Forming Coalitions under Sharing Disagreements*

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Abstract

Agents endowed with powers compete for a divisible resource by forming coalitions with other agents. The winning coalition divides this resource among its members according to a given sharing rule. We investigate the case where the sharing rule satisfies a property we call *consistent ranking*. Sharing rules that satisfy consistent ranking ensures that agents' ranking of competing coalitions coincide. Sharing rules such as equal sharing and proportional sharing satisfy this property. We also examine a particular sharing rule that violates consistent ranking, which we call *combination sharing* that is a convex combination of equal and proportional sharing.

For these different sharing rules, we characterize rules on choosing coalitions (called *transition correspondence*) that satisfy two main axioms: *self-enforcement*, which requires that no further deviation happens after a coalition has formed, and *rationality*, which requires that agents pick the coalition that gives them their highest payoff.

We find that a transition correspondence that satisfies self-enforcement and rationality always exists for sharing rules that satisfy consistent ranking. However, this is not true for combination sharing. For combination sharing, the maximal domain of games under which there exists transition correspondences that satisfy self-enforcement and rationality involves restrictions on the configuration of powers in the society or restrictions on the convex combination parameter between equal sharing and proportional sharing.

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1 Introduction

Forming groups confers to its members certain advantages that cannot be appropriated had its members acted individually. However, conflicts may arise within the group when there are disagreements in dividing the prize or effort among its members. Thus, specifying the appropriate sharing rule to divide an economic surplus (or cost) provides the incentive structure that encourages cooperation or discourages conflict within coalitions (Moulin [29]).[1] These incentives to maintain stability in coalitions, however, are altered when some agents in the society are more influential or powerful than others (Jandoc and Juarez [23, 24]; Piccione and Razin [31]; Jordan [26]).

While there has been a large and growing literature on coalition formation with externalities focusing on static equilibria (e.g., Bogomolnaia and Jackson [7], Papai [30], Ehlers [15], Bloch and Dutta [6], Chatterjee et al. [12], Pycia [32], Romero-Medina [9], Banerjee et al. [4]) and dynamic equilibria (e.g., Greenberg [20], Chwe [13], Bloch [5], Xue [36], Arnold and Schwalbe [3], Diamantoudi and Xue [14], Ray and Vohra [33], Iñarra et al. [22]), there is little work on examining how the interconnectedness of sharing-rules (payoffs) and the power of agents (ability to form winning coalitions) affect the stability of coalitions in dynamic settings. An important advance in the literature to address this gap stems from Acemoglu, Egorov and Sonin [1,2] (henceforth, AES). AES provides a coalition formation model where agents endowed with different powers compete for a divisible prize by forming coalitions with other agents for infinitely many rounds. In each round, a "winning coalition" will form where agents inside this coalition have combined powers that are higher than the combined powers of the rest of the society. Agents who are not part of this winning coalition are "killed" in the sense that they cannot participate in succeeding rounds of the game.[2]. If there is a winning coalition that forms without subsequent deviations into subcoalitions after several rounds, this winning coalition is deemed as the "ultimate ruling coalition". [3] Members of the ultimate ruling coalition will divide this prize among themselves according to a fixed and predetermined sharing rule.

AES focused on the case when agents in the ultimate ruling coalition divide the prize in proportion to their power inside the ultimate ruling coalition. We call this type of sharing rule *proportional sharing.* AES characterizes which coalitions will form at the

[1] The applications are wide-ranging, for instance, appropriating gains and costs from managing global public goods (Carraro [8]), cost and revenue sharing in irrigation networks (Jandoc, Juarez and Roumasset [25]), and market design (Roth [34]), to name a few.

[2] A real world example of this setting is the purge in the Russian Politburo documented in Acemoglu, Egorov and Sonin [1].

[3] Elsewhere in this paper, we interchange the term ultimate ruling coalition with the term *limit coalition.*

very first round of the game that will never disintegrate into subcoalitions in subsequent rounds. However, this characterization by AES does not extend to different types of sharing rules–such as when the prize is divided equally among coalition members (we call this type of sharing rule *equal sharing*).[4] Jandoc and Juarez [23] extends the characterization to equal sharing and to the case when power accumulates over time.

These two sharing rules, equal and proportional sharing, share the property that they do not create disagreements among agents when faced between a choice of joining competing coalitions. We call this property *consistent ranking* because these sharing rules induce agents to have the same ordinal ranking over coalitions in which they could possibly join.[5] Equal sharing satisfies consistent ranking because agents in the intersection of competing coalitions will unanimously prefer to join the coalition with the smaller size. Proportional sharing satisfies consistent ranking because agents in the intersection of competing coalitions will unanimously prefer to join the coalition with the least total power. In both cases, agents increase their share of the prize by joining these coalitions.

However, there are also sharing rules that create disagreements among agents on their preferred coalitions. For instance, a convex combination of equal and proportional sharing (call this *combination sharing*) may induce some agents in the intersection of competing coalitions to prefer the smaller-sized coalition, while others may prefer to be in the lower-powered coalition. This situation then creates tension between coalition size and power. This sharing rule has its roots in the *Sen share* (Sen [35]) popular in the literature of team production and moral hazard.[6] Combination sharing does not satisfy the consistent ranking property.

The main objective of this paper is to examine how different sharing rules affect the manner in which coalitions form when agents have heterogeneous power. "Power" can emanate from various sources–for instance wealth, military might or political influence–and is exogenously given in the model.[7]

[4]Equal sharing is widely used not only on theoretical grounds but also for practical purposes such as inheritance bequest (Erixson and Ohlsson [16]).

[5]Consistent ranking is related to the notion of *pairwise alignment in preference profiles* (Pycia [32]) which says that agents' preferences are pairwise aligned if any two agents rank coalitions that contain both of them in the same way. Pycia [32] shows that pairwise alignment is a necessary and sufficient condition to establish the existence of a stable coalition structure for all preference profiles. It is also related to the condition of *common ranking property* by Farrell and Scotchmer [19] that guarantees stability of the core in hedonic games.

[6]The Sen share is a convex combination of an equal share of the surplus among productive agents and a proportional share according to an agent's labor to the total labor supplied in the economy. Fabella [17, 18], shows that under increasing returns to scale the Sen share of the production surplus can support Pareto optimal production.

[7]Since agents are given exogenous power, we avoid the "joint-bargaining paradox" where, in a pure

Similar to AES, we employ an axiomatic approach to find rules to choose coalitions (which we call "transition correspondences") that satisfy two main axioms: *self-enforcement* and *rationality*. Since our model is dynamic, we allow for the possibility of factions within the winning coalition to secede from this coalition and form their own winning coalition in the next round. Since agents are forward-looking, they prefer to be part of the ultimate ruling coalition to get a share of the prize. Our axiom of self-enforcement requires that no subcoalition of the winning coalition be powerful enough to encourage further deviations. Self-enforcement is a robustness property that ensures that the winning coalition that forms never disintegrates afterwards. The notion of self-enforcement is related to the concept of *farsighted stability* attributed to Chwe [13]. Stability under this concept requires that the coalition obtained in equilibrium should be robust to current deviations, and any future deviation should be robust to further deviations themselves. An upshot of this is that agents are farsighted enough to evaluate their payoffs to the eventual coalition that will form. In our context, although agents potentially increase their expected payoffs from deviating into a subcoalition, the possibility that they will eventually be excluded from the ultimate ruling coalition (and get no share of the prize) will discourage them from certain types of deviations.[8] In addition, rationality requires that agents choose to join the coalition that gives them their highest payoff among self-enforcing coalitions. Rationality is related to the traditional axiom of coalitional stability, where no coalition will have the incentive to deviate.[9]

In the next subsection, we provide an illustrative example of how our coalition formation game works and how sharing of the prize shapes the coalitions that form.

1.1 An illustrative example

Suppose our society is composed of 6 agents $\{1, 2, 3, 4, 5, 6\}$ with power profile $\pi = [29, 26.5, 21, 2.49, 2.4, 2.39]$. The game is played for an infinite number of periods and for each round the prize to be divided by the winning coalition is $I = 3$. A coalition's power is simply the sum of the agents' powers inside it. A winning coalition should have more than 50% of the total power in the society at any given round. Since agents are farsighted, they only care about their payoffs by being a member of the ultimate

bargaining situation, an individual can be made worse-off by negotiating as a member of a group than by negotiating alone (Harsanyi [21]; Chae and Heidhues [10]; Chae and Moulin [11]).

[8]Using a laboratory experiment, Jandoc and Juarez [24] show that agents do not display farsighted behavior when playing a simplified version of the game informed by the model of this paper.

[9]This is related to immunity to group manipulations in models discussed by Bogomolnaia and Jackson [7], Ehlers [15], Juarez [27], Papai [30].

ruling coalition. If an agent is part of a coalition S, his share of the prize under equal sharing will be $\frac{1}{|S|}$, under proportional sharing it is his power divided by the power of coalition S (that is, $\frac{\pi_i}{\pi(S)}$), under combination sharing it is $\lambda\frac{1}{|S|} + (1-\lambda)\frac{\pi_i}{\pi(S)}$ for some $\lambda \in (0, 1)$.

Case 1: Sharing rule satisfies consistent ranking.

First, no coalition of size 1 is winning and therefore is incapable of forming. Although there are winning coalitions of size 2, we argue that this will not be self-enforcing. To see this, suppose coalition $\{1, 2\}$ forms. This coalition is winning since their combined power (55.5) is higher than the rest of society composed of agents $3, 4, 5$ and 6 (with combined power of 28.28). After coalition $\{1, 2\}$ forms, the non-winning agents $3, 4, 5$ and 6 are killed and agents 1 and 2 continue on the next round. Left alone to themselves, agent 1 can now kill agent 2 since he has the higher power. Since agent 2 is farsighted, he will never agree to form $\{1, 2\}$. This will be true for any coalition of size 2.

