On the Catenary Degrees of Numerical Monoids Generated by Generalized Arithmetic Sequences

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Introduction
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Periodicity
On the Catenary Degrees of Numerical Monoids Generated by Generalized Arithmetic Sequences
Let $S$ be a numerical semigroup generated by the generalized arithmetic sequence $\langle a, ha + d, \ldots, ha + kd \rangle$ where $\gcd(a, d) = 1$ and $1 \leq k \leq a - 1$.

What is known about the catenary degree?
Let $S$ be a numerical semigroup generated by the generalized arithmetic sequence $\langle a, ha + d, \ldots, ha + kd \rangle$ where $\gcd(a, d) = 1$ and $1 \leq k \leq a - 1$.

**What is known about the catenary degree?**

- From Omidali [1] we know that $c(S) = \lceil a/k \rceil h + d$.
- From SDSU REU, [2] we know that the minimum and maximum nonzero catenary degrees are achieved at Betti elements of $S$. 
\[ S = \langle 9, 34, 41 \rangle \text{ with } a = 9, h = 3, d = 7 \]
Remaining Questions:

- What other catenary degrees are there?
- Exactly what elements have which catenary degrees?
Research Problem

Remaining Questions:

- What other catenary degrees are there?
- Exactly what elements have which catenary degrees?

Our research problem was to characterize $c(s)$ for all $s \in S$ in embedding dimension 3.
\[ S = \langle 9, 34, 41 \rangle \text{ with } a = 9, h = 3, d = 7 \]
$S = \langle 9, 34, 41 \rangle$ with $a = 9$, $h = 3$, $d = 7$
Recall

**Definition**

Let $\nabla_s$ be the factorization graph for a given element $s \in S$. If $\nabla_s$ is not connected, then $s$ is called a *Betti element* of $S$. We write

$$\text{Betti}(S) = \{ s \in S : \nabla_s \text{ is disconnected} \}$$

for the set of Betti elements of $S$.

**Definition**

The *dissonance* is the largest value $s \in S$ such that there is periodicity of catenary degree for all elements greater than the dissonance.
$S = \langle 9, 34, 41 \rangle$ with $a = 9, h = 3, d = 7$
Proposition

The Betti elements for \( \langle a, ha + d, ha + 2d \rangle \) are:

- \( a \cdot c(S) \) with catenary degree \( c(S) \)
- \( 2(ha + d) \) with catenary degree \( h + 1 \)
- \( \left\lceil \frac{a}{2} \right\rceil (ha + 2d) \) with catenary degree \( a + d + \left\lfloor \frac{a}{2} \right\rfloor (h - 2) \)
From the Betti elements, we can find the following fundamental moves as long as all three coordinates remain non-negative:

1. \((\alpha_0, \alpha_1, \alpha_2) = (\alpha_0 - h, \alpha_1 + 2, \alpha_2 - 1) = (\alpha_0 + h, \alpha_1 - 2, \alpha_2 + 1)\)
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**For example:** Given the monoid \(\langle 9, 34, 41 \rangle\) where \(a = 9, h = 3, d = 7,\)

\((0, 6, 0) = 6 \cdot 34 = 204\)
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\((h, 4, 1) = h \cdot 9 + 4 \cdot 34 + 1 \cdot 41\)  
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For example: Given the monoid \(\langle 9, 34, 41 \rangle\) where \(a = 9, h = 3, d = 7\),

\[
(0, 6, 0) = 6 \cdot 34 = 204 \\
(h, 4, 1) = h \cdot 9 + 4 \cdot 34 + 1 \cdot 41 = 204 \\
(2h, 2, 2) = 2h \cdot 9 + 2 \cdot 34 + 2 \cdot 41 = 204
\]
From the Betti elements, we can find the following fundamental moves as long as all three coordinates remain non-negative:

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   \[= (\alpha_0 + h, \alpha_1 - 2, \alpha_2 + 1)\]

**For example:** Given the monoid \(\langle 9, 34, 41 \rangle\) where \(a = 9, h = 3, d = 7\),

\[(0, 6, 0) = 6 \cdot 34\]
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\[(h, 4, 1) = h \cdot 9 + 4 \cdot 34 + 1 \cdot 41\]
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\[(2h, 2, 2) = 2h \cdot 9 + 2 \cdot 34 + 2 \cdot 41\]
\[= 204\]

\[(3h, 0, 3) = 3h \cdot 9 + 0 \cdot 34 + 3 \cdot 41\]
\[= 204\]
From the Betti elements, we can find the following fundamental
moves as long as all three coordinates remain non-negative:

1. \((\alpha_0, \alpha_1, \alpha_2) = (\alpha_0 - h, \alpha_1 + 2, \alpha_2 - 1)\)
   \(= (\alpha_0 + h, \alpha_1 - 2, \alpha_2 + 1)\)

2. \((\alpha_0, \alpha_1, \alpha_2) = (\alpha_0 + c(S), \alpha_1 - a + 2 \left\lfloor \frac{a}{2} \right\rfloor, \alpha_2 - \left\lfloor \frac{a}{2} \right\rfloor)\)
   \(= (\alpha_0 - c(S), \alpha_1 + a - 2 \left\lfloor \frac{a}{2} \right\rfloor, \alpha_2 + \left\lfloor \frac{a}{2} \right\rfloor)\)

3. \((\alpha_0, \alpha_1, \alpha_2) = (\alpha_0 - \left\lfloor \frac{a}{2} \right\rfloor h - d, \alpha_1 - a + 2 \left\lfloor \frac{a}{2} \right\rfloor, \alpha_2 + a - \left\lfloor \frac{a}{2} \right\rfloor)\)
   \(= (\alpha_0 + \left\lfloor \frac{a}{2} \right\rfloor h + d, \alpha_1 + a - 2 \left\lfloor \frac{a}{2} \right\rfloor, \alpha_2 - a + \left\lfloor \frac{a}{2} \right\rfloor)\)
Let $S = \langle 9, 34, 41 \rangle$.

The function $c(s)$ for $s \in S$ is shown in the graph.

With $a = 9$, $h = 3$, and $d = 7$.
Let $S = \langle a, ha + d, ha + 2d \rangle$ where $gcd(a, d) = 1$. The set of catenary degrees is $C(S) = \{0\}$. 
Catenary Set

Let $S = \langle a, ha + d, ha + 2d \rangle$ where $\gcd(a, d) = 1$. The set of catenary degrees is $C(S) = \{0, h + 1\}$. 

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Let $S = \langle a, ha + d, ha + 2d \rangle$ where $gcd(a, d) = 1$. The set of catenary degrees is $C(S) = \{0, h + 1, a + d + x(h - 2)\}$. We will now show which elements obtain these catenary degrees and propose a closed form solution for any element the catenary degree of any element $s \in S$. 

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$C(S) = \{0, h + 1, a + d + x(h - 2), c(S)\}$ for $x \in \{1, \ldots, \left\lfloor \frac{a}{2} \right\rfloor\}$
Catenary Set

Let $S = \langle a, ha + d, ha + 2d \rangle$ where $\gcd(a, d) = 1$.

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\[ S = \langle 9, 34, 41 \rangle \] with \( a = 9, \ h = 3, \ d = 7 \)
The following elements have only a unique factorization and thus have catenary degree zero.

If $a$ is an even number:

1. $(\alpha_0, \alpha_1, 0)$, $\alpha_0 \in \{0, \ldots, c(S) - 1\}$, $\alpha_1 \in \{0, 1\}$
2. $(\alpha_0, \alpha_1, \alpha_2)$, $\alpha_0 \in \{0, \ldots, h - 1\}$, $\alpha_1 \in \{0, 1\}$, $\alpha_2 \in \{0, \ldots, \left\lfloor \frac{a}{2} \right\rfloor - 1\}$

If $a$ is an odd number:

1. $(\alpha_0, \alpha_1, 0)$, $\alpha_0 \in \{0, \ldots, c(S) - 1 - \alpha_1 h\}$, $\alpha_1 \in \{0, 1\}$
2. $(\alpha_0, \alpha_1, \alpha_2)$, $\alpha_0 \in \{0, \ldots, h - 1\}$, $\alpha_1 \in \{0, 1\}$, $\alpha_2 \in \{0, \ldots, \left\lfloor \frac{a}{2} \right\rfloor - \alpha_1\}$

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\[ S = \langle 9, 34, 41 \rangle \text{ with } a = 9, \ h = 3, \ d = 7 \]
Proposition

The element \((a - x)(ha + 2d)\) has catenary degree \(a + d + x(h - 2)\) for \(x \in \{1, \ldots, \left\lfloor \frac{a}{2} \right\rfloor \} \).
$S = \langle 9, 34, 41 \rangle$ with $a = 9$, $h = 3$, $d = 7$
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Proposition

The element $(a - x)(ha + 2d)$ has catenary degree $a + d + x(h - 2)$ for $x \in \{1, \ldots, \left\lfloor \frac{a}{2} \right\rfloor \}$.

