$ and $j \in \tilde{T}$.

Take i, j such that $i, j \in S^i$, then there exists an $\tilde{S}^i$ such that $i \in \tilde{S}^i$ and $j \notin \tilde{S}^i$. Suppose that there exists an S^z such that $z \in S^z$, and for all elements $k \in S^z$ any set $S \in \mathbb{F}$, we have $k \in S$, where $S \subseteq S^z$. This means that any coalition that contains any player in S^z but do not contain z is a subset of S^z (See Figure). We can thus describe two sets:

$$\bar{\mathbb{F}} = \{S \cap \{(N \setminus S^z \cup z\}|\ S \in \mathbb{F}\}$$
$$\mathbb{F}^* = \{S \cap S^z|\ S \in \mathbb{F}\}$$

Since $\bar{\mathbb{F}}$ is a restriction of $\mathbb{F}$ and $\mathbb{F}$ always has an NTE, then $\bar{\mathbb{F}}$ has an NTE. Hence, by induction there exists a network $\bar{N}$ such that $\bar{\mathbb{F}} \subseteq C(\bar{N})$. Similarly, since $\mathbb{F}^*$ is a restriction of $\mathbb{F}$ and $\mathbb{F}$ always has an NTE, then $\mathbb{F}^*$ has an NTE. Hence, by induction there exists a network N^* such that $\mathbb{F}^* \subseteq C(N^*)$.

Consider the network in $\bar{\mathbb{F}} \cup \mathbb{F}^*$ which is just the original network N. Thus $\mathbb{F} \subseteq C(N)$.

Finally, assume that we are able to find a cycle as in Figure. Take a set of preference such that:

$$S_1 \succ_1 S_2 \succ_1 S^1, \quad \forall S^1 \neq S_1, S_2$$
$$S_2 \succ_2 S_3 \succ_2 S^2, \quad \forall S^2 \neq S_2, S_3$$
$$\vdots$$
$$S_1 \succ_n S_{n-1} \succ_2 S^n, \quad \forall S^n \neq S_1, S_{n-1}$$

then we know that NTE does not exists, which is a contradiction of our assumption. ■