Furthermore, note that there are several coalitions of size 3 that are winning but only $\{1, 2, 3\}$ is self-enforcing. This is because if $\{1, 2, 3\}$ forms and agents 4,5, and 6 are killed, then there can be no dictator among them and a deviation to a 2-person coalition is not feasible following the argument outlined above. On the other hand, take the case of winning coalition $\{2, 3, 5\}$. This will not be self-enforcing since when agents 1,4 and 6 are killed, agent 2 can be a dictator and can deviate from $\{2, 3, 5\}$.

Following the same arguments, it is straightforward to show that the coalitions $\{1, 2, 4, 5\}$, $\{1, 2, 5, 6\}$, $\{1, 2, 4, 6\}$ and $\{2, 3, 4, 5, 6\}$ are self-enforcing since they don't contain subcoalitions that are self-enforcing. In the same manner, the grand coalition $\{1, 2, 3, 4, 5, 6\}$ is not self-enforcing since it can deviate to either one of the self-enforcing coalitions mentioned.

With proportional sharing, the coalition $\{2, 3, 4, 5, 6\}$ will be preferred by everyone inside this coalition over any other possible self-enforcing coalitions. For instance, given the choice between $\{2, 3, 4, 5, 6\}$ and $\{1, 2, 3\}$ agents 2 and 3 (the agents common in these coalitions) will prefer the former since their share of the prize ($\frac{26.5}{54.78}$ and $\frac{21}{54.78}$ for agents 2 and 3, respectively) is higher than the latter coalition ($\frac{26.5}{76.5}$ and $\frac{21}{76.5}$ for agents 2 and 3, respectively). Hence, under proportional sharing, rationality implies that the coalition $\{2, 3, 4, 5, 6\}$ should form.

On the other hand, under equal sharing the coalition $\{1, 2, 3\}$ should form because the agents in this coalition will get a higher share $\frac{1}{3}$ than in any other possible self-enforcing coalition.

Case 2: Combination sharing.

Recall that combination sharing is a convex combination of equal sharing and proportional sharing, $\lambda\frac{1}{|S|}+(1-\lambda)\frac{\pi_i}{\pi(S)}$ for some combination sharing parameter $\lambda \in (0,1)$, We will show in Example 1 later that when the combination sharing parameter $\lambda = 0.5$, there will be disagreement among agents on the preferred coalition. For instance, comparing coalitions $\{1,2,3\}$ and $\{2,3,4,5,6\}$, the share of agent 2 is higher in $\{2,3,4,5,6\}$ and the share of agent 3 is higher in $\{1,2,3\}$. Unlike in the previous case of equal or proportional sharing where these agents agree on the preferred coalition, in the case of combination sharing there may be no such agreement.

1.2 Overview of the results

Section 3 of the paper presents the main results. Proposition 1 characterizes the unique transition correspondence that is rational and self-enforcing under consistent ranking sharing rules.[10] Our proof is based on the key Lemma 1, which highlights the importance of transition correspondences that do not necessarily satisfy rationality but meet a weaker requirement, *minimalistic*, introduced in Section 3. Under consistent ranking, the transition correspondence chooses the highest ranked coalition among self-enforcing and winning coalitions. In particular, for equal sharing, the smallest-sized self-enforcing coalition will be chosen, whereas for equal sharing, the least-powered self-enforcing coalition is chosen.

Under combination sharing, Propositions 2 and 3 characterize the unique transition correspondences that are self-enforcing and rational for restricted classes of coalition formation games. Indeed, we show that the class of games with power vectors that meet size-power monotonicity (where larger coalitions have higher power) is the maximalist domain that works for any combination sharing rule (Proposition 2). Alternatively, for a fixed combination sharing rule, Proposition 3 characterizes the maximalist domain of games where the unique transition correspondence that satisfies self-enforcing and rationality exist. Such a domain is substantially larger than the domain found in Proposition 2. Furthermore, such a correspondence may select the smallest-sized self-enforcing coalition for some games, the least-powered self-enforcing coalition for other games, and (perhaps surprisingly) just the right “compromise” coalition for even other games.

[10] A transition correspondence is a mapping that defines which coalitions form over time. The precise definition is given in Section 2.1.

2 The model

Consider the set $N = \{1, \ldots, n\}$ of initial agents who are endowed with powers $\pi = [\pi_1, \ldots, \pi_n]$, respectively.[11] A coalition S is a subset of N, that is, $S \subseteq N$. The set of coalitions are all possible subsets of N, denoted by 2^N. A coalition formation game is a pair (S, π) where $S \subseteq N$ and $\pi \in \mathbb{R}_+^S$.[12] The set of coalition formation games is denoted by $\mathbf{G}$. We assume that power is additive, that is, the power of coalition S is the sum of all powers of the agents inside the coalition, $\pi(S) = \sum_{i \in S} \pi_i$.[13] We denote by π_S the restriction of the vector $\pi \in \mathbb{R}_+^N$ over coalition S.

Definition 1 *Given a game (T, π), the **set of winning coalitions** is*

$$W_{(T,\pi)} = \{S \subset T | \pi(S) > \pi(T \setminus S)\}.$$

This definition requires winning coalitions to have relative power larger than 50%.[14]

There is a prize $I \in \mathbb{R}_+$ that will be divided by the agents according to a sharing rule that is fixed throughout the game.

Definition 2 *A **sharing rule** is a function $\xi : \mathbf{G} \to \mathbb{R}_+^N$ such that:*

i. If $k \notin S$, then $\xi_k(S, \pi) = 0$.

ii. $\sum_{i \in S} \xi_i(S, \pi) = 1$, and

iii. (Cross-Monotonicity) If $(S, \pi) \in \mathbf{G}$, $T \subset S$, $i \in T$ and $\pi_i > 0$, then $\xi_i(T, \pi_T) > \xi_i(S, \pi)$.

Cross-monotonicity of the sharing rule requires that the share of the prize of agent i in coalition S would be higher if he is part of any subcoalition of S that deviates compared to the share he will get if he stayed in coalition S.

Throughout the paper we devote special attention to commonly used sharing rules such as equal sharing, proportional sharing, or a convex combination of the two.[15]

[11] For convenience, the power vector $\pi \in \mathbb{R}_+^N$ can be normalized such that $\sum_{i \in N} \pi_i = 1$.

[12] It is convenient to restrict the game (S, π_S) such that π_S has no ties in the power of any pair of coalitions, that is, $\pi(V) \neq \pi(T)$ for any $V, T \subset S$. However, this is a weak condition because games that do not satisfy this property has a Lebesgue measure equal to zero (See Jandoc and Juarez [23], Acemoglu et al. [1]).

[13] Juarez and Vargas [28] considers a more general version where power can be any arbitrary monotonic function.

[14] Our results below can be easily adapted to require winning coalitions to have relative power larger than α, where $\alpha \geq 50\%$.

[15] Note that these three sharing rules are cross-monotonic.

That is, if $i \in S$, then the share of agent i when S is winning and the power profile is π equals:

$$\xi_i(S,\pi) = \begin{cases} \dfrac{1}{|S|} & \text{if equal sharing (ES)} \\ \dfrac{\pi_i}{\pi(S)} & \text{if proportional sharing (PR)} \\ \lambda \cdot \dfrac{1}{|S|} + (1-\lambda) \cdot \dfrac{\pi_i}{\pi(S)},\ \lambda \in (0,1) & \text{if combination sharing (CS}^{\lambda}\text{)} \end{cases}$$

Suppose agents i and j belong to the intersection of coalitions S and T. A sharing rule satisfies consistent ranking if whenever agent i prefers S over T, then agent j also prefers S over T. In other words, between competing coalitions, a coalition S is picked if all agents in the intersection unanimously pick S over a competing coalition.

Definition 3 *(Consistent Ranking (CR)) The sharing rule ξ satisfies **consistent ranking (CR)** if for any two agents i and j, and coalitions S and T such that $i, j \in S \cap T$, if $\xi_i(S,\pi) > \xi_i(T,\pi)$, then $\xi_j(S,\pi) > \xi_j(T,\pi)$.*

If the sharing rule ξ satisfies consistent ranking, then there exists a ranking R^{ξ} : $\mathbf{G} \to \mathbb{R}$ for the society that coincides with individual rankings. That is, for any coalitions S and T such that $S \cap T \neq \emptyset$, we have that $R^{\xi}(S,\pi) > R^{\xi}(T,\pi) \Leftrightarrow \xi_i(S,\pi) > \xi_i(T,\pi)$ for any $i \in S \cap T$.

Equal sharing and proportional sharing satisfy consistent ranking. Under equal sharing, agents' share increases as they move to coalitions of smaller sizes; therefore, $R^{ES}(S,\pi) = \frac{1}{|S|}$ is an example of a consistent ranking for equal sharing. Similarly, under proportional sharing, agents' share increases as they move to coalitions of lower power; therefore, $R^{PR}(S,\pi) = \frac{1}{\pi(S)}$ is an example of a consistent ranking for proportional sharing. On the other hand, combination sharing does not satisfy consistent ranking (see Example 1).

2.1 Dynamic Coalition Formation

Let t, where $t = 0, 1, \ldots$, denote the discrete rounds of the game. A *transition correspondence* is a mapping from the set of coalition formation games to the set of winning coalitions.

Definition 4 *A **transition correspondence** is a continuous correspondence ϕ : $\mathbf{G} \to 2^N$ such that $\forall$ $(X,\pi) \in \mathbf{G}$: $\phi(X,\pi) \subset W_{(X,\pi)}$.*[16]

[16] A correspondence is continuous if for any sequence of power vectors $\pi^1, \pi^2, \cdots \to \pi^*$ where $S \in \phi(N,\pi^i)$ $\forall i$ and S is winning in π^*, then $S \in \phi(N,\pi^*)$.

The transition correspondence ϕ selects all coalitions that could be winning at a given game (S, π). The evolution of the coalition formation games at every round depends on the transition correspondence. Our goal is to axiomatize this correspondence.