Proposition

The element $ha + d + aah - xah + 2ad - 2xd = (0, 1, a - x)$ has catenary degree $a + d + x(h - 2)$ for $x \in \{1, \ldots, \left\lfloor \frac{a}{2} \right\rfloor \}$. 
Proposition

The element \((a - x)(ha + 2d)\) has catenary degree \(a + d + x(h - 2)\) for \(x \in \{1, \ldots, \lfloor \frac{a}{2} \rfloor \} \).

Proposition

The element \(ha + d + aah - xah + 2ad - 2xd = (0, 1, a - x)\) has catenary degree \(a + d + x(h - 2)\) for \(x \in \{1, \ldots, \lfloor \frac{a}{2} \rfloor \} \).

Proposition

The element \((m, \alpha_1, a - x)\) for \(\alpha_1 \in \{0, 1\}, m \in \mathbb{N}\) also has catenary degree \(a + d + x(h - 2)\).
\[ S = \langle 9, 34, 41 \rangle \text{ with } a = 9, h = 3, d = 7 \]
Catenary Degree $c(S)$

**Proposition**

Let $S$ be a numerical semigroup generated by a generalized arithmetic sequence. Then, $c(m \cdot a) = c(S)$ for $m \geq c(S)$.
\[ S = \langle 9, 34, 41 \rangle \text{ with } a = 9, h = 3, d = 7 \]
Catenary Degree $h + 1$

**Theorem**

$c(n) = h + 1$ if and only if $n$ minus the Betti element $2(ha + d)$ is in the monoid $S$. 
Consider the element 272 in the monoid $S = \langle 9, 34, 41 \rangle$. 
Consider the element 272 in the monoid $S = \langle 9, 34, 41 \rangle$. 
For $272$ in $S = \langle 9, 34, 41 \rangle$.

1. $272 - 2(ha + d) = 272 - 68 = 204$
   
   $204 \in S$ since $6 \cdot 34 = 204$. 

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For 272 in $S = \langle 9, 34, 41 \rangle$.

1. $272 - 2(ha + d) = 272 - 68 = 204$
   
   $204 \in S$ since $6 \cdot 34 = 204$.

2. $272 - \left\lceil \frac{a}{2} \right\rceil (ha + 2d) = 272 - 205 = 67$
   
   $67 \not\in S$ since $9 \nmid 67$ and $67 < 2 \cdot 34$. 

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For 272 in $S = \langle 9, 34, 41 \rangle$.

1. $272 - 2(ha + d) = 272 - 68 = 204$
   
   $204 \in S$ since $6 \cdot 34 = 204$.

2. $272 - \left\lceil \frac{a}{2} \right\rceil (ha + 2d) = 272 - 205 = 67$
   
   $67 \notin S$ since $9 \nmid 67$ and $67 < 2 \cdot 34$.

3. $272 - a \left( \left\lceil \frac{a}{2} \right\rceil h + d \right) = 272 - 198 = 74$
   
   $74 \notin S$. 
Periodicity

\[ S = \langle 9, 34, 41 \rangle \text{ with } a = 9, h = 3, d = 7 \]
Periodicity

\[ S = \langle 9, 34, 41 \rangle \text{ with } a = 9, \ h = 3, \ d = 7 \]
Periodicity

Proposition

Recall that $x \in \{1, \ldots, \left\lfloor \frac{a}{2} \right\rfloor \}$. The elements $(m, 0, a - x)$, $(m, 1, a - x)$, and $(m, 0, 0)$ have different equivalence classes mod $a$. Thus there are $a$ distinct equivalence classes mod $a$.

Theorem

The dissonance is given by $a \cdot c(S) + F(S)$, where $F(S) = \left\lceil \frac{a-1}{2} \right\rceil ha + ad - a - d$, and $c(S) = \left\lceil \frac{a}{2} \right\rceil h + d$. The catenary degree of the Dissonance is $h + 1$. After the Dissonance we have periodicity $a$. 
Closed Form Solution

Theorem

Let \( S = \langle a, ha + d, ha + 2d \rangle \) and \( x \in S \). Define
\( y_1 = x - b_1, y_2 = x - b_2, \) and \( y_3 = x - b_3 \) where \( b_1, b_2, \) and \( b_3 \) are the Betti elements of \( S \). We can write \( x = qa + id \) with \( q, i \in \mathbb{N}_0 \) and \( i \in \{ 0, \ldots, a - 1 \} \), then

\[
c(x) = \begin{cases} 
0 & \text{if } y_j \notin S, \forall j \\
h + 1 & \text{if only } y_1 \in S \\
a + d + \left\lceil \frac{a-i}{2} \right\rceil (h - 2) + a(\text{mod}2)\left\lfloor \frac{a-i}{a} \right\rfloor & \text{if } y_2 \in S \text{ or } y_3 \in S
\end{cases}
\]
Thank you for listening!

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References


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