**Question 1.** Let \( a, b \in \mathbb{Z} \). Show that \( 4 \mid a^2 - b^2 \) if and only if \( a \) and \( b \) are of the same parity.

**Discussion 1.** This is a biconditional statement \( p \iff q \) with \( p \) being “\( 4 \mid a^2 - b^2 \)” and \( q \) being “\( a \) and \( b \) have the same parity”. Thus, we have two statements to prove:

\[
p \Rightarrow q: \text{ “If } 4 \mid a^2 - b^2, \text{ then } a \text{ and } b \text{ have the same parity.” For this statement, will instead prove the contrapositive statement } \neg q \Rightarrow \neg p \text{ given by “If } a \text{ and } b \text{ have opposite parity, then } 4 \nmid a^2 - b^2."
\]

\[
q \Rightarrow p: \text{ “If } a \text{ and } b \text{ have the same parity, then } 4 \mid a^2 - b^2. \text{ This can be proven directly.}
\]

**What we know:**

\[
\begin{align*}
&\cdot \text{ For } \neg q \Rightarrow \neg p, \text{ we know that } a \text{ and } b \text{ have opposite parity. Thus, } a \text{ is odd and } b \text{ is even or } a \text{ is even and } b \text{ is odd. Since the statement is symmetric in } a \text{ and } b, \text{ we only need a proof for one of the cases. So, we will assume that } a \text{ is odd and } b \text{ is even.} \\
&\cdot \text{ For } q \Rightarrow p, \text{ we know that } a \text{ and } b \text{ have the same parity. Thus, they are both even or both odd. We will proceed with a proof by cases.}
\end{align*}
\]

**What we want:**

\[
\begin{align*}
&\cdot \text{ For } \neg q \Rightarrow \neg p, \text{ we wish to conclude that } 4 \nmid a^2 - b^2. \text{ Thus, if we can write } a^2 - b^2 \text{ as one of the three possibilities: } 4x + 1, 4x + 2, \text{ or } 4x + 3 \text{ for some } x \in \mathbb{Z}, \text{ then it is not divisible by 4}. \\
&\cdot \text{ For } q \Rightarrow p, \text{ we wish to conclude that } 4 \mid a^2 - b^2. \text{ Thus, we want to write } a^2 - b^2 \text{ as } 4x \text{ for some } x \in \mathbb{Z}.
\end{align*}
\]

**Proof 1.** To prove this biconditional statement, we will prove the two conditional statements “If \( 4 \mid a^2 - b^2 \), then \( a \) and \( b \) have the same parity” and “If \( a \) and \( b \) have the same parity, then \( 4 \mid a^2 - b^2 \).”

For the first conditional statement, we will instead prove the contrapositive: “If \( a \) and \( b \) have opposite parity, then \( 4 \nmid a^2 - b^2 \).” So, if \( a \) and \( b \) have opposite parity, then \( a \) is odd and \( b \) is even or \( a \) is even and \( b \) is odd. By the symmetry of the statement we wish to prove, we can consider only the first case; the second case will be almost identical. Thus, we assume that \( a \) is odd and \( b \) is even. So, there exist integers \( k_1, k_2 \in \mathbb{Z} \) such that \( a = 2k_1 + 1 \) and \( b = 2k_2 \). Thus,

\[
a^2 - b^2 = (2k_1 + 1)^2 - (2k_2)^2 = (4k_1^2 + 4k_1 + 1) - 4k_2^2 = 4k_1^2 - 4k_2^2 + 4k_1 + 1 = 4(k_1^2 - k_2^2 + k_1) + 1.
\]

Since \( k_1, k_2 \in \mathbb{Z} \), then \( k_1^2 - k_2^2 + k_1 \in \mathbb{Z} \). So, since \( a^2 - b^2 = 4(k_1^2 - k_2^2 + k_1) + 1 \), then \( 4 \nmid a^2 - b^2 \). Thus, the contrapositive is true and the original statement “If \( 4 \mid a^2 - b^2 \), then \( a \) and \( b \) have the same parity” is a true statement.
For the second conditional statement, we assume that $a$ and $b$ have the same parity. Thus, $a$ and $b$ are even or $a$ and $b$ are odd, giving us two cases. In the first case, $a$ and $b$ are even, so there exist integers $k_1, k_2 \in \mathbb{Z}$ such that $a = 2k_1$ and $b = 2k_2$. Thus,

