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Concepts, Theory, and Techniques

SHAREHOLDER LIMITED LIABILITY AND MEAN-VARIANCE MODELS OF CAPITAL STRUCTURE*

S. Ghon Rhee
Department of Finance and Insurance, University of Rhode Island, Kingston, RI 02881

ABSTRACT

This paper demonstrates that shareholder limited liability imposes a restriction on corporate borrowing and that failure to incorporate this restriction into the analysis yields the "reductio ad absurdum" argument against mean-variance models of optimal capital structure. With corporate income taxes and costless bankruptcy, the firm's value is a monotonically increasing function of debt as long as the amount of debt does not exceed the upper limit imposed by shareholder limited liability. As a result, the introduction of costly bankruptcy into the mean-variance framework is justified.

Subject Areas: Capital Asset Pricing Model, Corporate Finance, Financial Planning and Modeling, and Risk and Uncertainty.

INTRODUCTION

In recent years, the capital asset pricing model (CAPM) has been extended to corporate financing decisions, particularly those dealing with the optimal capital structure of a firm. Hamada [8] proved that the Modigliani and Miller [11] [12] propositions hold in the CAPM framework with or without corporate income taxes. Bierman and Oldfield [2] extended these results to include risky debt. Kim [9] and Chen [4] introduced "ex ante" bankruptcy costs into the CAPM to determine the optimal debt ratio of the firm, which balances the present value of tax subsidies against the "ex ante" bankruptcy costs.

Gonzalez, Litzenberger, and Rolfo [6], however, contend that the CAPM is not a suitable means for analyzing the optimal capital structure of the firm. Specifically, they point out that the market value of a levered firm is not a monotonically increasing function of its financial leverage when its end-of-period cash flows are positively correlated with the market portfolio's return, given an economy where interest expenses are tax deductible and corporate bankruptcy is costless. They demonstrate that the firm's value may attain a maximum at a finite level of debt or it may be a decreasing function of its debt even before costly bankruptcy is introduced. Subsequently, their contention effectively questions the validity of an approach toward optimal capital structure which introduces ex ante bankruptcy costs into the CAPM.

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This paper shows that if bankruptcy is costless and debt is not allowed to drive the value of the shareholder's claim below zero, i.e., that the restriction of limited shareholder liability is observed, the value of the firm increases monotonically with debt. If the assumption of costless bankruptcy is replaced with costly bankruptcy, however, then the value of the firm becomes a concave function of debt and monotonicity may no longer hold. Therefore, to obtain the Gonzalez et al. result [6], it is necessary that the shareholder limited-liability constraint be violated. If the constraint is respected, then an interior optimum is obtained only if there are bankruptcy costs. This resolves the inconsistency between Gonzalez et al. [6] and Kim [9] and Chen [4].

VALUATION MODELS

This section derives initially the market value of the firm in the CAPM framework in order to examine the effect of corporate leverage on the firm's value. Under the assumption that (1) a risk-free rate of return exists in a perfect capital market, (2) investors have homogeneous expectations with regard to the probability distribution of future cash flows of risky assets, and (3) investors are risk averse and single-period-expected-utility-of-terminal-wealth maximizers, the CAPM market value of any risky asset can be expressed as

$$V_j = [E(\tilde{Y}_j) - \lambda \mathrm{Cov}(\tilde{Y}_j, \tilde{R}_m)]/r, \tag{1}$$

where V_j = the equilibrium value of asset j,
$\tilde{Y}_j$ = the end-of-period cash flows to the owner of asset j,
$\lambda = [E(\tilde{R}_m) - R_f]/\sigma^2(\tilde{R}_m)$ = the market price of risk,
$\tilde{R}_m$ = the rate of return on the market portfolio,
R_f = the risk-free rate of interest,
$r = 1 + R_f$, and
$E(\cdot)$, $\sigma^2(\cdot)$, and $\mathrm{Cov}(\cdot)$ denote the expected value, variance, and covariance operators, respectively.

The firm issues only common equity and debt, both of which have limited liability. The end-of-period cash flows, $\tilde{X}$, are taxed at a constant corporate income tax rate, t, and we assume that the total amount of debt obligations, D, including principal and interest expense, is tax deductible. This is not a realistic assumption, but the analysis is simplified without loss of generality. When common stockholders are unable to meet the debt obligations with the end-of-period cash flows, $\tilde{X} < D$, the firm is declared bankrupt. Upon bankruptcy, the stockholders' claim on $\tilde{X}$ is lost and its ownership is transferred to bondholders. Therefore, bondholders have the claim on $\tilde{X}$ in a state in which $D > \tilde{X} > 0$ but receive nothing in an extreme case in which $\tilde{X} < 0$. In this sense, bondholders are protected by their limited liability in that they are not responsible for the negative cash flows although ownership of $\tilde{X}$ has been transferred to them. Transfer of ownership from stockholders to bondholders under bankruptcy is assumed to be costless. Thus, the respective cash flow distributions to common stockholders and bondholders are defined as:

$$\tilde{Y}_s = \begin{cases} (\tilde{X}-D)(1-t) & \text{if } \tilde{X}>D \\ 0 & \text{if } D\geq\tilde{X}, \end{cases} \tag{4}$$

$$\tilde{Y}_d = \begin{cases} D & \text{if } \tilde{X}\geq D \\ \tilde{X} & \text{if } D>\tilde{X}>0 \\ 0 & \text{if } 0\geq\tilde{X}, \end{cases} \tag{3}$$

where $\tilde{Y}_s$ = the cash flows to common stockholders and $\tilde{Y}_d$ = the cash flows to bondholders.

The end-of-period cash flow distribution to the firm is obtained by combining (2) and (3) as shown below:

$$\tilde{Y}_l = \tilde{Y}_s + \tilde{Y}_d = \begin{cases} \tilde{X}(1-t)+tD & \text{if } \tilde{X}\geq D \\ \tilde{X} & \text{if } D>\tilde{X}>0 \\ 0 & \text{if } 0\geq\tilde{X}, \end{cases} \tag{4}$$

where $\tilde{Y}_l$ denotes the cash flows to the firm.

Because both common equity and debt are traded in a perfect capital market, they are priced according to the CAPM as defined by Equation (1). The expected value of $\tilde{Y}_s$ from (2), assuming that $\tilde{X}$ is normally distributed, can be expressed as

$$\begin{aligned} E(\tilde{Y}_s) &= \int_D^{\infty}(1-t)(\tilde{X}-D)f(X)\mathrm{d}X \\ &= (1-t)\{E(\tilde{X})[1-F(D)]+\sigma^2(\tilde{X})f(D)-D[1-F(D)]\}, \end{aligned} \tag{5}$$

where $f(\cdot)$ and $F(\cdot)$ denote the normal density function and the cumulative normal distribution function, respectively. We can also define the covariance between $\tilde{Y}_s$ and $\tilde{R}_m$ when $\tilde{X}$ and $\tilde{R}_m$ are assumed to be jointly normally distributed as

$$\begin{aligned} \mathrm{Cov}(\tilde{Y}_s,\tilde{R}_m) &= \int_D^{\infty}\int_{-\infty}^{\infty}[(1-t)(\tilde{X}-D)-E(\tilde{Y}_s)][\tilde{R}_m-E(\tilde{R}_m)]f(X,R_m)\mathrm{d}X\mathrm{d}R_m \\ &\quad -\int_{-\infty}^{D}\int_{-\infty}^{\infty}E(\tilde{Y}_s)[\tilde{R}_m-E(\tilde{R}_m)]f(X,R_m)\mathrm{d}X\mathrm{d}R_m \\ &= (1-t)\mathrm{Cov}(\tilde{X},\tilde{R}_m)[1-F(D)], \end{aligned} \tag{6}$$

where $f(\ ,\ )$ denotes the bivariate normal density function. By substituting Equations (5) and (6) into (1), we express the market value of common stock as

$$V_s = (1-t)\{[E(\tilde{X})-\lambda\mathrm{Cov}(\tilde{X},\tilde{R}_m)][1-F(D)]+\sigma^2(\tilde{X})f(D)-D[1-F(D)]\}/r. \tag{7}$$

The entire numerator of (7) represents the risk-adjusted cash flows, or certainty equivalent of risky cash flows, to stockholders. Since stockholders are protected by limited liability as specified by (2), the risk-adjusted cash flows to stockholders cannot be negative, which, in turn, implies that the risk-adjusted present value of their cash flows cannot be negative. (See [2, p. 952] for a discussion of shareholder limited liability.) Thus,

$$V_s \geq 0. \tag{8}$$

From (7) and (8), we can specify the amount of debt the firm may borrow without violating shareholder limited liability as

$$D < [E(\tilde{X}) - \lambda \mathrm{Cov}(\tilde{X},\tilde{R}_m)] + \sigma^2(\tilde{X}) \frac{f(D)}{1 - F(D)}. \tag{9}$$

If the firm is not subject to bankruptcy risk, shareholder limited liability allows the firm to borrow only up to the certainty equivalent of its end-of-period cash flows. In the presence of bankruptcy risk, the right-hand side of (9) represents the upper limit of debt obligations the firm can incur.[1] Denoting this upper limit by D^*, we obtain:

$$D^* = [E(\tilde{X}) - \lambda \mathrm{Cov}(\tilde{X},\tilde{R}_m)] + \sigma^2(\tilde{X}) \frac{f(D)}{1 - F(D)}. \tag{10}$$