We start with the game (S^0, π^0) at time 0 and look at the potential evolution of the game throughout time, $(S^0, \pi^0), (S^1, \pi^1), (S^2, \pi^2), \dots$, where $S^t \in \phi(S^{t-1}, \pi^{t-1})$ is a chosen coalition at time t and $\pi^t = \pi^{t-1}_{S^t}$ is their respective power. Furthermore, we assume that agents are killed, $S^t \subseteq S^{t-1}$ for any $t \geq 1$. That is, only agents who were part of the winning coalition at time $t-1$ could participate at time t.[17]

Since coalition sizes do not increase over time, the sequence $(S^0, \pi^0), (S^1, \pi^1), (S^2, \pi^2), \dots$ converges in at most n steps. We call this limit (S^∞, π^∞). We assume that agents only care about $\xi(S^\infty, \pi^\infty)$. Thus, the prize is only given to the limit coalition (S^∞, π^∞), or alternatively, agents are infinitely forward-looking.

3 Axioms

The first main axiom, self-enforcement, ensures that a transition correspondence maps to coalitions that do not have the incentive nor the power to deviate in future rounds of the game.

Axiom 1 (Self-enforcement (SE)) *The transition correspondence ϕ is **self-enforcing (SE)** if for any game $(X, \pi) \in \mathbf{G}$ and $S \in \phi(X, \pi)$, then $S \in \phi(S, \pi_S)$.*

When there is no confusion, given a transition correspondence ϕ and a game (S, π), we say that the coalition S is self-enforcing if $S \in \phi(S, \pi_S)$.

Self-enforcement requires that if a coalition S is chosen by the transition correspondence $\phi(X, \pi)$ given a starting game (X, π), then it should be chosen again by the same transition correspondence in a game where a coalition S playing (that is, $\phi(S, \pi_S)$). This means that the rule or transition correspondence will not ensue further deviations into subcoalitions of S once S forms.

Since the sharing rule is cross-monotonic, we expect that in the presence of self-enforcing and winning coalitions that are strict subsets of the grand coalition, the grand coalition will not be chosen, since all of the agents gain by choosing its subset. This is reflected in the definition of a minimalistic transition correspondence.

[17] We impose no restriction in which coalition from $\phi(S^{t-1}, \pi^{t-1})$ will be selected. This allows our results to be more robust, since the evolution of the game includes any potential path of coalitions such that $S^t \in \phi(S^{t-1}, \pi^{t-1})$ for all t.

Axiom 2 (Minimalistic (MIN)) *The transition correspondence ϕ is **Minimalistic (MIN)** if for the game $(S,\pi) \in \mathbf{G}$ such that there exists $T \subsetneq S$, where $T \in \phi(T,\pi_T)$ and $T \in W_{(S,\pi)}$, then $S \notin \phi(S,\pi)$.*

Next, we focus on comparing different transition correspondences based on the coalitions that they choose.

Axiom 3 (Superiority) *Consider two transition correspondences ϕ and $\hat{\phi}$. We say that ϕ is **superior** to $\hat{\phi}$ if for any game (X,π), $T \in \hat{\phi}(X,\pi)$ and $S \in \phi(X,\pi)$ such that $\xi_i(T,\pi_T) \geq \xi_i(S,\pi_S)$ for some $i \in T \cap S$ if and only if $T \in \phi(X,\pi)$.*

If a transition correspondence is superior to another, then it always picks outcomes that are preferred by common agents being chosen.

Axiom 4 (Rationality (RAT)) *The transition correspondence ϕ is **rational (RAT)** if for any $S \in 2^N$, for any $T \in \phi(S,\pi)$ and for any $Z \subset S$ such that $Z \in W_{(S,\pi)}$ and $Z \in \phi(Z,\pi_Z)$, we have that $Z \notin \phi(S,\pi) \Leftrightarrow \xi_i(T,\pi_T) > \xi_i(Z,\pi_Z)\ \forall i \in T \cap Z$.*

Rationality implies that agents prefer to join self-enforcing coalitions that give them a larger share of the resource. This is similar to other notions of coalitional stability previously discussed in the literature, where a coalition is chosen if it cannot be blocked by another coalition that is winning and self-enforcing. Note that the cross-monotonicity of the sharing rule implies that if a transition correspondence satisfies RAT, then it also satisfies MIN.

4 Results

4.1 Result with Consistent Ranking

Let the transition correspondence ϕ^* be defined as:

$$\phi^*(S,\pi) = \underset{M \in Q(S,\pi) \cup \{S\}}{\arg\max} R^{\xi}(M,\pi_M) \tag{1}$$

where $Q(S,\pi) = \{T \subsetneq S \mid T \in W_{(S,\pi)}, T \in \phi^*(T,\pi_T)\}$

This transition correspondence defines for the game (S,π) a set $Q(S,\pi)$ of proper subcoalitions, which are both winning in S and self-enforcing. It picks the coalition that yields the highest rank for the agents in the intersection of all the coalitions contained in $Q(S,\pi)$. If $Q(S,\pi)$ is empty then it picks coalition S itself. Thus, $\phi^*(S,\pi_S) \neq \emptyset$ and ϕ^* is well defined.

Proposition 1 *Consider a sharing rule that satisfies consistent ranking. Then, the following conditions are equivalent for the transition correspondence ϕ that is self-enforcing:*

i. *ϕ is superior to any other transition correspondence that is self-enforcing and minimalistic,*

ii. *ϕ is rational,*

iii. $\phi = \phi^*$.

We note that AES's main result has a similar characterization to parts *ii* and *iii* under proportional sharing. This proposition shows that AES's result actually extends to a larger class of sharing rules that satisfy consistent ranking. The proof of this result is provided in Appendix A.2. The proof relies on using a key observation that any two self-enforcing and minimalistic transition correspondences have the same sets of self-enforcing coalitions (see Lemma 1 in Appendix A.1). This observation greatly simplifies the proof provided by AES even for proportional sharing.

In Appendix A.1 we provide the necessary and sufficient conditions on the sharing rule that allow for the existence of a self-enforcing and rational transition correspondence. In particular, we show that the assumption of consistent ranking in Proposition 1 can be extended to an even larger class of sharing rules that satisfy a property called *generalized consistent ranking.*

4.2 Results under Combination Sharing

4.2.1 Size-Power Monotonic Games

An example of a sharing rule that does not satisfy consistent ranking is combination sharing.[18] This is illustrated in the next example:

Example 1 *Consider the game*

$$(N, \pi) = (\{1, 2, 3, 4, 5, 6\}, [29, 26.5, 21, 2.49, 2.4, 2.39]).$$

The self enforcing coalitions that are contained in (N, π) *are* $\{1, 2, 3\}$, $\{1, 2, 5, 6\}$, $\{1, 2, 4, 5\}$, $\{1, 2, 4, 6\}$, *and* $\{2, 3, 4, 5, 6\}$

To see this, note that the power of agents 3, 4, 5 and 6 together is less than the power of agent 1. Therefore, any self-enforcing coalition that contains agent 1 should also contain agent 2. In order for 1 not to be a dictator in a game that contains

[18]Combination sharing does not satisfy generalized consistent ranking as defined in Appendix A.1.

agents 1 and 2, the game must contain agents whose combined powers should exceed 2.5. Therefore, any self-enforcing coalition that contains agents 1 and 2 should include either agent 3 alone or any two agents from agents 4, 5 and 6.

On the other hand, note that if a self-enforcing coalition does not contain agent 1, then it is a subset of the coalition $\{2,3,4,5,6\}$. *The coalition* $\{2,3,4,5,6\}$ *is self-enforcing because any subcoalition of size 2, 3 or 4 has a dictator. Therefore,* $\{2,3,4,5,6\}$ *is the only self-enforcing coalition that does not contain agent 1.*

Let ξ *be the combination sharing rule with* $\lambda = .5$. *Notice that comparing coalitions* $\{2,3,4,5,6\}$ *and* $\{1,2,3\}$ *we have that*

$$\xi_2(\{2,3,4,5,6\},[26.5,21,2.49,2.4,2.39]) = 0.34;$$

$$\xi_3(\{2,3,4,5,6\},[26.5,21,2.49,2.4,2.39]) = 0.29;$$

and

$$\xi_2(\{1,2,3\},[29,26.5,21]) = 0.33;$$

$$\xi_3(\{1,2,3\},[29,26.5,21]) = 0.30;$$

The share of agent 2 is higher in the coalition $\{2,3,4,5,6\}$ *while the share of agent 3 is higher in the coalition* $\{1,2,3\}$. *Hence, combination sharing does not satisfy the property of consistent ranking.*

There is a class of games, however, where combination sharing will yield consistent ranking for any value λ (and hence satisfies GC for the restriction to this games).

Definition 5 *(Size-Power Monotonicity) A game* (N,π) *is size-power monotonic (SPM) if for any* $A, B \subset N$ *such that* $|B| > |A|$, *we have that* $\pi(B) > \pi(A)$. *The set of SPM games is denoted by* $\bar{G}$.

What the SPM condition does is to take away the tension between coalition size and power (since power increases with size) and thus coalitions with smaller sizes (which, by definition, have lower power) will always give a higher share for any value of λ.

Since there is no disagreement with coalition size and power, then there will exist a ranking over these coalitions to which agents consistently agree.

Proposition 2 *Under combination-sharing* CS^{λ}, *the transition correspondence* ϕ *defined on the domain of SPM games* $\bar{G}$ *satisfies SE and RAT if and only if* $\phi = \phi^*$. *Furthermore, if the transition correspondence* ϕ *defined on the domain of games* G *satisfies SE and RAT for any* $\lambda \in (0,1)$, *then* $\phi = \phi^*$ *and* $G \subseteq \bar{G}$.

This results shows that if we want to find a transition correspondence that satisfies SE and RAT for all possible values of $\lambda \in (0,1)$, then the domain of games should be a subset of $\bar{G}$. However, this is not true for a fixed λ, as there might exists games that are not size-power monotonic where a SE and RAT transition correspondence exists, as we see below.

