

Chapter 7

The Peano Axioms

7.1 An Axiomatic Approach to Mathematics

In our previous chapters, we were very careful when proving our various propositions and theorems to only use results we knew to be true. However, many of the statements that we take to be true had to be proven at some point. Those proofs, of course, relied on other true statements. If we continue to “trace back” our mathematics proofs, we begin to notice that mathematics must have some initial set of true statements that cannot be proven. These statements, known as **axioms**, are the starting point for any mathematical theory.

In this chapter, we will axiomatically define the natural numbers \mathbb{N} . As we move through the various axioms, we will see that each one is crucial in defining \mathbb{N} in such a way that they are equal to $\{0, 1, 2, 3, \dots\}$, which are the natural numbers that we know and love. After establishing this, we will establish the basic arithmetic operations on \mathbb{N} by defining addition and multiplication.

7.2 Axiomatizing the Natural Numbers

In this section, we will develop the Peano Axioms and use them to provide a completely formal definition of the natural numbers \mathbb{N} . In what follows, it is best to train yourself to assume nothing and use only statements that are known to be true via axioms or statements that follow from these axioms. We will formalize the notion of equality and then present the Peano axioms.

7.2.1 The Notion of Equality

When approaching mathematics axiomatically, it is important to not assume anything at all, including something as rudimentary as how equality behaves. After all, “=” is merely a symbol until we declare it to have some important properties. Below, we will define the natural numbers \mathbb{N} axiomatically. Before we get deep into this, let’s establish the properties that “=” should have. First, every natural number should be equal to itself; this is known as the *reflexivity axiom*.

Axiom 1. For every $x \in \mathbb{N}$, $x = x$.

Next, if one natural number equals a second one, then that second one should equal the first one. This is called the *symmetry axiom*.

Axiom 2. For every $x, y \in \mathbb{N}$, if $x = y$, then $y = x$.

The next property allows us to say that if one natural number is equal to a second, and that second natural number is equal to a third, then the first and third are equal to each other. This is known as the *transitivity axiom*.

Axiom 3. For every $x, y, z \in \mathbb{N}$, if $x = y$ and $y = z$, then $x = z$.

The above three properties of reflexivity, symmetry, and transitivity come up numerous times in essentially every mathematical field. Equality is an example of what is called a *relation*, and relations that enjoy the above three properties are known as *equivalence relations*.

There remains one last axiom related to equality. In each of the above axioms, whenever we used a symbol (like x, y , or z), we always had an assumption that these elements were in the natural numbers. If we don't make this assumption, then we may run into problems. To help get around this, the fourth axiom, called the *closure of equality axiom*, says that if you have a natural number that is equal to something, then that "something" also has to be a natural number.

Axiom 4. For all x and y , if $x \in \mathbb{N}$ and $x = y$, then $y \in \mathbb{N}$.

In other words, the only way for something to be equal to a natural number is for it to be a natural number itself.

In different versions of the Peano axioms, the above four axioms are excluded, as they these properties of equality are frequently assumed to be true as part of that logic system. For completeness, though, we include them in these notes. Furthermore, their inclusion here highlights the importance of questioning even the most basic mathematical assumptions.

7.2.2 The Peano Axioms

Now, we are ready to present the main Peano axioms. It is important to keep in mind that when Peano and others constructed these axioms, their goal was to provide the fewest axioms that would generate the natural numbers that everyone was familiar with. The insight is that this could be done by asserting the existence of at least one natural number, and then defining a function, called *successor function*, that can be used to construct the remaining natural numbers.

An obvious element to axiomatically include in the natural numbers is zero.

Axiom 5. 0 is a natural number. That is, $0 \in \mathbb{N}$.

In alternate versions of the Peano axioms, Axiom 5 actually replaces 0 with 1. This creates an almost identical set of natural numbers, which correspond to "positive whole numbers" (as we known them now). Whether a mathematician includes 0 in the natural numbers or not depends on the context. We use the convention of including 0 as a natural number.