Likewise, from (3) we define the following expressions: $E(\tilde{Y}_d) = D[1 - F(D)] - \sigma^2(\tilde{X})[f(D) - f(0)] + E(\tilde{X})[F(D) - F(0)]$ and $\mathrm{Cov}(\tilde{Y}_d,\tilde{R}_m) = \mathrm{Cov}(\tilde{X},\tilde{R}_m)[F(D) - F(0)]$. Substitution of these expressions into (1) yields the market value of debt:

$$\begin{aligned} V_d = \{&D[1 - F(D)] + [E(\tilde{X}) - \lambda \mathrm{Cov}(\tilde{X},\tilde{R}_m)][F(D) - F(0)] \\ &- \sigma^2(\tilde{X})[f(D) - f(0)]\}/r. \end{aligned} \tag{11}$$

The entire numerator of (11) represents the risk-adjusted cash flows, or certainty equivalent of risky cash flows, to bondholders. The leftmost term in the outer brackets is the total payment to bondholders when the firm remains nonbankrupt. The last two terms represent the risk-adjusted cash flows to bondholders when the firm is bankrupt but its end-of-period cash flows are positive.

By adding (7) and (11), we can express the market value of the firm as

[1]As a referee correctly pointed out, the firm's use of debt financing can be restricted not only by shareholder limited liability but also by other factors such as credit rationing, managerial risk aversion, and imperfect information. In fact, the firm cannot borrow without limit even with unlimited liability.

$$V_l = (1-t)\{[E(\tilde{X}) - \lambda \mathrm{Cov}(\tilde{X},\tilde{R}_m)][1-F(0)] + \sigma^2(\tilde{X})f(0)\}/r$$
$$+ t\{D[1-F(D)] + [E(\tilde{X}) - \lambda \mathrm{Cov}(\tilde{X},\tilde{R}_m)][F(D)-F(0)]$$
$$- \sigma^2(\tilde{X})[f(D)-f(0)]\}/r. \tag{12}$$

(The derivation of Equation (12) is shown in the Appendix.)

When corporate debt is risky but bankruptcy is costless, Equation (12) can be simplified into the familiar tax-correction model of Modigliani and Miller [12]:

$$V_l = V_u + tV_d, \tag{13}$$

where $V_u = (1-t)\{[E(\tilde{X}) - \lambda \mathrm{Cov}(\tilde{X},\tilde{R}_m)][1-F(0)] + \sigma^2(\tilde{X})f(0)\}/r$ = the market value of the unlevered firm,[2] and
V_d = the market value of risky debt.

The market value of the firm as defined by Equations (12) and (13) is used to examine the functional relationship between V_l and D.

THE MARKET VALUE OF THE FIRM AS A FUNCTION OF DEBT

To examine the relationship between the market value of the firm and its debt, we obtain the first- and second-order conditions of (13) with respect to D:

$$r[\mathrm{d}V_l/\mathrm{d}D] = t\{[1-F(D)] - \lambda \mathrm{Cov}(\tilde{X},\tilde{R}_m)f(D)\}, \tag{14}$$

and

$$r[\mathrm{d}^2 V_l/\mathrm{d}D^2] = tf(D)\{\lambda \mathrm{Cov}(\tilde{X},\tilde{R}_m)[D - E(\tilde{X})]/\sigma^2(\tilde{X}) - 1\}, \tag{15}$$

respectively. (The derivation of Equation (14) is shown in the Appendix.)

When $\mathrm{Cov}(\tilde{X},\tilde{R}_m) \leq 0$, the first-order condition as expressed by (14) is strictly positive and the firm's value is an increasing function of its debt. When $\mathrm{Cov}(\tilde{X},\tilde{R}_m) > 0$, the first-order condition is positive, negative, or zero. Based upon this first-order condition, Gonzalez et al. [6] argue that a local interior maximum exists for the firm's value when $\mathrm{Cov}(\tilde{X},\tilde{R}_m) > 0$ and that its market value is not a

---

[2]If the firm was unlevered, its end-of-period cash flows would be

$$\tilde{Y}_u = \begin{cases} (1-t)\tilde{X} & \text{if } \tilde{X} > 0 \\ 0 & \text{if } \tilde{X} \leq 0. \end{cases}$$