4.2.2 Maximalist Domain

Consider the sharing rule CS^{λ}. Even if the game is not SPM (that is, there is a tension between coalition size and power), it is still possible to find conditions wherein a transition correspondence that satisfies SE and RAT will exist. To illustrate this, recall from the definition of combination sharing that agents in the intersection of two coalitions S and T will agree on coalition S if and only if their share is higher in S than in T, that is:

$$\lambda \frac{1}{|S|} + (1-\lambda)\frac{\pi_i}{\pi(S)} > \lambda \frac{1}{|T|} + (1-\lambda)\frac{\pi_i}{\pi(T)} \ \forall\ i \in S \cap T$$

or, after rearranging,

$$\lambda \frac{|T|-|S|}{|T||S|} + (1-\lambda)\pi_i \frac{\pi(T)-\pi(S)}{\pi(T)\pi(S)} > 0 \ \forall\ i \in S \cap T \tag{2}$$

For instance, if the size of coalition S were smaller than T ($|S| < |T|$) but the coalition power were higher ($\pi(S) > \pi(T)$), the the only way that S will be preferred is when the first term on the left-hand side of Equation 2 is higher than the second term.

Since all the parameters (coalition sizes, coalition powers, and the agent's power) in Equation 2 are known, in principle we can find values of λ where we can make all agents in the intersection of two coalitions prefer one over the other. That is, we can find a λ high enough for agents to prefer the smaller-sized coalition (since higher λ puts more weight towards equal sharing) and a λ low enough for the agents to prefer lower-powered coalitions (where proportional sharing dominates). Thus, for any two coalitions $S, T \in (N, \pi)$ we can define:

$$\underline{\lambda}^{(S,T)}(N,\pi) = \frac{\underline{\pi}_i^{(S,T)} \left(\frac{\pi(S)-\pi(T)}{\pi(S)\pi(T)}\right)}{\underline{\pi}_i^{(S,T)} \left(\frac{\pi(S)-\pi(T)}{\pi(S)\pi(T)}\right) + \left(\frac{|T|-|S|}{|T||S|}\right)} \tag{3}$$

where $\underline{\pi}_i^{(S,T)}$ is the power of agent i in the intersection S and T with the lowest power.

and

$$\bar{\lambda}^{(S,T)}(N,\pi) = \frac{\bar{\pi}_i^{(S,T)}\left(\frac{\pi(S)-\pi(T)}{\pi(S)\pi(T)}\right)}{\bar{\pi}_i^{(S,T)}\left(\frac{\pi(S)-\pi(T)}{\pi(S)\pi(T)}\right) + \left(\frac{|T|-|S|}{|T||S|}\right)} \tag{4}$$

where $\bar{\pi}_i^{(S,T)}$ is the power of agent i in the intersection of S and T with the highest power.

In Equation 3, any $\lambda \leq \underline{\lambda}$ will convince agents in the intersection to choose S over T when S has the lower power by putting more weight on proportional sharing. Note that we only need to convince the agent with the lowest power $\underline{\pi}_i$ since he will have the least to gain under proportional sharing. If this lowest-powered agent's share is higher under this level of λ, then any other agent in the intersection of S and T with a higher power will also have a higher share in coalition S. In the same manner, Equation 4 provides the incentives to choose a smaller-sized coalition over a larger coalition (but with a lower power) by putting more weight on equal sharing. Note that only the agent in the intersection with the largest power must be convinced to choose the smaller-sized coalition since he has the most to lose in moving to a level of λ that puts more weight on equal sharing.

As a corollary, λ values between $\underline{\lambda}^{(S,T)}(N,\pi)$ and $\bar{\lambda}^{(S,T)}(N,\pi)$ create disagreement between the agents in the intersection of S and T. Particularly, when $\lambda > \underline{\lambda}^{(S,T)}(N,\pi)$ the lowest powered agent in the intersection prefers the coalition with the smaller size (and higher power) while for $\lambda < \bar{\lambda}^{(S,T)}(N,\pi)$ the highest powered agent in the intersection prefers the lower powered coalition (with larger size).

In order to define the transition correspondence over these non-SPM games, we first note that the set of self-enforcing coalitions coincide for all sharing rules, even those that does not satisfy consistent ranking. This is shown in Lemma 1 in Appendix A. What this means is that the set of self-enforcing coalitions does not depend on the sharing rule but only on the vector of powers in the game (N,π). Hence, we can construct the class of "stable games" generated by the self-enforcing coalitions in the game (N,π).[19]

From these stable games we can define the set

[19]The concept of stable games and the sets $SEC(N,\pi)$ and $UDC(N,\pi)$ are introduced in Appendix A.1.

$$SEC(N,\pi) = \{T \subsetneq N \mid (T,\pi_T) \text{ is stable and } T \in W_{(N,\pi)}\}$$

as the set of coalitions that generate a stable game and are winning in the game (N,π).

Finally, we can define the set of *undominated coalitions*.

$$UDC(N,\pi) = \{T \subseteq N \,|\, T \in SEC(N,\pi) \; \nexists S \in SEC(N,\pi), \xi_i(S,\pi_S) > \xi_i(T,\pi_T) \forall i \in S \cap T\}$$

The set $UDC(N,\pi)$ contains coalitions that can have potential disagreements between agents given a sharing rule ξ. To illustrate these sets, refer to Appendix B.1.

We are now in the position to calculate for the game $(N,\pi) \notin \bar{G}$ two threshold values $\underline{\Lambda}(N,\pi)$ and $\bar{\Lambda}(N,\pi)$ over the set $UDC(N,\pi)$. The threshold value $\underline{\Lambda}(N,\pi)$ ensures that for any $\lambda \leq \underline{\Lambda}(N,\pi)$ the lowest-powered coalition (with the largest size) in $UDC(N,\pi)$ will be unanimously preferred by all agents over all other coalitions in $UDC(N,\pi)$ to which they could possibly belong. On the other hand, for $\lambda \geq \bar{\Lambda}(N,\pi)$ the smallest-sized coalition in $UDC(N,\pi)$ will be unanimously preferred by all agents. These values are:

$$\underline{\Lambda}(N,\pi) = \min_{T} \underline{\lambda}^{(\underline{S},T)}(N,\pi) \tag{5}$$

where $\underline{S}, T \in UDC(N,\pi)$ and $|\underline{S}| > |T| \;\forall\; T \in UDC(N,\pi)$

and

$$\bar{\Lambda}(N,\pi) = \max_{T} \bar{\lambda}^{(\bar{S},T)}(N,\pi) \tag{6}$$

where $\bar{S}, T \in UDC(N,\pi)$ and $|\bar{S}| < |T| \;\forall\; T \in UDC(N,\pi)$

Refer to Appendix B.1 for an illustration of these threshold values.

In some games we may also find an interval within $\lambda \in (0,1)$ where a "compromise coalition" (with size between the smallest-sized and largest-sized cooalition) will be unanimously preferred by agents in the intersection. These compromise coalitions will exist whenever there exists a coalition Q where $|S| < |Q| < |T|$ and $S, Q, T \in UDC(N,\pi)$ such that:

$$\max_{T} \bar{\lambda}^{(T,Q)}(N,\pi) < \min_{S} \underline{\lambda}^{(S,Q)}(N,\pi) \tag{7}$$

Label these values as $\bar{\Lambda}^{Q}(N,\pi) = \max_{T} \bar{\lambda}^{(T,Q)}(N,\pi)$ and $\underline{\Lambda}^{Q}(N,\pi) = \min_{S} \underline{\lambda}^{(S,Q)}(N,\pi)$. (Please refer to Appendix B.1 for an illustration on the existence of these compromise coalitions.)

The next proposition establishes the largest class of non-SPM games for which a transition correspondence will satisfy SE and RAT. In these games we basically need the combination sharing parameter λ to be either low enough for agents in the agents to unanimously pick the coalition with the smallest power (but larger size), high enough for them to pick the coalition with the smallest size, or just right for a compromise coalition to be picked.

Formally, consider the classes of games such that

$$G^*(\lambda) = \{(S,\pi_S)|\lambda \in [0, \underline{\Lambda}(S,\pi_S)] \cup \left[\bar{\Lambda}(S,\pi_S), 1\right] \cup \bigcup_{Q \in UDC(S,\pi_S)} \left[\bar{\Lambda}^Q(S,\pi_S), \underline{\Lambda}^Q(S,\pi_S)\right]\}$$

Proposition 3 *Consider the sharing rule CS^λ and let $\breve{\phi}$ be a transition correspondence defined over the class of games in $\breve{G}$ that satisfies CR. Then,*

a. $\breve{G} \subset G^*(\lambda)$

b. The transition correspondence $\breve{\phi} : \breve{G} \to 2^N$ satisfies SE and RAT, if and only if:

i. if $\lambda \in \left[\bar{\Lambda}(S,\pi_S), 1\right]$,

$$\breve{\phi}(S,\pi_S) = \underset{V \in UDC(S,\pi_S)}{\arg\min} |V|$$

ii. if $\lambda \in [0, \underline{\Lambda}(S,\pi_S)]$,

$$\breve{\phi}(S,\pi_S) = \underset{V \in UDC(S,\pi_S)}{\arg\min} \pi(V)$$

iii. if $\bar{\Lambda}^Q(S,\pi_S) \leq \lambda \leq \underline{\Lambda}^Q(S,\pi_S)$

$$\breve{\phi}(S,\pi_S) = \left\{Q \mid Q \in UDC(S,\pi_S), \bar{\Lambda}^Q(S,\pi_S) < \underline{\Lambda}^Q(S,\pi_S)\right\}$$

Parts (i), (ii) and (iii) ensures that the chosen coalition will give the agents in the intersection of competing coalitions their highest possible share of the prize. This is achieved by choosing the coalition with the smallest size in the case where λ is high enough, choosing the coalition of the least power in the case where λ is low enough, and choosing all compromise coalitions whenever λ is just right for every agent in the intersection to agree on these coalitions. The next Remark follows readily from Proposition 3.