At this point, we are only guaranteed the existence of a single natural number, 0. The next axiom uses the successor function to generate other natural numbers. As its name implies, the successor function is a function S that has as its domain \mathbb{N} . The next axiom simply states that the co-domain of S is also \mathbb{N} .

Axiom 6. If $x \in \mathbb{N}$, then $S(x) \in \mathbb{N}$. That is, if x is a natural number, then so is its successor.

As the above axiom implies, we will commonly refer to $S(x)$ as the *successor of x* . Intuitively, we should think of $S(x)$ as $x+1$. Of course, we cannot formally define it this way yet since we do not know what $+$ means! At this point, we are still quite far away from having the natural numbers as we know them. For example, if we have $\mathbb{N} = \{0\}$ and define $S(0) = 0$, then all of the above axioms are satisfied. We, of course, want to avoid this. One way to ensure this is to insist that 0 is not the successor of any natural number (including itself, of course).

Axiom 7. For every natural number $x \in \mathbb{N}$, $S(x) = 0$ is false.

Rephrasing this using our knowledge of functions, we can say that the pre-image of 0 under S in the natural numbers is the empty set. At this point, we are slightly closer to having \mathbb{N} look more like the natural numbers we know. In particular, we know that $S(0)$ is not equal to 0, and thus it must equal some other natural number. We can denote this natural number by 1. Thus, we can define 1 by $S(0) = 1$.

At this point, we know that \mathbb{N} contains at least two natural numbers, 0 and 1. If we were to stop here, though, we could not be guaranteed that all the rest of the natural numbers (as we know them) exist. For example, we could define $\mathbb{N} = \{0, 1\}$ where $S(0) = 1$ and $S(1) = 1$. Again, using our knowledge of functions, we recognize that the fact that $S(0) = 1$ and $S(1) = 1$ means that the S is not an injective function. We would like this successor function to be injective, so we have the following axiom.

Axiom 8. For all $x, y \in \mathbb{N}$, if $S(x) = S(y)$, then $x = y$.

The above axiom has some very important ramifications. First, it excludes the possibility of defining \mathbb{N} to be just $\{0, 1\}$. To see why, notice that we already have that $S(0) = 1$ and, by injectivity, we cannot have that $S(1) = 1$. Axiom 6 excludes the possibility that $S(1) = 0$. Thus, $S(1)$ must be some other natural number, which we denote as 2. Thus, we can define $2 = S(1)$. A similar argument gives that $S(2)$ cannot be 0, 1, or 2. Thus, it must be some other natural number, which we call 3. Continuing in this pattern, we see that \mathbb{N} must contain all the natural numbers that we know!

At this point, we have established that \mathbb{N} must include 0, its successor $1 = S(0)$, its successor's successor $2 = S(1)$, and so on. Thus, we formally have that \mathbb{N} must include $0, S(0), S(S(0)), S(S(S(0))), \dots$. Of course, to avoid so many nested applications of S , we use the numerals 1, 2, 3 to denote $S(0), S(S(0))$, and $S(S(S(0)))$, respectively.

We are, however, not done. These first eight axioms have pushed our formal definition of \mathbb{N} to include all of the “usual” natural numbers that we know and love. That is, we know now that

$$\{0, 1, 2, \dots\} \subset \mathbb{N}.$$

However, what disallows our axiomatic \mathbb{N} from containing more? So far, nothing does.

To see this, let's consider this version of \mathbb{N} that satisfies all the above axioms, but is not the usual natural numbers we know:

$$\mathbb{N} = \{0, 1, 2, 3, \dots\} \cup \{a, b\}.$$

That is, this version of \mathbb{N} contains all the natural numbers *and* also includes two other symbols, a and b . We also need to describe the successor function.

