Math 431 - Real Analysis I
Solutions to Test 1

Question 1. Below, you are given an *open* set S and a point $\mathbf{x} \in S$. Thus, by definition of openness, there exists an $\varepsilon > 0$ such that

$$B(\mathbf{x}; \varepsilon) \subset S.$$

Your job is to do the following:

(i) Provide such an $\varepsilon > 0$ that "works".

(ii) Show that your ε is actually positive.

Note: There is no need to *prove* that $B(\mathbf{x}; \varepsilon) \subset S$...that would take too long!

(a) Let $x \in (a, b)$ [the open interval in $\mathbb{R}$ from a to b]. What is an $\varepsilon > 0$ such that $B(x; \varepsilon) \subset (a, b)$?

(b) Let $(u, v) \in T$, where $T = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1\}$. What is an $\varepsilon > 0$ such that $B((u, v); \varepsilon) \subset T$?

(c) Let $(u, v, w) \in (0, 1) \times (0, 1) \times (0, 1)$ [the "unit open cube" in $\mathbb{R}^3$] . What is an $\varepsilon > 0$ such that $B((u, v, w); \varepsilon) \subset (0, 1) \times (0, 1) \times (0, 1)$?

Solution 1.

(a) Let $\varepsilon = \min\{x - a, b - x\}$. Since $x \in (a, b)$, then $a < x < b$. Thus, $x - a > 0$ and $b - x > 0$. So, since the minimum of two positive numbers is positive, then $\varepsilon > 0$.

(b) Let $\varepsilon = \min\{u, 1 - u\}$. Since $(u, v) \in T$, then $0 < u < 1$. Thus, $u > 0$ and $1 - u > 0$. Since the minimum of two positive numbers is positive, $\varepsilon > 0$.

(c) Let $\varepsilon = \min\{u, 1-u, v, 1-v, w, 1-w\}$. Since $(u, v, w) \in (0, 1)\times(0, 1)\times(0, 1)$. Then $0 < u < 1$, $0 < v < 1$, and $0 < w < 1$. Thus, $u, 1 - u, v, 1 - v, w$, and $1 - w$ are all positive. Since the minimum of positive numbers is positive, $\varepsilon > 0$.

Question 2. Let $\{x_n\}$ be a real sequence such that $x_n > 0$ for all n. Show that if x_n converges, then its limit is non-negative. A proof by contradiction might be helpful here.

Solution 2. Assume, to the contrary, that $x_n \to p$ where $p < 0$. Note that $-p/2 > 0$. Then, since $x_n \to p$, there exists some N such that for all $n > N$, $|x_n - p| < -p/2$. Thus, for all $n > N$,

$$p/2 < x_n - p < -p/2.$$

In particular, using the second inequality and adding p to both sides, we get that $x_n < p/2$. Since $p < 0$, $p/2 < 0$ so $x_n < 0$ for all $n > N$. This contradicts that x_n is a positive sequence. Thus, $x_n \to p$ where $p \geq 0$.

Question 3. Let S be a non-empty *open* subset of $\mathbb{R}$ that is bounded above. Show that $\sup S$ exists but that $\max S$ does not exist.

Solution 3. Since S is a non-empty open set that is bounded, by the Completeness Axiom, $\sup S$ exists. We will show that $\max S$ does not exist. Assume, to the contrary, that it does; then, $\max S \in S$. Since S is open, $\max S$ is interior and there exists an $\varepsilon > 0$ such that $B(\max S; \varepsilon) \subset S$. In particular, $\max S + \varepsilon/2 \in B(\max S; \varepsilon)$ and so $\max S + \varepsilon/2 \in S$. However, since $\max S + \varepsilon/2 > \max S$, this contradicts that $\max S$ is the maximal element. Thus, $\max S$ does not exist.

Question 4. Consider the real sequence $\{x_n\}$ given by

$$x_n = \frac{1}{\ln n}$$

for $n \geq 2$. Provide an $\varepsilon - N$ proof that $x_n \to 0$.

Solution 4. Given $\varepsilon > 0$, let $N = e^{1/\varepsilon}$. Then, for all $n > N = e^{1/\varepsilon}$, we have that

$$e^{1/\varepsilon} < n.$$

Taking the natural logarithm of both sides, we have that

$$\frac{1}{\varepsilon} < \ln n.$$

Since $n \geq 2$, $\ln n > 0$ and $\varepsilon > 0$. So we can cross divide to get that

$$\frac{1}{\ln} < \varepsilon.$$

This is equivalent to

$$\left| \frac{1}{\ln n} - 0 \right| < \varepsilon.$$

So, $1/\ln n \to 0$.

Question 5. Consider the sequence in $\mathbb{R}^2$ given by

$$x_n = \left(\frac{1}{n} \cos\left(\frac{\pi n}{2}\right), \frac{1}{n} \sin\left(\frac{\pi n}{2}\right) \right).$$

Recall that we can equip $\mathbb{R}^2$ with several metrics, including the following:

- The L^2 metric, given by

$$d_2((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

- The L^1 metric, given by

$$d_1((x_1, y_1)) = |x_1 - x_2| + |y_1 - y_2|.$$

- The L^∞ metric, given by $d_\infty((x_1, y_1)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$
- The discrete metric, given by

$$d_d((x_1, y_1), (x_2, y_2)) = \begin{cases} 1 & \text{if } (x_1, y_1) \neq (x_2, y_2) \\ 0 & \text{if } (x_1, y_1) = (x_2, y_2) \end{cases}$$

(a) Compute the L^2 distance between x_n and $(0,0)$. That is, compute $d_2(x_n,(0,0))$. Your answer should be written in terms of n.

