Question
What is a series-parallel graph?
Definition

A graph is **series-parallel** if it can be obtained from a single edge by a sequence of the following operations:
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**Parallel Composition**

\[
\begin{align*}
\Gamma_1 & \quad \Gamma_2 \\
\text{Parallel Composition} & \\
 s & \quad t
\end{align*}
\]

**Series Composition**

\[
\begin{align*}
\Gamma_1 & \quad \Gamma_2 \\
\text{Series Composition} & \\
 s_1 & \quad t_2
\end{align*}
\]
Example: Series-Composition
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Example: Series-Composition

\[ P_2^4 \bigoplus_s P_2^4 = P_3^4 \]
Example: Parallel-Composition

\[
\bigoplus_p \mathbb{P}^4 \bigoplus_p \mathbb{P}^4 = \mathbb{C}^4
\]
Example: Parallel-Composition

\[ P_3^4 \oplus_p P_3^4 = \]
Example: Parallel-Composition

\[ P_3^4 \bigoplus_p P_3^4 = C_4^4 \]
Definition

A thick cycle graph, denoted as $C[a_1, a_2, \ldots, a_n]$, is a cycle graph on $n$ vertices $\{v_1, \ldots, v_n\}$ with $wt(v_i, v_{i+1}) = a_i$ for $i \in [n-1]$ and $wt(v_n, v_1) = a_n$. If $a_i = a_j$ for all $i, j \in \{1, 2, \ldots, n\}$, then we denote the graph $C_n^a$. 
Question

What is the sandpile group of a thick cycle?
Theorem 1
Let $C_n^a$ be a thick cycle with $n$ vertices and $a$ edges between adjacent vertices. Then,

$$S(C_n^a) \cong \mathbb{Z}_{a}^{n-2} \times \mathbb{Z}_{na}.$$
Theorem 1
Let $C_n^a$ be a thick cycle with $n$ vertices and $a$ edges between adjacent vertices. Then,

$$S(C_n^a) \cong \mathbb{Z}_{a}^{n-2} \times \mathbb{Z}_{na}.$$ 

Sketch of Proof:

$$L = \begin{bmatrix} 2a & -a & 0 & \cdots & 0 & -a \\ -a & 2a & -a & \cdots & 0 & 0 \\ 0 & -a & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 2a & -a & 0 \\ 0 & 0 & \cdots & -a & 2a & -a \\ -a & 0 & \cdots & 0 & -a & 2a \end{bmatrix} \rightarrow \begin{bmatrix} a & 0 & 0 & \cdots & 0 & 0 \\ 0 & a & 0 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a & 0 & 0 \\ 0 & 0 & \cdots & 0 & a & 0 \\ 0 & 0 & \cdots & 0 & 0 & na \end{bmatrix}$$
Theorem 2

Let $C[a, a, ..., a, b]$ be a thick cycle with $n \geq 2$. Then

$$S(C[a, a, ..., a, b]) \cong \mathbb{Z}_d \times \mathbb{Z}_{a}^{n-2} \times \mathbb{Z}_{\frac{a}{d}}(a+nb),$$

where $d = \gcd(a, b)$. 
Proof of Theorem

\[
L = \begin{bmatrix}
  a + b & -a & 0 & \cdots & 0 & -b \\
  -a & 2a & -a & 0 & \cdots & 0 \\
  0 & -a & 2a & \ddots & \ddots & \vdots \\
  \vdots & 0 & \ddots & \ddots & -a & 0 \\
  \vdots & 0 & \ddots & \ddots & \ddots & \ddots \\
  0 & \ddots & -a & 2a & -a \\
  -b & 0 & \cdots & 0 & -a & a + b
\end{bmatrix}
\]
Proof of Theorem