Remark 1 *i. For any coalition formation game* (N,π)*, there is always a compromise sharing* λ *such that* $(N,\pi) \in G^*(\lambda)$*. Thus, the transition correspondence* $\breve{\phi}$ *satisfies SE and RAT at* $G^*(\lambda)$*, and there is a stable solution at game* (N,π)*.*

ii. For any coalition formation game (N,π) *and sharing rule* CS^{λ}*, there is always a coalition* $S \subset N$ *who can give up their power to* $\tilde{\pi}_S \leq \pi_S$ *such that the game* $(N,(\tilde{\pi}_S,\pi_{-S})) \in G^*(\lambda)$*. Thus, the transition correspondence* $\breve{\phi}$ *satisfies SE and RAT at* $G^*(\lambda)$*, and there is a stable solution at game* $(N,(\tilde{\pi}_S,\pi_{-S}))$*.*

Appendix B has the example illustrating this remark. Two lesson from this remark are that potential disagreements among agents on the choice of coalitions need not lead to a complete breakdown of cooperation. Regardless of the initial configuration of power, it is always possible to find some suitable 'stable' compromises if only agents agree to either set the correct method of sharing the prize (λ) or alternatively give up power.[20] This is an optimistic observation, and can very well be applied to any activity that requires cooperation. Problems from climate change to water scarcity may be solved through collective action if only the cost and benefits are appropriately shared by everyone concerned.

5 Conclusion

This paper develops an axiomatic approach to a coalition formation model by focusing on two main axioms: self-enforcement and rationality. We investigate the effect of different sharing rules on the existence of transition correspondences that satisfy these axioms. Our results show that the existence of these transition correspondences is very sensitive to the choice of sharing rules.

We find that when the sharing rule satisfies the property of consistent ranking (where agents have the same ordinal rank over coalitions) then we can always find a transition correspondence that satisfies these axioms. Under combination sharing, however, these transition correspondences do not exist in general. We have to restrict the domain of games either to the case where coalition size and power move in the same direction or by allowing the sharing parameter λ to be high enough for agents to agree on the smallest-sized undominated coalition, low enough for agents to agree on the least-powered undominated coalition, or just enough for a compromise coalition to exist.

[20]Giving up power is just one way to change the power profile. Another way of changing such a profile includes the transferring of power between agents. An open question that we leave for future studies is to find the 'minimal conditions' on the change of the power to guarantee the stability of the sharing rule CS^{λ}.

A Appendix

A.1 General Results without Consistent Ranking: extension of Proposition 1

In this Appendix we discuss the necessary and sufficient conditions on the sharing rule that allows for the existence of a self-enforcing and rational transition correspondence. In particular, we show that the assumption of consistent ranking in Proposition 1 can be extended to a larger class of sharing rules that satisfies a property which we call "generalized consistent ranking".

First, we show that the set of self-enforcing coalition coincide even for sharing rules that do not satisfy consistent ranking.

Lemma 1 *Consider the self-enforcing and minimalistic transition correspondences* ϕ *and* $\tilde{\phi}$ *for the sharing rules* ξ *and* $\tilde{\xi}$*, respectively. Then, for a given power vector* π*, the sets of coalitions that are self-enforcing coincide. That is,*

$$\{S|S \in \phi(S, \pi)\} = \{T|T \in \tilde{\phi}(T, \pi)\}.$$

Proof. Consider the sets $A^u = \{S|S \in \phi(S, \pi), |S| \leq u\}$ and $B^u = \{T|T \in \tilde{\phi}(T, \pi), |T| \leq u\}$.

We will prove by induction on the size of u that $A^u = B^u$.

This is clearly true if $u = 1$, because any singleton coalition is self-enforcing.

For the induction hypothesis, assume that $A^{u-1} = B^{u-1}$.

Consider $S \in A^u$. Then $S \in \phi(S, \pi)$. Therefore, since ϕ is minimalistic, there is no $Q \subsetneq S$ such that $Q \in W_{(S,\pi)}$ and $Q \in \phi(Q, \pi_Q)$.

Therefore, since $A^{u-1} = B^{u-1}$, there is no $Q \subsetneq S$ such that $Q \in W_{(S,\pi)}$ and $Q \in \tilde{\phi}(Q, \pi_Q)$.

Hence, $S \in \tilde{\phi}(S, \pi)$ and $S \in B^u$. Thus $A^u \subset B^u$.

We can similarly prove that $B^u \subset A^u$. ■

The game (S, π) is *stable* if the coalition S is self-enforcing at the vector of power π under any other self-enforcing and minimalistic transition correspondence (for instance, under PR or ES). By Lemma 1, this is well defined.

The class of stable games can be easily constructed. First, note that any coalition of size one is stable. Second, note that a game of size larger than 1 is stable if and only if no subgame that is winning is stable. Hence, the construction of the stable games is made inductively on the size of the coalition. Example 2 shows the class of coalitions that generate a stable game for a power profile with 8 agents.

Example 2 *Consider an 8-agent society with power profile*

$$\pi = [12, 11.5, 11, 10.5, 10, 9, 9.5, 8].$$

The table below gives the coalitions from the set of stable games for different coalition sizes:

Coalition size	Coalitions that generate a stable game for the power profile π
1	$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}$
2	None, a coalition of size 1 can deviate
3	$\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,2,6\}, \{1,2,7\}, \{1,2,8\}, \{2,3,4\}, \{2,3,5\}, \{2,3,6\}, \{2,3,7\}, \{2,3,8\}, \{3,4,5\}, \{3,4,6\}, \{3,4,7\}, \{3,4,8\}, \{4,5,6\}, \{4,5,7\}, \{4,5,8\}, \{5,6,7\}, \{5,6,8\}, \{6,7,8\}, \{1,3,4\}, \{1,3,5\}, \{1,3,6\}, \{1,3,7\}, \{1,3,8\}, \{1,4,5\}, \{1,4,6\}, \{1,4,7\}, \{1,4,8\}, \{1,5,6\}, \{1,5,7\}, \{1,5,8\}, \{1,6,7\}, \{1,6,8\}, \{1,7,8\}, \{2,4,5\}, \{2,4,6\}, \{2,4,7\}, \{2,4,8\}, \{2,5,6\}, \{2,5,7\}, \{2,5,8\}, \{2,6,7\}, \{2,6,8\}, \{2,7,8\}, \{3,5,6\}, \{3,5,7\}, \{3,5,8\}, \{3,6,7\}, \{3,6,8\}, \{3,7,8\}, \{4,6,7\}, \{4,6,8\}, \{4,7,8\}, \{5,7,8\}$
4	None, a coalition of size 3 can deviate
5	None, a coalition of size 3 can deviate
6	None, a coalition of size 3 can deviate
7	$\{1,2,3,4,5,6,7\}, \{2,3,4,5,6,7,8\}, \{1,3,4,5,6,7,8\}, \{1,2,4,5,6,7,8\}$ $\{1,2,3,5,6,7,8\}, \{1,2,3,4,6,7,8\}, \{1,2,3,4,5,7,8\}, (\{1,2,3,4,5,6,8\}$
8	None, a coalition of size 7 can deviate

All singleton coalitions generate a stable game since there are no other possible deviations. Coalitions of size 2 do not generate a stable game because the agent with the highest power can deviate and form a singleton coalition. Coalitions of size 3 generate a stable game because no single agent has enough power to deviate. Furthermore, coalitions of size 2 cannot deviate from a coalition of size 3 since they do not generate a stable game. Coalitions of size 4 do not generate a stable game since the three highest powered agents can deviate from this coalition and form a stable game. This is the same for coalitions of size 5 or 6. Coalitions of size 7 generate a stable game since a winning subcoalition needs to be at least of size 4, and we know that coalitions of sizes 4, 5, and 6 do not generate a stable game. The grand coalition does not generate a stable game because any coalition of size 7 can deviate and form a stable game.

Definition 6 • *Consider the game (N, π) and let*

$$SEC(N, \pi) = \{T \subsetneq N \mid (T, \pi_T) \text{ is stable and } T \in W_{(N,\pi)}\}$$

be the set of coalitions that generate a stable game and are winning in the game (N, π).

- *Consider the game* (N, π) *and let*

$$UD(N,\pi) = \{(T,\pi_T) \mid T \in SEC(N,\pi)\ \nexists S \in SEC(N,\pi), \xi_i(S,\pi_S) > \xi_i(T,\pi_T) \forall i \in S \cap T\}$$

be the set of undominated stable games that are winning in (N, π).

Definition 7 (Generalized consistent ranking) *A sharing rule* ξ *satisfies generalized consistent ranking (GC) if for any game* (N, π)*:*

a. *The set* $UD(N, \pi) \neq \emptyset$.

b. *For the game* $(S, \pi_S) \in UD(N, \pi)$ *and* $(V, \pi_V) \notin UD(N, \pi)$ *such that* $V \in SEC(N, \pi)$*, we have that* $\xi_i(S, \pi_S) > \xi_i(V, \pi_V)$ *for all* $i \in S \cap V$.

Generalized consistent ranking requires that for every game, there exists a winning and stable subgame that is not dominated[21] by any other winning and stable subgame. Therefore, it cannot be that every game in $SEC(N, \pi)$ is dominated by another game in the same set. Moreover, it requires that every undominated game dominates every other game in $SEC(N, \pi)$ that is not undominated.

Note that if a sharing rule satisfies consistent ranking, then it satisfies GC. This is because the agents share the same ordinal ranking R^ξ over games. Therefore, $UD(N, \pi)$ coincides with the set of games that maximize R^ξ.

Example 3 *The class of sharing rules that satisfy generalized consistent ranking contains a large class of important sharing rules not covered by consistent ranking.*

The definition of generalized consistent ranking only imposes restrictions on the class of stable games. Thus, for instance, a sharing rule will meet GC if it satisfies consistent ranking within the class of stable games (and does not necessarily satisfy consistent ranking for games that are not stable). One example of such sharing rule might split the resource in proportion (or equally) within the class of stable games, and use combination sharing outside the class of stable games.[22]

Alternatively, note that the sharing rule does not need to satisfy consistent ranking within the class of stable games. For instance, consider a convex combination of dictatorial sharing and proportional, where 90% of the resource is allocated to a single agent following a priority ordering and the remaining 10% of the resource is allocated to all the agents in proportion to their power. For instance, if the priority ordering is $1 \succ 2 \succ \cdots \succ n$*, then for all the stable games that contain agent 1, 90% of the resource is given to agent 1 and the remaining 10% is split between all the agents in proportion*

[21] Coordinate by coordinate for the agents in the intersection.