Since $E(\tilde{Y}_u) = (1-t)[E(\tilde{X})[1-F(0)] + \sigma^2(\tilde{X})f(0)]$ and $\mathrm{Cov}(\tilde{Y}_u,\tilde{R}_m) = (1-t)\mathrm{Cov}(\tilde{X},\tilde{R}_m)[1-F(0)]$, the market value of the unlevered firm (V_u) can be obtained after substitution of these expressions into Equation (1).

monotonically increasing function of its debt obligations.[3] The necessary and sufficient conditions for the relative extremum are:

$$\frac{1}{\lambda \mathrm{Cov}(\tilde{X},\tilde{R}_m)} = \frac{f(D')}{1-F(D')} \qquad \text{where } r\left[\frac{\mathrm{d}V_l}{\mathrm{d}D}\right]\Bigg|_{D'} = 0. \tag{16}$$

$$D < E(\tilde{X}) + \frac{\sigma^2(\tilde{X})}{\lambda \mathrm{Cov}(\tilde{X},\tilde{R}_m)} \qquad \text{when } r\left[\frac{\mathrm{d}^2V_l}{\mathrm{d}D^2}\right] < 0. \tag{17}$$

From the second-order condition, the unique inflection point is obtained at

$$\overline{D} = E(\tilde{X}) + \frac{\sigma^2(\tilde{X})}{\lambda \mathrm{Cov}(\tilde{X},\tilde{R}_m)}, \tag{18}$$

where $r\left[\frac{\mathrm{d}^2V_l}{\mathrm{d}D^2}\right]\Bigg|_{\overline{D}} = 0$ and $r\left[\frac{\mathrm{d}^3V_l}{\mathrm{d}D^3}\right]\Bigg|_{\overline{D}} = t\lambda \mathrm{Cov}(\tilde{X},\tilde{R}_m)f(\overline{D})/\sigma^2(\tilde{X}) \neq 0.$

As pictured in Figure 1, the first-order condition is positive for $0<D<D'$ and negative for $D>D'$ while the second-order condition is negative for $0<D<\overline{D}$ and positive for $D>\overline{D}$.

Theoretically, we can always determine the level of debt (D') which maximizes the market value of the firm in the presence of corporate income taxes and costless bankruptcy. This conclusion raises a serious problem in determining an optimal leverage ratio, which results from a trade-off of the present value of tax subsidies and the present value of bankruptcy costs in the CAPM. One critical question has yet to be resolved: What are the implications of the restriction on corporate borrowing imposed by shareholder limited liability on the above analysis and conclusion? The following section addresses this question.

---

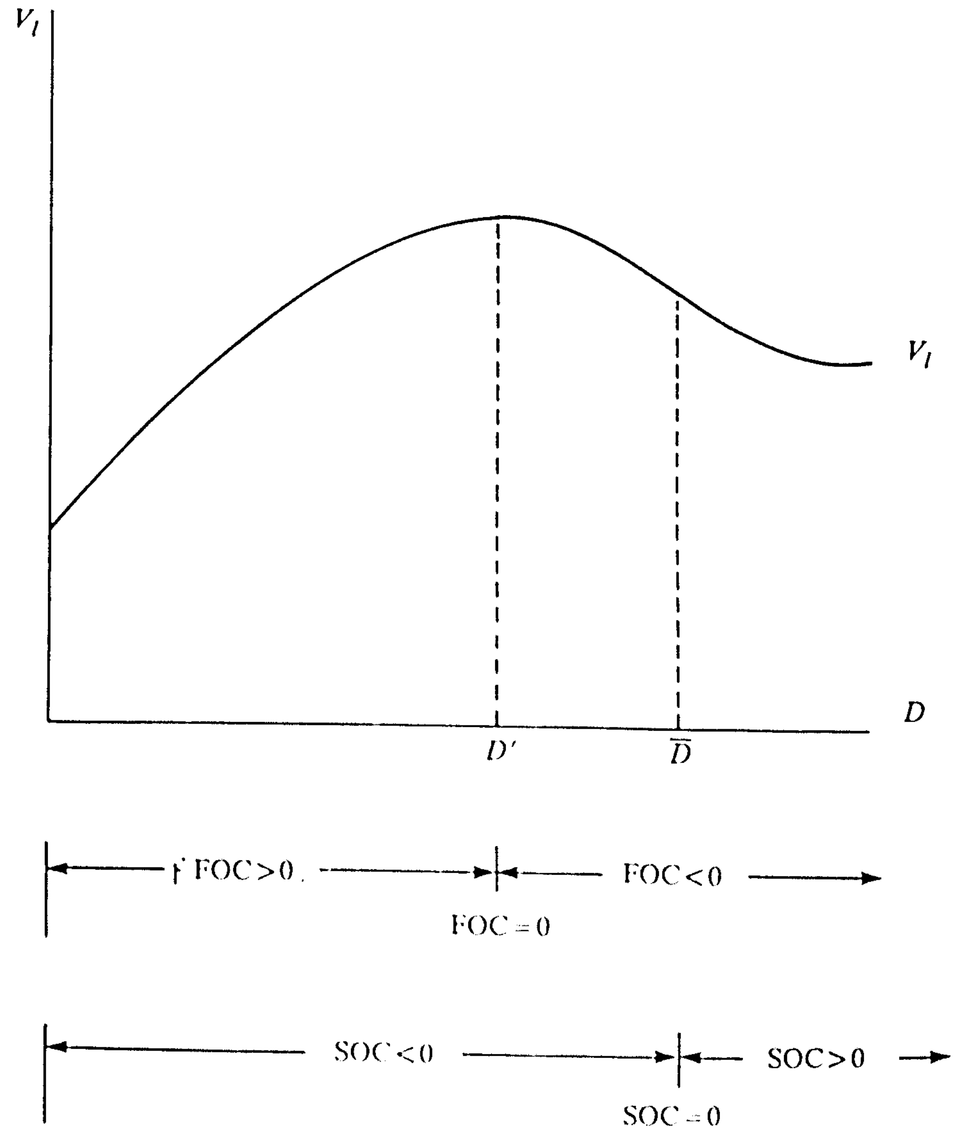
[3]A similar conclusion was obtained by Brennan and Schwartz [3]. They investigated the effect of financial leverage on the value of the firm based upon the relationship between V_l/V_u and D/V_l, where the latter variable is the ratio of the book value of debt to the market value of the firm. It is suspected that their choice of this variable produces an interior maximum for the levered firm. Had the market value leverage ratio been used, the firm-value curve would never have attained a local interior maximum. The reasoning is as follows:

Since $V_l/V_u = 1 + tV_d/V_u = 1 + t\theta/(1-t\theta)$ where $\theta = V_d/V_l$ = the market value leverage ratio, we know that

$$\frac{\mathrm{d}(V_l/V_u)}{\mathrm{d}\theta} = \frac{t}{(1-t\theta)^2} > 0 \text{ and } \frac{\mathrm{d}^2(V_l/V_u)}{\mathrm{d}\theta^2} = \frac{2t^2}{(1-t\theta)^3} > 0.$$

Because both the first and second derivatives are positive, V_l/V_u is an increasing function of V_d/V_l, which is rather consistent with our results.

FIGURE 1
The Market Value of the Firm as a Function of Debt



Note: FOC = first-order condition. SOC = second-order condition.

SHAREHOLDER LIMITED LIABILITY AND LOCAL INTERIOR MAXIMUM OF THE FIRM VALUE

In this section we demonstrate that the upper limit of debt obligations (D^*) imposed by shareholder limited liability is not greater than the amount of debt (D')

that maximizes the firm value when corporate bankruptcy is costless. This conclusion implies that the local interior maximum of the firm's value is obtained at a phantom level of debt which exceeds the upper limit of debt restricted by shareholder limited liability. This conclusion further implies that the value of the firm is an increasing function of its debt obligations when bankruptcy is costless.

After the amount of debt (D) is standardized, we can rewrite Equation (16) as follows:

$$\frac{\phi(Z')}{1-\Phi(Z')}=\frac{1}{K} \tag{19}$$

where $Z'=[D'-E(\tilde{X})]/\sigma(\tilde{X})$,

$K=\dfrac{E(\tilde{R}_m)-R_f}{\sigma(\tilde{R}_m)}\cdot\varrho(\tilde{X},\tilde{R}_m)$

$\varrho(\cdot)$ denotes the coefficient of correlation, and

$\phi(\cdot)$ and $\Phi(\cdot)$ are the standard normal variate density and cumulative distribution functions, respectively.

Equation (10) can also be rewritten as

$$\frac{\phi(Z^*)}{1-\Phi(Z^*)}-Z^*=K, \tag{20}$$

where $Z^*=[D^*-E(\tilde{X})]/\sigma(\tilde{X})$. It is obvious from (19) and (20) that we can estimate Z' and Z^* by trial and error once the value of the positive constant K is known. Given estimates of Z' and Z^*, we can investigate whether $Z^*<Z'$ or, equivalently, $D^*<D'$.