On the portion $\{0, 1, 2, 3, \dots\}$, we define S in the way described above, where $S(0) = 1$, $S(1) = 2$, $S(2) = 3$, and so on. On the portion $\{a, b\}$, we can define $S(a) = b$ and $S(b) = a$. This version of \mathbb{N} with this successor function satisfies all the axioms, but is “larger” than we want our natural numbers to be.

The next (and final) axiom will exclude versions of \mathbb{N} that are “too large” from occurring. Before we begin, we need a definition that is inspired by our previous chapter on induction. A set V is called **inductive** if the following two conditions are satisfied:

- $0 \in V$
- If $x \in V$, then $S(x) \in V$.

Of course, the name comes from the fact that the first condition is similar to our “base case” from induction, and the second condition is analogous to the induction step. The last axiom, many times called the *Axiom of Induction* says that if V is an inductive set, then V contains the set of natural numbers.

Axiom 9. If V is an inductive set, then $\mathbb{N} \subset V$.

As stated above, the first 8 axioms ensure that $\{0, 1, 2, 3, \dots\} \subset \mathbb{N}$. Furthermore, notice that the set $\{0, 1, 2, 3, \dots\}$ is an inductive set! Thus, by Axiom 9, it must be true that $\mathbb{N} \subset \{0, 1, 2, 3, \dots\}$. Thus, we finally have the set equality that we were after:

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}.$$

7.3 Axiomatic Arithmetic

As we know, the importance of \mathbb{N} does not stem from its set-theoretic properties. Rather, \mathbb{N} is of crucial importance because of the arithmetic that we can perform on it. Specifically, addition $+$ and multiplication \cdot touch every single aspect of our academic (and non-academic) lives. Given that so much care was taken to axiomatically define the set of natural numbers, we should be sure to carefully define what is meant by addition and multiplication. The Peano axioms and the successor function allow us to do precisely that.

7.3.1 Addition

We will now, using only the information provided in the Peano Axioms, define the operation $+$ of addition. In what follows, we let $a, b \in \mathbb{N}$. Axiom 5 guarantees that $0 \in \mathbb{N}$, so we begin by defining what it means to add 0. Thus, we define

$$a + 0 = a.$$

To define the sum of any two natural numbers, we use the following recursive definition:

$$a + S(b) = S(a + b).$$

Therefore, if we want to compute $1 + 1$, we note that $1 = S(0)$ and get $1 + 1 = 1 + S(0) = S(1 + 0) = S(1) = 2$. We can proceed further to compute $1 + 2$. To do so, we note that $2 = S(1)$ and therefore that

$$1 + 2 = 1 + S(1) = S(1 + 1) = S(2) = 3.$$

Notice that this latter computation relied on the previous computation of $1 + 1 = 2$ and the fact that we defined 3 to be $S(2)$. Similarly, the first computation of $1 + 1$ relied on using the given fact that $1 + 0 = 1$. Iterating this process enough times will generate the usual addition that we use.

7.3.2 Multiplication

When we first learned multiplication in early grade school, it was taught by asking us to perform addition in an iterative manner. Similarly, axiomatic multiplication is built upon the previously defined axiomatic addition. As with addition, axiomatic multiplication \cdot is given recursively:

$$\begin{aligned}a \cdot 0 &= 0 \\ a \cdot S(b) &= a + (a \cdot b)\end{aligned}$$

Thus, we can easily show that $a \cdot 1 = a$ by noting that $1 = S(0)$ and therefore

$$a \cdot 1 = a \cdot S(0) = a + (a \cdot 0) = a + 0 = a.$$

We can use this to multiply $3 \cdot 2$. Of course, we know that $2 = S(1)$ and therefore

$$3 \cdot 2 = 3 \cdot S(1) = 3 + (3 \cdot 1) = 3 + 3 = 6.$$

Notice that in the last two steps of the computation, we use that $3 \cdot 1 = 3$ (which we proved above) and that $3 + 3 = 6$. This latter statement, of course, can be shown using enough iterations of addition from the previous section.