[22] As long as cross-monotonicity is satisfied.

to their power. For all the stable games that contain agent 2 but do not contain agent 1, 90% of the resource is given to agent 2 and the remaining 10% is split between all the agents in proportion to their power, and so forth. This rule satisfies generalized consistent ranking for the game $(N, \pi) = (\{1, 2, 3, 4, 5\}, (18.5, 21, 20, 19, 18.6))$ *but does not satisfy consistent ranking. To see this, note that any game for three agents is stable. However, the only undominated game is the one with coalition* $\{1, 4, 5\}$ *because this coalition is prefered by agents 1,4 and 5 over any other coalition that contains three agents. Clearly consistent ranking is not satisfied, for instance, for the games with coalitions* $\{1, 2, 3\}$ *and* $\{2, 3, 4\}$*, agent 3 prefers the former whereas agent 2 prefers the latter.*

Note that the sharing rule does not need to be asymmetric. For instance, consider the sharing rule that allocates 90% of the resource to the agent with the largest power and 10% of the resource is split to all the agents in proportion to their power. Then, for the starting game $(N, \pi) = (\{1, 2, 3, 4, 5\}, (18.5, 21, 20, 19, 18.6))$*, only coalitions of size 3 are stable. Similarly as above,* $\{1, 4, 5\}$ *is the only undominated coalition. This rule does not satisfy consistent ranking.*[23]

The following result provides the complete class of sharing rules that allow the compatibility of SE and RAT. It also provides the unique transition correspondence that meets SE and RAT. Proposition 1(*ii-iii*) is a straighforward consequence of this result.

Proposition 4 *There exists a transition correspondence* ϕ *that satisfies SE and RAT under the sharing rule* ξ *if and only if* ξ *meets GC. Moreover, if* ϕ *satisfies SE and RAT then*

$$\phi(N, \pi) = \{T | (T, \pi_T) \in UD(N, \pi)\}.$$

Proof.

We prove this result in four steps.

Step 1. Suppose ϕ and $\tilde{\phi}$ are transition correspondences that satisfy SE and RAT under the sharing function ξ. Then, $\phi = \tilde{\phi}$.

Proof. Suppose there is a game (N, π) such that $\phi(N, \pi) \neq \tilde{\phi}(N, \pi)$. Without loss of generality, let $T \in \tilde{\phi}(N, \pi) \setminus \phi(N, \pi)$ and let $S \in \phi(N, \pi)$. Note that S and T are self-enforcing under ϕ and $\tilde{\phi}$ by Lemma 1. Hence, by the rationality of ϕ, $\xi_i(S, \pi_S) > \xi_i(T, \pi_T)$ for all $i \in T \cap S$.

[23] Note that the sharing rules provided in this example only work locally for this specific game. The extension of the above rules to the entire class of games requires more details, but they are omitted due to space constraints.

If $S \in \tilde{\phi}(N,\pi)$, then by the rationality of $\tilde{\phi}$, $T \notin \tilde{\phi}(N,\pi)$. This is a contradiction.

On the other hand, if $S \notin \tilde{\phi}(N,\pi)$, then by the rationality of $\tilde{\phi}$, we have that $\xi_i(S,\pi_S) < \xi_i(T,\pi_T)$ for all $i \in T \cap S$. This is a contradiction.

Therefore, $\tilde{\phi}(N,\pi) \setminus \phi(N,\pi) = \emptyset$. Hence, $\tilde{\phi}(N,\pi) \subset \phi(N,\pi)$. By a similar argument we can show that $\phi(N,\pi) \subset \tilde{\phi}(N,\pi)$.

Step 2. Let

$$DOM(N,\pi) = \{S \in SEC(N,\pi) | \exists T \in SEC(N,\pi) \text{ s.t. } \xi_i(T,\pi_T) > \xi_i(S,\pi_S) \forall i \in S \cap T\}$$

be the set of dominated coalitions in (N,π). Then, for any ϕ that satisfies RAT and SE, we have that

$$\phi(N,\pi) \cap DOM(N,\pi) = \emptyset.$$

Proof. We prove it by contradiction. Suppose that $S \in \phi(N,\pi) \cap DOM(N,\pi)$. Then, there exists $T \in SEC(N,\pi)$ such that $\xi_i(T,\pi_T) > \xi_i(S,\pi_S)$ for all $i \in S \cap T$. Since ϕ satisfies rationality, then it also satisfies minimalistic. Therefore, by Lemma 1, $T \in \phi(T,\pi_T)$.

If $T \in \phi(N,\pi)$, then $S \notin \phi(N,\pi)$ by rationality, which is a contradiction.

On the other hand, if $T \notin \phi(N,\pi)$, then $\xi_i(S,\pi_S) > \xi_i(T,\pi_T)$ for all $i \in S \cap T$. This is a contradiction.

Hence, $\phi(N,\pi) \cap DOM(N,\pi) = \emptyset$.

Step 3. We prove necessity.

Consider the transition correspondence ϕ that satisfies SE and RAT. First, note that RAT implies minimalistic; therefore, ϕ satisfies the conditions of Lemma 1. Let (N,π) be a coalition formation game and let $Q \in \phi(N,\pi)$. By SE, $Q \in \phi(Q,\pi_Q)$. Thus, by Lemma 1, the game (Q,π_Q) is stable. Therefore, $(Q,\pi_Q) \in SEC(N,\pi)$.

By step 2, $(Q,\pi_Q) \in UD(N,\pi)$. Therefore, $UD(N,\pi) \neq \emptyset$.

Let $(T,\pi_T) \in UD(N,\pi)$. Then, $T \in SEC(N,\pi)$. Thus, by Lemma 1, $T \in \phi(T,\pi_T)$.

If $T \notin \phi(N,\pi)$, then $\xi_i(Q,\pi_Q) > \xi_i(T,\pi_T)$ for all $i \in T \cap Q$. Thus, $(T,\pi_T) \notin UD(N,\pi)$. This is a contradiction. Therefore, $T \in \phi(N,\pi)$. Hence,

$$\{T | (T,\pi_T) \in UD(N,\pi)\} \subset \phi(N,\pi).$$

Let $S \in \phi(N,\pi)$. Then, S is self-enforcing. Thus, $S \in SEC(N,\pi)$. By step 2, $S \notin DOM(N,\pi)$. Hence, $(S,\pi_S) \in UD(N,\pi)$. Therefore,

$$\phi(N,\pi) \subset \{T | (T,\pi_T) \in UD(N,\pi)\}.$$

Hence,

$$\phi(N, \pi) = \{T | (T, \pi_T) \in UD(N, \pi)\}.$$

Step 4. We prove sufficiency by constructing a transition correspondence $\hat{\phi}$ that satisfies SE and RAT. Note that, by step 2, this will be unique.

Let $\hat{\phi}$ be the set of coalitions generated by an undominated game. That is,

$$\hat{\phi}(N, \pi) = SEC(N, \pi) \setminus DOM(N, \pi).$$

We show that $\hat{\phi}$ is self-enforcing. To see this, notice that if $S \in \hat{\phi}(N, \pi)$, then (S, π_S) is stable. Therefore, no game that is stable can be winning in (S, π_S). Thus, $SEC(S, \pi_S) = \{S\}$. Thus, $\hat{\phi}(S, \pi_S) = \{S\}$.

We show that $\hat{\phi}$ is rational. To see this, suppose that $Z \notin \hat{\phi}(N, \pi)$, where $Z \in \hat{\phi}(Z, \pi_Z)$ and $Z \in W_{(N,\pi)}$. Since $Z \in \hat{\phi}(Z, \pi_Z)$, then (Z, π_Z) is stable. Thus, $Z \in SEC(N, \pi)$. Since $Z \notin \hat{\phi}(N, \pi)$, then $Z \in DOM(N, \pi)$.

Let $\tilde{Q} \in \hat{\phi}(N, \pi)$. Then, $(Q, \pi_Q) \in UD(N, \pi)$. Therefore, by generalized consistent ranking, $\xi_i(Q, \pi_Q) > \xi_i(Z, \pi_Z)$ for all $i \in Q \cap Z$. Therefore, rationality is satisfied.

■

A.2 Proofs

Proof of Proposition 1

Proof.

Note that, except for step 5, Proposition 1 and its proof is a trivial consequence of Proposition 4. Therefore, the proofs of Steps 1-4 will be removed from the final version of this paper. We leave this proof for the referees only, as an alternative of the main differences with the proof provided by AES in the special case of PR.

Step 1. ϕ^* is SE and minimalistic.

Proof. To show SE, take any $X \in \phi^*(S, \pi_S)$. There are two cases: either $X = S$ or $X \in Q$. If $X = S$, then $X \in \phi^*(S, \pi_S) = \phi^*(X, \pi_X)$. If $X \in Q$, then $X \in \phi^*(X, \pi_X)$ by definition of Q.

On the other hand, ϕ^* is minimalistic because ξ is cross-monotonic. That is, at the coalition formation game (S, π_S), the set S is chosen only if $Q(S, \pi_S) = \emptyset$.

Step 2. ϕ^* satisfies RAT.

Proof. Take $T \in \phi^*(S, \pi_S)$ and consider a coalition Z such that $Z \in W_{(S,\pi)}$ such that $Z \in \phi^*(Z, \pi_Z)$.
($\Rightarrow$) First assume that $Z \notin \phi^*(S, \pi_S)$. Since $T \in \phi^*(S, \pi_S)$ we have that

$$T \in \underset{M \in Q(S,\pi) \cup \{S\}}{\arg\max} R^{\xi}(M, \pi_M)$$

Notice that Z is winning and self-enforcing within S, therefore $Z \in Q(S, \pi) \cup \{S\}$. Moreover, since $Z \notin \phi^*(S, \pi_S)$, then $Z \notin \underset{M \in Q(S,\pi) \cup \{S\}}{\arg\max} R^{\xi}(M, \pi_M)$. Hence, $R^{\xi}(T, \pi_T) > R^{\xi}(Z, \pi_Z)$.