\[ L = \begin{bmatrix}
a + b & -a & 0 & \cdots & 0 & -b \\
-a & 2a & -a & 0 & \cdots & 0 \\
0 & -a & 2a & \ddots & & \vdots \\
\vdots & 0 & \ddots & \ddots & -a & 0 \\
0 & \vdots & -a & 2a & -a \\
-b & 0 & \cdots & 0 & -a & a + b
\end{bmatrix} \rightarrow \begin{bmatrix}
a & 0 & 0 & \cdots & \cdots & 0 \\
0 & a & 0 & \cdots & \cdots & 0 \\
0 & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & a & 0 & 0 \\
\vdots & \vdots & \vdots & 0 & na & -(n - 1)a \\
0 & 0 & \cdots & 0 & -a & a + b
\end{bmatrix} \]
Proof of Theorem

\[ L = \begin{bmatrix}
    a + b & -a & 0 & \cdots & 0 & -b \\
    -a & 2a & -a & 0 & \cdots & 0 \\
    0 & -a & 2a & \ddots & \ddots & \vdots \\
    \vdots & \ddots & \ddots & \ddots & -a & 0 \\
    0 & \cdots & -a & 2a & -a \\
    -b & 0 & \cdots & 0 & -a & a + b
\end{bmatrix} \rightarrow \begin{bmatrix}
    a & 0 & 0 & \cdots & \cdots & 0 \\
    0 & a & 0 & \cdots & \cdots & 0 \\
    0 & 0 & \cdots & \ddots & \ddots & \vdots \\
    \vdots & \ddots & \ddots & \ddots & a & 0 \\
    \vdots & \cdots & \cdots & 0 & a & b \\
    0 & 0 & \cdots & 0 & 0 & a + nb
\end{bmatrix} \]
Proof

Let \( d = (a, b) \). Then there exists \( x, y, \alpha, \beta \in \mathbb{Z} \) such that

\[
ax + by = d, \quad \alpha d = a \quad \beta d = b
\]
Let \( d = (a, b) \). Then there exists \( x, y, \alpha, \beta \in \mathbb{Z} \) such that

\[
ax + by = d, \quad \alpha d = a \quad \beta d = b
\]

\[
\begin{bmatrix}
a & b \\
0 & a + nb
\end{bmatrix}
\begin{bmatrix}
x & -\beta \\
y & \alpha
\end{bmatrix}
= 
\begin{bmatrix}
d & 0 \\
(a + nb)y & \alpha(a + nb)
\end{bmatrix}
\rightarrow 
\begin{bmatrix}
d & 0 \\
0 & \alpha(a + nb)
\end{bmatrix}
\]
Proof

\[
\begin{bmatrix}
    a & 0 & 0 & \cdots & \cdots & 0 \\
    0 & a & 0 & \cdots & \cdots & 0 \\
    0 & 0 & \ddots & \cdots & \ddots & \ddots \\
    \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    \vdots & \ddots & \ddots & a & 0 & 0 \\
    \vdots & \ddots & \ddots & 0 & d & 0 \\
    0 & 0 & \cdots & 0 & 0 & \frac{a}{d}(a + nb)
\end{bmatrix}
\]
\[ S(C[a, a, ..., a, b]) \cong \mathbb{Z}_d \times \mathbb{Z}_a^{n-2} \times \mathbb{Z}_{\frac{a}{d}}(a+nb), \]
Question

Are the sandpile groups of $C[a, a, a, b, b, b]$ and $C[a, b, a, b, a, b]$ isomorphic?
Theorem (Corri, Rossin)

The sandpile groups of a graph $G$ and its dual graph $G^*$ are isomorphic.

Figure: $C[2, 4, 3]$ with Dual Graph shown in red
Theorem 3
Let $C[a_1, \ldots, a_n]$ be a thick cycle and $\sigma \in S_n$ be the group of permutations on the set $\{1, \ldots, n\}$. Then,

$$S(C[a_1, \ldots, a_n]) \cong S(C[a_{\sigma(1)}, \ldots, a_{\sigma(n)}]).$$
Theorem 3

Let $C[a_1, \ldots, a_n]$ be a thick cycle and $\sigma \in S_n$ be the group of permutations on the set $\{1, \ldots, n\}$. Then,

$$S(C[a_1, \ldots, a_n]) \cong S(C[a_{\sigma(1)}, \ldots, a_{\sigma(n)}]).$$
Theorem 4