(b) Use (a) to show that $x_n \to (0,0)$ with the L^2 metric.

(c) Compute the L^1 distance between x_n and $(0,0)$. That is, compute $d_1(x_n,(0,0))$. Your answer should be written in terms of n.

(d) Use (c) to show that $x_n \to (0,0)$ with the L^1 metric.

(e) Compute the L^∞ distance between x_n and $(0,0)$. That is, compute $d_\infty(x_n,(0,0))$. Your answer should be written in terms of n.

(f) Use (d) to show that $x_n \to (0,0)$ in the L^∞ metric.

(g) Show that x_n does not converge to $(0,0)$ in the discrete

Solution 5.

(a) We compute the L^2 distance as

$$d_2(x_n,(0,0)) = \sqrt{\left(\frac{1}{n}\cos\left(\frac{n\pi}{2}\right)\right) + \left(\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\right)^2} = \frac{1}{n}.$$

(b) Since the distance is $1/n$ and $1/n \to 0$ in $\mathbb{R}$, then $x_n \to (0,0)$.

(c) We compute the L^1 distance as

$$d_1(x_n,(0,0)) = \left|\frac{1}{n}\cos\left(\frac{n\pi}{2}\right) - 0\right| + \left|\frac{1}{n}\sin\left(\frac{n\pi}{2}\right) - 0\right| =$$

$$\left|\frac{1}{n}\cos\left(\frac{n\pi}{2}\right)\right| + \left|\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\right| =$$

$$\left|\frac{1}{n}\right|\left(\left|\cos\left(\frac{n\pi}{2}\right)\right| + \left|\sin\left(\frac{n\pi}{2}\right)\right|\right).$$

Notice that $\cos(n\pi/2)$ and $\sin(n\pi/2)$ take on the values $-1, 0$, or 1. In particular, when one is 0, the other is ± 1 and vice versa. Thus, in absolute values, exactly one is 0 and one is 1. So, our distance reduces to

$$d_1(x_n,(0,0)) = 1/n.$$

(d) From (c), $d_1(x_n,(0,0)) = 1/n \to 0$. Thus, $x_n \to (0,0)$ in the L^1 metric.

(e) We compute the L^∞ distance as

$$d_\infty(x_n,(0,0)) = \max\left\{\left|\frac{1}{n}\cos\left(\frac{n\pi}{2}\right) - 0\right|, \left|\frac{1}{n}\sin\left(\frac{n\pi}{2}\right) - 0\right|\right\} =$$

$$\max\left\{\left|\frac{1}{n}\cos\left(\frac{n\pi}{2}\right)\right|, \left|\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\right|\right\} =$$

$$\left|\frac{1}{n}\right|\max\left\{\left|\cos\left(\frac{n\pi}{2}\right)\right|, \left|\sin\left(\frac{n\pi}{2}\right)\right|\right\}.$$

As mentioned above, the two terms are either 0 or 1 at opposite times. Thus, the maximum is equal to 1. So, the distance is given by

$$d_\infty(x_n,(0,0)) = \frac{1}{n}.$$

(f) From (e), we have that $d_\infty(x_n, (0,0)) = \frac{1}{n} \to 0$. Thus, $x_n \to (0,0)$.

(g) Notice that for all n, $|\cos(n\pi/2)| \neq |\sin(n\pi/2)|$. Thus, in particular, they are never both 0 at the same time. So, they are never equal to $(0,0)$. So, $d_d(x_n, (0,0)) = 1$ for all n. This does not converge to 0, so $x_n \not\to (0,0)$ (or any other point).

Question 6.

(a) Show that every compact subset $S \subset \mathbb{R}$ contains a maximum and a minimum element.

(b) Show that the statement "If $S \subset \mathbb{R}$ contains a maximum and a minimum element, then S is compact" is false by finding a counterexample.

Solution 6.

(a) Since S is compact, it is closed and bounded. Since it is bounded and non-empty, the supremum exists. Since the supremum is an adherent point and S is closed, the supremum is an element of the set. Thus, $\max S$ exists. A similar argument shows that $\min S$ exists.

(b) Consider the set $S = [0,1) \cup (2,3]$. Clearly 0 and 3 are the max and min, respectively, of S. However, notice that S is not compact since it is not closed. In particular, its complement is $(-\infty,0) \cup [1,2] \cup (3,\infty)$, which has 1 as a non-interior point. Thus, since the complement is not open, the set S is not closed. Thus, by the Heine-Borel Theorem, S is not compact.

Extra Credit:

(a) Give the equation for the L^p metric on $\mathbb{R}^2$.

Extra Credit Solution.

(a) The L^p metric on $\mathbb{R}^2$ is given by

$$d((x_1, y_1), (x_2, y_2)) = \left(|x_1 - x_2|^p + |y_1 - y_2|^p\right)^{1/p}$$