$$a^2 - b^2 = (2k_1)^2 - (2k_2)^2 = 4k_1^2 - 4k_2^2 = 4(k_1^2 - k_2^2).$$

Since $k_1, k_2 \in \mathbb{Z}$, then $k_1^2 - k_2^2 \in \mathbb{Z}$. Since $a^2 - b^2 = 4(k_1^2 - k_2^2)$, then $4 \mid a^2 - b^2$. For the second case, assume that both $a$ and $b$ are odd. Thus, there exist integers $k_1, k_2 \in \mathbb{Z}$ such that $a = 2k_1 + 1$ and $b = 2k_2 + 1$. Thus,

$$a^2 - b^2 = (2k_1 + 1)^2 + (2k_2 + 1)^2 =$$

$$(4k_1^2 + 4k_1 + 1) - (4k_2^2 + 4k_2 + 1) = 4(k_1^2 - k_2^2 + k_1 - k_2).$$

Since $k_1, k_2 \in \mathbb{Z}$, then $k_1^2 - k_2^2 + k_1 - k_2 \in \mathbb{Z}$. Since $a^2 - b^2 = 4(k_1^2 - k_2^2 + k_1 - k_2)$, then $4 \mid a^2 - b^2$. Since, in either case, $4 \mid a^2 - b^2$, then our conditional statement is true.

Since we have proven both conditional statements, our biconditional statement “$4 \mid a^2 - b^2$ if and only if $a$ and $b$ have the same parity” is true. 

□

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**Question 2.**

(a) Let $a \in \mathbb{Z}$. Show that $3 \mid a$ if and only if $3 \mid a^2$.

(b) Use (a) to show that $\sqrt{3}$ is irrational.

**Discussion 2a.** This is a biconditional statement $p \leftrightarrow q$ with $p$ being “$3 \mid a$” and $q$ being “$3 \mid a^2$.” Thus, we will break this up into its two conditional statements:

- $p \Rightarrow q$: “If $3 \mid a$, then $3 \mid a^2$. This will be a direct proof.

- $q \Rightarrow p$: “If $3 \mid a^2$, then $3 \mid a$.” We will instead prove the contrapositive statement $\neg p \Rightarrow \neg q$ given by “If $3 \nmid a$, then $3 \nmid a^2$.”

**What we know:**

- For $p \Rightarrow q$, we will assume that $3 \mid a$. Thus, there exists a $k \in \mathbb{Z}$ such that $a = 3k$.

- For $\neg p \Rightarrow \neg q$, we will assume that $3 \nmid a$. Thus, there exists a $k \in \mathbb{Z}$ such that $a = 3k + 1$ or $a = 3k + 2$. This will give us two cases to consider.

**What we want:**

- For $p \Rightarrow q$, we will need to conclude that $3 \mid a^2$. Thus, we will want to show that $a^2 = 3x$ for some $x \in \mathbb{Z}$.

- For $\neg p \Rightarrow \neg q$, we will need to conclude that $3 \nmid a^2$. Thus, we will want to show that, for some $x \in \mathbb{Z}$, $a^2 = 3x + 1$ or $a^2 = 3x + 2$. 

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Proof 2a. To prove this biconditional statement, we will prove the two conditional statements: “If $3 \mid a$, then $3 \mid a^2$” and “If $3 \mid a^2$, then $3 \mid a$.”

To prove the first conditional statement, we assume that $3 \mid a$. Thus, $a = 3k$ for some $k \in \mathbb{Z}$. Thus,

$$a^2 = (3k)^2 = 9k^2 = 3(3k^2).$$

Since $k \in \mathbb{Z}$, then $3k^2 \in \mathbb{Z}$. Since $a^2 = 3(3k^2)$, then we can conclude that $3 \mid a^2$, as desired.