Note that both (19) and (20) contain the term $\phi(\cdot)/[1-\Phi(\cdot)]$, which is known as the hazard function (also known as the conditional failure function). This hazard function has been widely used for life testing and reliability in engineering statistics. (See [1, pp. 73-80] and [7, pp. 103-120].) One interesting feature of this hazard function for a normally distributed variable is $\phi(Z)/[1-\Phi(Z)]-Z>0$ but $\phi(Z)/[1-\Phi(Z)]-Z\cong 0$ as Z becomes large [5, pp. 175-176]. This particular property of the hazard function becomes useful in comparing Z' and Z^* using Equations (19) and (20). Upon examination of (19), we know that $1/K>0$ because $\phi(Z)/[1-\Phi(Z)]>0$. We further know that the value of $\phi(Z')/[1-\Phi(Z')]$ converges to Z' as the value of $1/K$ becomes extremely large. For our numerical simulation, we allow $1/K$ to change from .20 to 5.00. This range of $1/K$ will produce the largest estimates of Z' or Z^* in the neighborhood of $Z=4.60$. When $Z=4.60$, we obtain $1-\Phi(Z)=21\times 10^{-7}$ and $\phi(Z)=101\times 10^{-7}$ which approach zero. (These values were obtained from the *Biometrica Tables for Statisticians*, Vol. I (3rd ed.).)

The simulation results are presented in Table 1. Note that Z' is consistently greater than Z^* for $.20\leq 1/K\leq 5.00$. This result should also hold for extreme positive values of $1/K$. For example, if $1/K=50$, then $\phi(Z')/[1-\Phi(Z')]\cong 50\ (=Z')$. Therefore, $\phi(Z')/[1-\Phi(Z')]-Z'\cong 0$. In contrast, Equation (20) suggests that

TABLE 1
Estimation of Z' and Z^*

$1/K$	Z'	Z^*
5.00	4.8057	4.5000
4.60	4.3906	4.1718
4.20	3.9734	3.7446
3.80	3.5515	3.3020
3.40	3.1255	2.8491
3.00	2.6937	2.3843
2.60	2.2540	1.9028
2.20	1.8031	1.3969
1.80	1.3356	.8537
1.40	.8419	.2477
1.00	.3026	-.4811
.60	-.3306	-1.5359
.20	-1.2643	-5.0000

Note: $K = \{[E(\tilde{R}_m) - R_f]/\sigma(\tilde{R}_m)\}\varrho(\tilde{X},\tilde{R}_m)$.
$Z' = [D' - E(\tilde{X})]/\sigma(\tilde{X})$.
$Z^* = [D^* - E(\tilde{X})]/\sigma(\tilde{X})$.

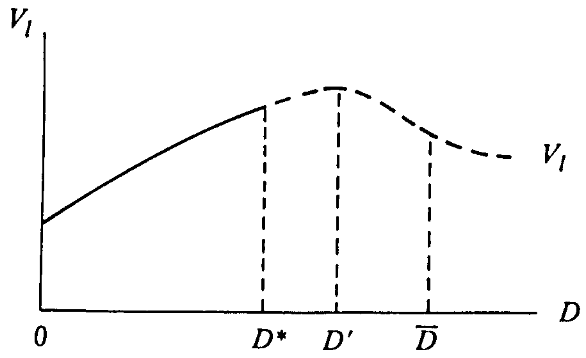
$\phi(Z^*)/[1 - \Phi(Z^*)] = .02$ $(=K)$. Unless $Z^* < Z'$, we cannot simultaneously satisfy the above two conditions as specified by Equations (19) and (20). This implies that the local interior maximum of firm value is obtained at a debt level that exceeds the upper limit of debt obligations (D^*) imposed by shareholder limited liability. Because $D^* < D'$ and $dV_l/dD > 0$ for $0 < D < D'$, we can conclude that the value of the firm is a monotonically increasing function of its debt obligations unless its borrowing policy jeopardizes shareholder limited liability. Figure 2A presents a graphical illustration of the firm's value as a function of debt. The meaningful portion of the firm's value curve is indicated by the solid line, while the dashed portion of the curve suggests feasible firm values only if shareholder limited liability is violated.

Our analysis so far requires no prior knowledge about $E(\tilde{X})$ and $\sigma(\tilde{X})$. As we introduce negative values for $E(\tilde{X})$, the firm's value curve shifts to the left. Two special cases are shown in Figures 2B and 2C where $D' = 0$ and $\overline{D} = 0$, respectively. A visual inspection of the firm's value curve where $D > 0$ may mislead us into concluding that the firm's value is a decreasing function of its debt obligations when $E(\tilde{X})$ is negative. As illustrated in these two figures, the dashed curves imply that these graphs are meaningless for all practical purposes because the amount of debt is negative or it exceeds the upper limit of debt (D^*) or both. The introduction of $\sigma(\tilde{X})$ will change the slope of the firm value curve but leave its overall shape unchanged.