($\Leftarrow$) Now, assume that $R^{\xi}(T, \pi_T) > R^{\xi}(Z, \pi_Z)$. Then, $Z \notin \underset{M \in Q(S,\pi) \cup \{S\}}{\arg\max} R^{\xi}(M, \pi_M)$. Hence, $Z \notin \phi^*(S, \pi_S)$

Step 3. Consider any cross-monotonic sharing rule and transition correspondences ϕ and $\tilde{\phi}$ that are self-enforcing and minimalistic. Then, the sets of coalitions that are self-enforcing coincide. That is,

$$\{S | S \in \phi(S, \pi)\} = \{T | T \in \tilde{\phi}(T, \pi)\}.$$

Proof. The proof of this result is a straighforward consequence of Lemma 1.

Step 4. There exists a unique transition correspondence that meets SE and RAT.

Proof. Consider a transition correspondence ϕ that is SE and RAT. Then, ϕ is minimalistic because the sharing rule is cross-monotonic. We will show that $\phi = \phi^*$.

Since ϕ and ϕ^* are SE and RAT, then by step 3,

$$\{T | T \in \phi(T, \pi_T)\} = \{T | T \in \phi^*(T, \pi_T)\} \tag{8}$$

Suppose $X \in \phi(X, \pi)$. Then, by equation 8, $X \in \phi^*(X, \pi)$. Thus, $\phi(X, \pi) \subset \phi^*(X, \pi)$. Similarly, $\phi^*(X, \pi) \subset \phi(X, \pi)$. Hence, $\phi(X, \pi) = \phi^*(X, \pi)$.

On the other hand, suppose $S \in \phi(X, \pi)$, where $S \neq X$. Then, by RAT, $\xi_i(S, \pi_S) \geq \xi_i(V, \pi_V)$ for any $V \in \{T | T \in \phi(T, \pi_T), T \in W_{(X,\pi)}\}$ and $i \in V \cap T$. Therefore, by consistent ranking, $R^{\xi}(S, \pi_S) \geq R^{\xi}(V, \pi_V)$ for any $V \in \{T | T \in \phi(T, \pi_T), T \in W_{(X,\pi)}\}$. Thus, $R^{\xi}(S, \pi_S) \geq R^{\xi}(V, \pi_V)$ for any $V \in \{T | T \in \phi^*(T, \pi_T), T \in W_{(X,\pi)}\}$. Hence, $S \in \phi^*(X, \pi)$ and $\phi(X, \pi) \subset \phi^*(X, \pi)$. Similarly, $\phi^*(X, \pi) \subset \phi(X, \pi)$. Hence, $\phi(X, \pi) = \phi^*(X, \pi)$.

Step 5. ϕ^* is superior to any transition correspondence that is SE and minimalistic.

Proof. We prove this step by contradiction. Suppose ϕ^* is not superior to the SE and minimalistic transition correspondence $\hat{\phi}$. Then, there exists a game (N, π) such that $S, T \subset N$ where $S \in \phi^*(N, \pi)$ and $T \in \hat{\phi}(N, \pi)$ such that $\xi_i(T, \pi_T) \geq \xi_i(S, \pi_S)$ for some $i \in T \cap S$, and $T \notin \phi^*(N, \pi)$.

By step 3, since T is self-enforcing for $\hat{\phi}$, then it is also self-enforcing for ϕ^*. Therefore, $T \in Q(N, \pi) = \{L \mid L \in W_{(N,\pi)}, L \in \phi(L, \pi_L)\}$.

Since $T \notin \phi^*(N, \pi)$, then $T \notin \underset{M \in Q(N,\pi) \cup \{N\}}{\arg\max} R^\xi(M, \pi_M)$. Since $S \in \phi^*(N, \pi)$, then $S \in \underset{M \in Q(N,\pi) \cup \{N\}}{\arg\max} R^\xi(M, \pi_M)$. Since S and T are winning within N (by the definition of a transition correspondence), we have that $T \cap S \neq \emptyset$. Therefore, $R^\xi(S, \pi_S) > R^\xi(T, \pi_T)$, which implies that $\xi_i(S, \pi_S) > \xi_i(T, \pi_T)$ for $i \in T \cap S$. This is a contradiction.

■

Proof of Proposition 2

Proof. Let the game be $(N, \pi) \in \bar{G}$. To prove the first statement, assume combination sharing and fix a λ. Hence, agent i's share in a coalition S where $S \in \phi(S, \pi_S)$ is:

$$\xi_i(S, \pi_S) = \lambda \cdot \frac{1}{|S|} + (1 - \lambda) \cdot \frac{\pi_i}{\pi(S)}$$

If there exists another coalition T such that $T \in W_{(N,\pi)}$, $T \in \phi(T, \pi_T)$ and $|S| < |T|$, it must be true that

$$\xi_i(S, \pi_S) > \xi_i(T, \pi_T)$$

for all agents in $i \in S \cap T$.

Let $R^{CS^\lambda}(S, \pi) = \lambda \cdot \frac{1}{|S|} + (1 - \lambda) \cdot \frac{\pi_i}{\pi(S)}$. Hence, the sharing rule satisfies consistent ranking and by Proposition 1, the transition correspondence ϕ^* is the unique transition correspondence satisfying SE and RAT.

Under combination sharing, if we want to find a transition correspondence that satisfies SE and RAT for any value of λ

Suppose for the game (S, π_S), we have coalitions $M, V \in UDC(S, \pi_S)$. Coalition M is preferred to coalition V for any value of $\lambda \in (0, 1)$ if and only if

$$\xi_i(M, \pi_M) = \lambda \frac{1}{|M|} + (1 - \lambda) \frac{\pi_i}{\pi(M)} > \lambda \frac{1}{|V|} + (1 - \lambda) \frac{\pi_i}{\pi(V)} = \xi_i(V, \pi_V)$$

This only happens when $|M| < |V|$ **and** $\pi(M) < \pi(V)$.

Furthermore, if $T \in \breve{\phi}(S, \pi_S)$ such that $T = \underset{V \in UD(S,\pi_S)}{\arg\min} \pi(V) = \underset{V \in UDC(S,\pi_S)}{\arg\min} |V|$ then it is guaranteed that $\xi_i(T, \pi_T) > \xi_i(M, \pi_M)$ for any other $M \in UDC(S, \pi_S)$.

■

Proof of Proposition 3

Proof. To prove part (a), we prove some ancillary steps. Suppose $(N,\pi) \notin \bar{G}$. Note that for any pair of coalitions $S, T \in UDC(N,\pi)$ where $|S| < |T|$, we have that $\underline{\lambda}^{(S,T)}(N,\pi) < \bar{\lambda}^{(S,T)}(N,\pi)$. This is because $\frac{\left(\frac{|T|-|S|}{|T||S|}\right)}{\underline{\pi}_i^{(S,T)}\left(\frac{\pi(S)-\pi(T)}{\pi(S)\pi(T)}\right)} > \frac{\left(\frac{|T|-|S|}{|T||S|}\right)}{\bar{\pi}_i^{(S,T)}\left(\frac{\pi(S)-\pi(T)}{\pi(S)\pi(T)}\right)} \Leftrightarrow$ $\underline{\lambda}^{(S,T)}(N,\pi) < \bar{\lambda}^{(S,T)}(N,\pi)$.

Case 1: $\bar{\Lambda}^Q(S,\pi_S) > \underline{\Lambda}^Q(S,\pi_S)$ for all $Q \in UDC(S,\pi_S)$.

This illustrates the case when there is no range for λ where we can find a compromise coalition for the game (S,π_S). Suppose $\bar{\Lambda}(S,\pi_S) > \lambda > \underline{\Lambda}(S,\pi_S)$ and $V \in \phi(S,\pi_S)$ such that $|M| > |V| > |T|$ where M is the coalition of the largest size and T is the coalition of the smallest size in the set $UDC(S,\pi_S)$.

Suppose $\bar{\Lambda}(S,\pi_S) \geq \bar{\lambda}^{(M,V)}(S,\pi_S) > \lambda > \underline{\Lambda}(S,\pi_S) = \underline{\lambda}^{(M,V)}(S,\pi_S)$. Then, for the lowest-powered agent in the intersection of V and M (call him $\underline{i}$), we have that $\xi_{\underline{i}}(V,\pi_V) > \xi_{\underline{i}}(M,\pi_M)$ and there exist some agent $j \in V \cap M$, $j \neq \underline{i}$ such that $\xi_j(V,\pi_V) < \xi_j(M,\pi_M)$. Hence, by RAT we have that $V \notin \phi(S,\pi_S)$, a contradiction.

Case 2.1: $\lambda \notin \bigcup_{Q \in UDC(S,\pi_S)} \left[\bar{\Lambda}^Q(S,\pi_S), \underline{\Lambda}^Q(S,\pi_S)\right]$ and $\lambda > \underline{\Lambda}(S,\pi_S)$.

This examines the case where λ lies out of the interval of any compromise coalition but is larger than $\underline{\Lambda}(S,\pi_S)$. Suppose $M \in \phi(S,\pi_S)$ where $|M| > |Q|$.

If $\lambda < \bar{\Lambda}^Q(S,\pi_S)$, then for the highest-powered agent in the intersection of M and Q (call him $\bar{i}$), we have that $\xi_{\bar{i}}(M,\pi_M) > \xi_{\bar{i}}(Q,\pi_Q)$ and there exist some agent $j \in M \cap Q$, $j \neq \bar{i}$ such that $\xi_j(M,\pi_M) < \xi_j(Q,\pi_Q)$. Hence, by RAT we have that $M \notin \phi(S,\pi_S)$, a contradiction.

Case 2.2: $\lambda \notin \bigcup_{Q \in UDC(S,\pi_S)} \left[\bar{\Lambda}^Q(S,\pi_S), \underline{\Lambda}^Q(S,\pi_S)\right]$ and $\lambda < \bar{\Lambda}(S,\pi_S)$.

This illustrates the case where λ lies out of the interval of any compromise coalition but is lower than $\bar{\Lambda}(S,\pi_S)$ Suppose $M \in \phi(S,\pi_S)$ where $|M| < |Q|$.

