Math 432 - Real Analysis II
Solutions Homework due February 4

Question 1. Recall that a function g is called *bounded* on S if there exists a positive number M such that $|f(x)| < M$ for all $x \in S$. Show that if each f_n is a bounded function that uniformly converges to f on S, then f is also bounded on S.

Solution 1. Since $f_n \to f$ uniformly, there exists an N such that for all $n > N$ and all $x \in S$, $|f_n(x) - f(x)| < 1$. (Here, we set $\varepsilon = 1$, but any positive ε would do). Consider $n = N + 1$. Since each f_n is bounded, there exists some $M > 0$ such that $-M < f_{N+1}(x) < M$ for all $x \in S$. Since $|f_{N+1}(x) - f(x)| < 1$, we know that $f_{N+1}(x) - 1 < f(x) < f_{N+1}(x) + 1$ for all $x \in S$. In particular, we have that

$$-M - 1 < f_{N+1}(x) - 1 < f(x) < f_{N+1}(x) + 1 < M + 1.$$

Thus,

$$|f(x)| < M$$

for all $x \in S$. Thus, $f(x)$ is bounded on S.

Question 2. Consider the function

$$f_n(x) = \frac{n + \cos x}{2n + \sin^2 x}.$$

(a) Compute the pointwise limit $f(x)$ for all $x \in \mathbb{R}$.

(b) Show that f_n converges to f uniformly on $\mathbb{R}$.

(c) Compute

$$\lim_{n\to\infty} \int_2^7 f_n(x)\,dx.$$

Solution 2.

(a) Notice that

$$f_n(x) = \frac{n + \cos x}{2n + \sin^2 x} = \frac{1 + \frac{\cos x}{n}}{2 + \frac{\sin^2 x}{n}}.$$

As $n \to \infty$, the fractions in the numerator and denominator approach 0 (by the Squeeze Theorem, for example). Thus, f_n converges pointwise to $f(x) = 1/2$, the constant function.

(b) Let $\varepsilon > 0$. Notice that

$$f_n(x) - f(x) = \frac{n + \cos x}{2n + \sin^2 x} - \frac{1}{2} = \frac{2\cos x + \sin^2 x}{2(2n + \sin^2 x)}$$

For all values of x, we have that $|2\cos x + \sin^2 x| < 3$ and that $2(2n + \sin^2 x) \geq 4n$. Thus, we have that

$$|f_n(x) - f(x)| < \frac{3}{4n}.$$

Thus, choose $N = \frac{3}{4\varepsilon}$. Thus, for all $n > N = \frac{3}{4\varepsilon}$, we have that $\frac{3}{4n} < \varepsilon$. Thus, for all $n > N$ and all $x \in \mathbb{R}$, we have that

$$|f_n(x) - f(x)| < \frac{3}{4n} < \varepsilon.$$

So, $f_n(x) \to f(x)$ uniformly.

(c) We will not integrate $f_n(x)$ on the interval $[2,7]$. Instead, since $f_n \to f$ uniformly on $\mathbb{R}$ (and therefore on $[2,7]$, we have that

$$\lim_{n\to\infty} \int_2^7 f_n(x)dx = \int_2^7 \lim_{n\to\infty} f_n(x)\,dx = \int_2^7 f(x)\,dx = \int_2^7 \frac{1}{2}\,dx = \frac{5}{2}.$$

Question 3. Consider the power series $\sum_{k=0}^{\infty} a_k x^k$. Show that if $\sum_{k=0}^{\infty} a_k$ converges absolutely as a series, then the power series $\sum_{k=0}^{\infty} a_k x^k$ converges uniformly on $[-1.1]$.

Solution 3. Since $\sum_{k=0}^{\infty} a_k$ converges absolutely, we have that $\sum_{k=0}^{\infty} |a_k| < \infty$. For all $x \in [-1,1]$, we have that $|x| \le 1$ and thus $|x|^k \le 1$. So, $|a_k x^k| = |a_k| \cdot |x|^k \le |a_k|$ for all $x \in [-1,1]$. Since $\sum_{k=0}^{\infty} |a_k| < \infty$ then $\sum_{k=0}^{\infty} a_k x^k$ converges uniformly on $[-1,1]$ by the Weierstrass M-test.

Question 4. Show that $\sum_{k=1}^{\infty} \frac{\cos kx}{k^2}$ converges uniformly on $\mathbb{R}$ to a continuous function.

Solution 4. Since $|\cos kx| \le 1$ for all $x \in \mathbb{R}$, we have that

$$\left|\frac{\cos kx}{k^2}\right| \le \frac{1}{k^2}.$$

Thus, by the Weierstrass M-test, we have that $\sum_{k=1}^{\infty} \frac{\cos kx}{k^2}$ converges uniformly. Since each partial sum is the sum of continuous functions, it is continuous. Thus, since the convergence is uniform, the limit is also continuous on all $\mathbb{R}$.

Question 5.

(a) Let $0 < a < 1$. Show that the series $\sum_{k=0}^{\infty} x^k$ converges uniformly to $1/(1-x)$ on $[-a,a]$.

(b) Does the series $\sum_{k=0}^{\infty} x^k$ converge uniformly on $(-1,1)$?

Solution 5.

(a) We have shown that $\sum_{k=0}^{\infty} x^k$ has a domain of convergence $(-1,1)$, on which it converges pointwise to the function $1/(1-x)$. By a theorem in class, since the radius of convergence is 1, then for any $0 < a < 1$, we have that the power series converges uniformly on $[-a,a]$.

(b) No. Our series does not converge uniformly on $(-1,1)$. To see this, consider

$$\sup\{|x^k| \mid x \in (-1,1)\} = 1.$$

Since $\lim_{k\to\infty} \sup\{|x^k| \mid x \in (-1,1)\} = 1 \neq 0$.