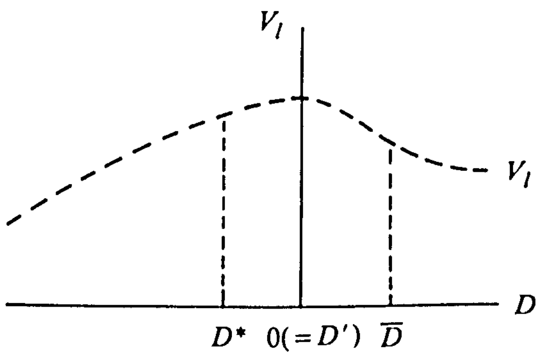
Our analysis so far proves that the firm's value is an increasing function of its debt as long as the amount of debt remains in the range of $0 < D \leq D^*$. Because

FIGURE 2
The Market Value of the Firm with Restriction on Borrowing

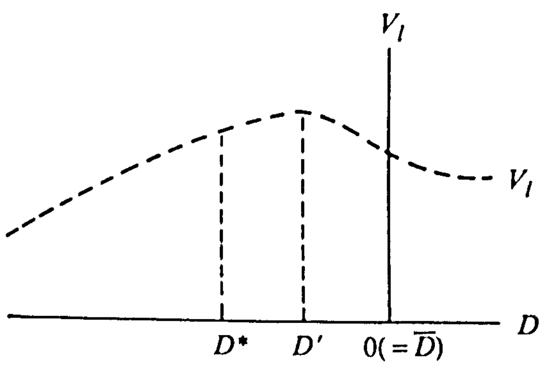
A. Firm's Value as a Function of Debt



B. $D' = 0$[a]



C. $\overline{D} = 0$[a]



[a]These graphs correspond to extended versions of Figures 1-B and 1-C in Gonzalez et al. [6, p. 169].

the firm-value curve never slopes downward in the presence of costless bankruptcy, 100 percent debt financing would remain optimal as in Modigliani and Miller's [12] tax-correction model which assumes no bankruptcy.[4] When costly bankruptcy is introduced, however, the firm value curve eventually slopes downward as the marginal bankruptcy costs of debt financing outweigh the marginal tax benefit of debt financing.[5] The theoretical analyses of Gonzalez et al. [6] are correct only when the firm is allowed to borrow without limit. Their prediction regarding the absurdity of the CAPM as a means for analyzing the firm's optimal capital structure, however, is avoided by shareholder limited liability. As a result, the introduction of costly bankruptcy into the mean-variance framework is justified for optimal capital structure.

SUMMARY AND A NOTE ON THE LIMITATIONS OF THIS STUDY

This study has reexamined the issue of the absurdity of the mean-variance model for optimal capital-structure decisions. The analysis indicates that the "reductio ad absurdum" argument does not necessarily hold when the restriction on corporate borrowing is explicitly considered in light of shareholder limited liability: The local interior maximum of the firm value is obtained at a phantom level of interest expense which is greater than the upper limit of interest expense allowable under the shareholder limited liability when corporate debt is subject to costless bankruptcy. Because the value of the firm is an increasing function of its debt obligations when bankruptcy is costless, the introduction of costly bankruptcy into the mean-variance model is justified in determining the optimal debt ratio of the firm.

A note on the limitations of this study is in order. First of all, the analysis does not attempt to rescue the CAPM from its well-known deficiency inherent in using a quadratic utility function. Because this quadratic function implies negative marginal utility when outcomes take extremely large values, the quadratic is at best an approximation to the true utility function of risk-averse individuals. Second, the analysis does not justify the use of mean and variance as the only characteristics of interest in the probability distribution of a relevant random variable. This entire issue remains open for future research. The study simply offers, in the traditional mean-variance framework, that shareholder limited liability effectively restricts corporate borrowing and that this neglected aspect of shareholder limited liability resolves the absurdity issue against the mean-variance model of optimal capital structure. Third, this conclusion is not meant to overlook Miller's [10] horse-and-rabbit stew criticism of the traditional corporate bankruptcy model.

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[4]When the firm's debt obligations are equal to D^*, which is the upper limit of corporate borrowing under shareholder limited liability, its debt ratio is unity. By substituting D^* as defined by Equation (10) into Equations (11) and (12), respectively, we find that $V_l = V_d$ as shown below:

$$V_l = V_d = \{[E(\tilde{X}) - \lambda \mathrm{Cov}(\tilde{X}, \tilde{R}_m)][1 - F(0)] + \sigma^2(\tilde{X})f(0)\}/r.$$

[5]Since Kim [9] and Chen [4] demonstrate the estimation of the firm's optimal level of debt in the presence of costly bankruptcy, the same demonstration is not repeated here.