If $\lambda > \underline{\Lambda}^Q(S,\pi_S)$, then for the lowest-powered agent in the intersection of M and Q (call him $\underline{i}$), we have that $\xi_{\underline{i}}(M,\pi_M) > \xi_{\underline{i}}(Q,\pi_Q)$ and there exist some agent $j \in M \cap Q$, $j \neq \underline{i}$ such that $\xi_j(M,\pi_M) < \xi_j(Q,\pi_Q)$. Hence, by RAT we have that $M \notin \phi(S,\pi_S)$, a contradiction.

Note that this set of games is well defined. That is, for any game $(S,\pi_S) \in \breve{G}(N,\pi)$ we can define $\underline{\Lambda}(S,\pi_S)$, $\bar{\Lambda}(S,\pi_S)$, $\underline{\Lambda}^Q(S,\pi_S)$ and $\bar{\Lambda}^Q(S,\pi_S)$ for $Q \in UDC(S,\pi_S)$. The intersection of the intervals $[0, \underline{\Lambda}(S,\pi_S)] \cup \left[\bar{\Lambda}(S,\pi_S), 1\right] \cup [\bar{\Lambda}^Q(S,\pi_S), \underline{\Lambda}^Q(S,\pi_S)]$ for all

$(S,\pi_S) \in \breve{G}(N,\pi)$ is nonempty.

Proof of part $(b.i)$

Suppose $\lambda \in \left[\bar{\Lambda}(S,\pi_S),1\right]$ and that $T \in \breve{\phi}(S,\pi_S)$. Assume that $T \neq \underset{V \in UDC(S,\pi_S)}{\arg\min} |V|$. Then for the coalition $M \in UDC(S,\pi_S)$ where $M = \underset{V \in UDC(S,\pi_S)}{\arg\min} |V|$ we have that for all the agents $i \in M \cap T$ we have that $\xi_i(M,\pi_M) > \xi_i(T,\pi_T)$. Hence, by RAT, $T \notin \breve{\phi}(S,\pi_S)$, a contradiction.

Proof of part $(b.ii)$

Suppose $\lambda \in [0,\underline{\Lambda}(S,\pi_S)]$ and that $T \in \breve{\phi}(S,\pi_S)$. Assume that $T \neq \underset{V \in UDC(S,\pi_S)}{\arg\min} \pi(V)$. Then for the coalition $M \in UDC(S,\pi_S)$ where $M = \underset{V \in UDC(S,\pi_S)}{\arg\min} \pi(V)$ we have that for all the agents $i \in M \cap T$ we have that $\xi_i(M,\pi_M) > \xi_i(T,\pi_T)$. Hence, by RAT, $T \notin \breve{\phi}(S,\pi_S)$, a contradiction.

Proof of part $(b.iii)$

Suppose $\bar{\Lambda}^Q(S,\pi_S) \leq \lambda \leq \underline{\Lambda}^Q(S,\pi_S)$ and that $T \in \breve{\phi}(S,\pi_S)$. Assume that $T \notin \left\{Q \mid Q \in UDC(S,\pi_S), \bar{\Lambda}^Q(S,\pi_S) < \underline{\Lambda}^Q(S,\pi_S)\right\}$. Then for the coalition $M \in UDC(S,\pi_S)$ where $M \in \left\{Q \mid Q \in UDC(S,\pi_S), \bar{\Lambda}^Q(S,\pi_S) < \underline{\Lambda}^Q(S,\pi_S)\right\}$ we have that for all the agents $i \in M \cap T$ we have that $\xi_i(M,\pi_M) > \xi_i(T,\pi_T)$. Hence, by RAT $T \notin \breve{\phi}(S,\pi_S)$, a contradiction.

■

B Appendix (Not For Publication)

B.1 Illustrative examples: Compromise λ may exist

Consider the game $(N, \pi) = (\{1,2,3,4,5,6\}, [29, 26.5, 21, 2.49, 2.4, 2.39])$ used in Example 1 in Section 4.2. In that game, all the self-enforcing coalitions are the following: the singletons $\{1\}$,$\{2\}$,$\{3\}$,$\{4\}$,$\{5\}$,$\{6\}$ as well as the coalitions $\{1,2,3\}$, $\{1,2,5,6\}$, $\{1,2,4,5\}$, $\{1,2,4,6\}$, and $\{2,3,4,5,6\}$. The set of stable games generated by these coalitions are the following:

$$(\{1\}, [29]), (\{2\}, [26.5]), (\{3\}, [21]), (\{4\}, [2.49]), (\{5\}, [2.4]), (\{6\}, [2.39])$$

$$(\{1,2,3\}, [29, 26.5, 21]),$$

$$(\{1,2,5,6\}, [29, 26.5, 2.4, 2.39]),$$

$$(\{1,2,4,5\}, [29, 26.5, 2.49, 2.4]),$$

$$(\{1,2,4,6\}, [29, 26.5, 2.49, 2.39]),$$

$$(\{2,3,4,5,6\}, [26.5, 21, 2.49, 2.4, 2.39])$$

In the same example,

$$SEC(N, \pi) = \{\{1,2,3\}, \{1,2,5,6\}, \{1,2,4,5\}, \{1,2,4,6\}, \text{ and } \{2,3,4,5,6\}\}.$$

Here, the singletons are excluded since they are not winning in the game (N, π).

Finally, the set of undominated coalitions under combination sharing are

$$UDC(N, \pi) = \{\{1,2,3\}, \{1,2,5,6\}, \{2,3,4,5,6\}\}$$

This set excludes the coalitions $\{1,2,4,5\}$ and $\{1,2,4,6\}$ since agents in the intersection of $\{1,2,5,6\}$ against either $\{1,2,4,5\}$ or $\{1,2,4,6\}$ prefer $\{1,2,5,6\}$ (since they have similar sizes but $\{1,2,5,6\}$ has the least power). Therefore, $\{1,2,4,5\}$ and $\{1,2,4,6\}$ are dominated by $\{1,2,5,6\}$.

Note that the game $(N, \pi) = (\{1,2,3,4,5,6\}, [29, 26.5, 21, 2.49, 2.4, 2.39])$ in our current example is not size-power monotonic since a two-person coalition like $\{4,5\}$ has less power than the singleton $\{1\}$. Recall in our example that the set UDC contains the winning and self-enforcing coalitions $\{1,2,3\}$, $\{1,2,5,6\}$ and $\{2,3,4,5,6\}$. We calculate the following values based on Equations 3, 4, 5 and 6:

$\underline{\lambda}^{(\{1,2,3\},\{1,2,5,6\})}(N,\pi) = 0.52$

$\underline{\lambda}^{(\{1,2,3\},\{2,3,4,5,6\})}(N,\pi) = 0.44$

$\underline{\lambda}^{(\{1,2,5,6\},\{2,3,4,5,6\})}(N,\pi) = 0.07$

$\bar{\lambda}^{(\{1,2,3\},\{1,2,5,6\})}(N,\pi) = 0.55$

$\bar{\lambda}^{(\{1,2,3\},\{2,3,4,5,6\})}(N,\pi) = 0.50$

$\bar{\lambda}^{(\{1,2,5,6\},\{2,3,4,5,6\})}(N,\pi) = 0.47$

$\underline{\Lambda}(N,\pi) = 0.07$

$\bar{\Lambda}(N,\pi) = 0.55$

If $\lambda \leq \underline{\Lambda}(N,\pi)$, we are assured that $\{2,3,4,5,6\}$ is preferred to either $\{1,2,5,6\}$ or $\{1,2,3\}$. On the other hand, when $\lambda \geq \bar{\Lambda}(N,\pi)$, the coalition $\{1,2,3\}$ is preferred over $\{1,2,5,6\}$ or $\{2,3,4,5,6\}$. Thus, we can always find a λ "low enough" to encourage agents to prefer the coalition with the lowest power and also a λ "high enough" to encourage agents to prefer the coalition with the smallest size.

In this particular example, however, we can also find an interval where a "compromise coalition" (with size between the smallest-sized and largest-sized cooalition) will be unanimously preferred by agents in the intersection. If λ is between 0.7 and 0.52, agents will prefer $\{1,2,5,6\}$ over $\{2,3,4,5,6\}$ and $\{1,2,3\}$. Suppose $\lambda = 0.48$, then:

Comparing $\{1,2,5,6\}$ and $\{1,2,3\}$, we have

$$\xi_1(\{1,2,5,6\},[29,26.5,2.4,2.39]) = 0.37;$$

$$\xi_2(\{1,2,5,6\},[29,26.5,2.4,2.39]) = 0.348;$$

$$\xi_1(\{1,2,3\},[29,26.5,21]) = 0.35;$$

$$\xi_2(\{1,2,3\},[29,26.5,21]) = 0.340;$$

Comparing $\{1,2,5,6\}$ and $\{2,3,4,5,6\}$, we have

$$\xi_2(\{1,2,5,6\},[29,26.5,2.4,2.39]) = 0.348;$$

$$\xi_5(\{1,2,5,6\},[29,26.5,2.4,2.39]) = 0.14;$$

$$\xi_6(\{1,2,5,6\},[29,26.5,2.4,2.39]) = 0.14;$$

$$\xi_2(\{2,3,4,5,6\},[26.5,21,2.49,2.4,2.39]) = 0.347;$$

$$\xi_5(\{2,3,4,5,6\},[26.5,21,2.49,2.4,2.39]) = 0.11;$$

$$\xi_6(\{2,3,4,5,6\},[26.5,21,2.49,2.4,2.39]) = 0.11;$$

In this example every agent in the intersection prefers $\{1,2,5,6\}$. Thus, aside from agreement in the extreme ends of the spectrum of λ, there can also be agreements where agents will prefer the coalition that has neither the smallest size nor the smallest power.

Of course there are games where we cannot find these compromise coalitions. For instance, suppose we decrease agent 3's power to 20 instead of 21 so that the game now becomes:

$$(N,\pi) = (\{1,2,3,4,5,6\},[29,26.5,20,2.49,2.4,2.39]).$$

In this example, the compromise coalition $\{1,2,5,6\}$ does not exist because $\bar{\Lambda}^{\{1,2,5,6\}}(N,\pi) = 0.5155 > \underline{\Lambda}^{\{1,2,5,6\}}(N,\pi) = 0.5151$.

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