To prove the second conditional statement, we will instead prove its contrapositive: “If $3 \nmid a$, then $3 \nmid a^2$.” Since $3 \nmid a$, then for some $k \in \mathbb{Z}$, we can write $a$ as $a = 3k + 1$ or $a = 3k + 2$, giving us two cases. In the first case, $a = 3k + 1$ and thus

$$a^2 = (3k + 1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1.$$

Since $k \in \mathbb{Z}$, then $3k^2 + 2k \in \mathbb{Z}$. Since $a^2 = 3(3k^2 + 2k) + 1$, then $3 \nmid a^2$, as desired. For the second case, $a = 3k + 2$. Thus,

$$a^2 = (3k + 2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1.$$

Since $k \in \mathbb{Z}$, then $3k^2 + 4k + 1 \in \mathbb{Z}$. Since $a^2 = 3(3k^2 + 4k + 1) + 1$, we can conclude that $3 \nmid a^2$, as desired. In either case, we conclude that $3 \nmid a^2$. So, the contrapositive statement is true and, thus, our original conditional statement “If $3 \mid a^2$, then $3 \mid a$” is also true.

Since we have proven both conditional statements, the biconditional statement “$3 \mid a$ if and only if $3 \mid a^2$” is true.

\[ \square \]

Discussion 2b. This proof will proceed in a similar fashion to the proof that $\sqrt{2}$ is irrational. Thus, this will be a proof by contradiction, where we assume that $\sqrt{3}$ is rational and can thus be written as $\frac{p}{q}$ in lowest term. Then, we will ultimately arrive at a contradiction by showing that both $p$ and $q$ are divisible by 3. We will use one direction of the above biconditional twice throughout our proof: “If $3 \mid a^2$, then $3 \mid a$.”

Proof 2b. Assume, to the contrary, that $\sqrt{3}$ is rational. Thus, we can write

$$\sqrt{3} = \frac{p}{q},$$

with $p, q \in \mathbb{Z}$, $q \neq 0$, and $p$ and $q$ have no common divisors. Squaring both sides of our equation, we obtain

$$3 = \frac{p^2}{q^2}$$

and, cross-multiplying, we obtain $3q^2 = p^2$. Thus, $3 \mid p^2$. Our previous proof shows us that if $3 \mid p^2$, then $3 \mid p$. Thus, we can write $p = 3r$ for some $r \in \mathbb{Z}$. Substituting, we obtain

$$3q^2 = (3r)^2 = 9r^2,$$

which is equivalent to $q^2 = 3r^2$. Thus, $3 \mid q^2$ and, by our previous proof, $3 \mid q$. Thus, $3 \mid p$ and $3 \mid q$, contradicting the fact that $p$ and $q$ have no common divisors. Thus, our initial assumption that $\sqrt{3}$ is rational is false and thus $\sqrt{3}$ is irrational.

\[ \square \]
**Question 3.** Let \( a, b \in \mathbb{R} \). Show that if \( a + b \) is rational, then \( a \) is irrational or \( b \) is rational.

**Discussion 3.** Our statement is the conditional statement \( p \Rightarrow q \) with \( p \) being “\( a + b \) is rational” and \( q \) being “\( a \) is irrational or \( b \) is rational.” We will instead prove the contrapositive statement \( \neg q \Rightarrow \neg p \). Notice that, to compute \( \neg q \), we must use DeMorgan’s Logic Law, which gives us that \( \neg q \) is “\( a \) is rational and \( b \) is irrational.” Thus, we will prove the contrapositive \( \neg q \Rightarrow \neg p \) given by “If \( a \) is rational and \( b \) is irrational, then \( a + b \) is irrational.

**What we know:** \( a \) is rational and \( b \) is irrational.

**What we want:** \( a + b \) is irrational. Since this is a negative statement, we will use proof by contradiction. Thus, we will assume that \( a + b \) is rational and arrive at a contradiction.

**Proof 3.** To prove our statement, we will instead prove its contrapositive: “If \( a \) is rational and \( b \) is irrational, then \( a + b \) is irrational.” Assume, to the contrary, that \( a + b \) is rational. Then, since \( a \) is rational, \( -a \) is also rational. Since the sum of two rational numbers is rational, then \((a + b) - a = b \) is also rational. This contradicts, however, that \( b \) is irrational. Thus, our initial assumption that \( a + b \) is rational is false, and thus \( a + b \) is irrational. So, we proven the contrapositive to be true and thus the original statement “If \( a + b \) is rational, then \( a \) is irrational or \( b \) is rational” is also true.

\( \square \)