The significance of bankruptcy costs for optimal capital structure is an empirical proposition which requires further research. Fourth, this paper does not introduce personal taxes into its analysis. The optimal capital structure theory in the presence of personal taxes is beyond the scope of this study. [Received: December 10, 1982. Accepted: September 7, 1983.]

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APPENDIX

Derivation of Equation (12)

The market value of the firm, V_l, is obtained by adding the market value of common stock, V_s, and the market value of debt, V_d. V_s and V_d are defined by Equations (7) and (11), respectively. Thus, we have

$$V_l = V_s + V_d \tag{A1}$$

$$\begin{aligned} &= \{(1-t)[E(\tilde{X}) - \lambda \mathrm{Cov}(\tilde{X},\tilde{R}_m)][1-F(D)] + (1-t)\sigma^2(\tilde{X})f(D) - (1-t)D[1-F(D)] \\ &\quad + D[1-F(D)] + [E(\tilde{X}) - \lambda \mathrm{Cov}(\tilde{X},\tilde{R}_m)][F(D)-F(0)] - \sigma^2(\tilde{X})[f(D)-f(0)]\}/r. \end{aligned} \tag{A2}$$

After adding and subtracting $(1-t)[E(\tilde{X}) - \lambda \mathrm{Cov}(\tilde{X},\tilde{R}_m)][1-F(0)] + (1-t)\sigma^2(\tilde{X})f(0)$ in the numerator of the above expression, we have

$$V_l = \{(1-t)[E(\tilde{X}) - \lambda \mathrm{Cov}(\tilde{X},\tilde{R}_m)][1-F(0)] + (1-t)\sigma^2(\tilde{X})f(0)$$

$$-(1-t)[E(\tilde{X}) - \lambda \mathrm{Cov}(\tilde{X},\tilde{R}_m)][1-F(0)] - (1-t)\sigma^2(\tilde{X})f(0)$$

$$+(1-t)[E(\tilde{X}) - \lambda \mathrm{Cov}(\tilde{X},\tilde{R}_m)][1-F(D)] + (1-t)\sigma^2(\tilde{X})f(D) - (1-t)D[1-F(D)]$$

$$+D[1-F(D)] + [E(\tilde{X}) - \lambda \mathrm{Cov}(\tilde{X},\tilde{R}_m)][F(D)-F(0)] - \sigma^2(\tilde{X})[f(D)-f(0)]\}/r. \quad \text{(A3)}$$

After simplification, we obtain

$$V_l = (1-t)\{[E(\tilde{X}) - \lambda \mathrm{Cov}(\tilde{X},\tilde{R}_m)][1-F(0)] + \sigma^2(\tilde{X})f(0)\}/r$$

$$+t\{D[1-F(D)] + [E(\tilde{X}) - \lambda \mathrm{Cov}(\tilde{X},\tilde{R}_m)][F(D)-F(0)] - \sigma^2(\tilde{X})[f(D)-f(0)]\}/r, \quad \text{(A4)}$$

which is Equation (12).

Derivation of Equation (14)

The first-order condition of Equation (13) with respect to D can be expressed as

$$r[\mathrm{d}V_l/\mathrm{d}D] = t\{[1-F(D)] - Df(D) + [E(\tilde{X}) - \lambda \mathrm{Cov}(\tilde{X},\tilde{R}_m)]f(D) - \sigma^2(\tilde{X})f'(D)\}, \quad \text{(A5)}$$

where $f'(D) = \mathrm{d}f(D)/\mathrm{d}D$. Since

$$f'(D) = \mathrm{d}f(D)/\mathrm{d}D = \frac{\mathrm{d}}{\mathrm{d}D}\{[1/\sqrt{2\pi}\sigma(\tilde{X})]e^{-[D-E(\tilde{X})]^2/2\sigma^2(\tilde{X})}\} \quad \text{(A6)}$$

$$= -[\frac{D-E(\tilde{X})}{\sigma^2(\tilde{X})}]\{[1\sqrt{2\pi}\sigma(\tilde{X})]e^{-[D-E(\tilde{X})]^2/2\sigma^2(\tilde{X})}\} \quad \text{(A7)}$$

$$= -[\frac{D-E(\tilde{X})}{\sigma^2(\tilde{X})}]f(D), \quad \text{(A8)}$$

the first-order condition can be simplified to that given in Equation (14).

S. Ghon Rhee is Associate Professor of Finance at the University of Rhode Island. His primary research interests include optimal capital structure, mergers and acquisitions, and inflation. He received his Ph.D. at the Ohio State University in 1978 and his M.B.A. from Rutgers University in 1974.