Let $C[a_1, \ldots, a_n]$ be a generalized thick cycle and $\sigma \in S_n$ be the group of permutations on the set $\{1, \ldots, n\}$. Then,

$$S(C[a_1, \ldots, a_n]) \cong S(C[a_{\sigma(1)}, \ldots, a_{\sigma(n)}]).$$

Figure: Dual Graphs
Theorem 4

For a thick cycle $C[a, a, \ldots, a, b, \ldots, b]$ with $n, m \geq 2$,

then $S(C[a, \ldots, a, b, \ldots, b]) \cong \mathbb{Z}_2^{\langle a, b \rangle} \times \mathbb{Z}_a^{n-2} \times \mathbb{Z}_b^{m-2} \times \mathbb{Z}_{\langle a, b \rangle^2}^{ab}(ma+nb)$. 
Theorem 5

The sandpile group of the thick cycle $C[a, b, c]$ is

$$\mathcal{S}(C[a, b, c]) \cong \mathbb{Z}_{(a,b,c)} \times \mathbb{Z}_{(ab+bc+ac) \bmod (a,b,c)}.$$
Theorem
The elementary divisors of a $2 \times 2$ matrix $M$ are
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$f_1$: The greatest common divisor of the entries in the matrix
$f_2$: $\frac{\text{Det}(M)}{f_1}$. 

Proof
The graph of the thick cycle $C[a, b, c]$ is shown in Figure 3:
Proof

The $3 \times 3$ Laplacian of $C[a, b, c]$ is

$$L = \begin{bmatrix}
a + c & -a & -c \\
-a & a + b & -b \\
-c & -b & b + c
\end{bmatrix}.$$ 

Letting $v_1$ be the distinguished vertex, the $2 \times 2$ reduced Laplacian is

$$\tilde{L} = \begin{bmatrix}
a + b & -b \\
-b & b + c
\end{bmatrix} = \tilde{\Delta}.$$
We now find the elementary divisors with the help of the Elementary Divisors Theorem.
Proof

\[
\begin{bmatrix}
  a + b & -b \\
  -b & b + c
\end{bmatrix} \overset{c_2 + c_1^*}{\longrightarrow} \begin{bmatrix}
  a & -b \\
  c & b + c
\end{bmatrix}.
\]

We now find the elementary divisors with the help of the Elementary Divisors Theorem. The first elementary divisor of \( \tilde{\Delta} \) is the greatest common divisor of the entries of the matrix.
Proof

\[
\begin{pmatrix}
  a + b & -b \\
  -b & b + c
\end{pmatrix}
\xrightarrow{c_2 + c_1^*}
\begin{pmatrix}
  a & -b \\
  c & b + c
\end{pmatrix}.
\]

We now find the elementary divisors with the help of the Elementary Divisors Theorem. The first elementary divisor of \( \tilde{\Delta} \) is the greatest common divisor of the entries of the matrix.

\[f_1 = \gcd(a, -b, c, b + c) = \gcd(a, b, c, b + c) = \gcd(a, b, c).\]
Proof

Now, since $\Delta$ is a $2 \times 2$ matrix, the second elementary divisor is
Proof

Now, since $\tilde{\Delta}$ is a $2 \times 2$ matrix, the second elementary divisor is

$$f_2 = \frac{\text{Det}(\tilde{\Delta})}{f_1} = \frac{a(b + c) - c(-b)}{f_1} = \frac{(ab + bc + ac)}{f_1} = \frac{(ab + bc + ac)}{\gcd(a, b, c)}.$$
Now, since $\widetilde{\Delta}$ is a $2 \times 2$ matrix, the second elementary divisor is

$$f_2 = \frac{\text{Det}(\widetilde{\Delta})}{f_1} = \frac{a(b + c) - c(-b)}{f_1} = \frac{(ab + bc + ac)}{f_1} = \frac{(ab + bc + ac)}{\gcd(a, b, c)}.$$ 

Therefore, since we know the elementary divisors of $\widetilde{\Delta}$, we have that

$$S(C[a, b, c]) \cong \mathbb{Z}_{(a,b,c)} \times \mathbb{Z}_{\frac{(ab+bc+ac)}{(a,b,c)}}.$$
Theorem 6
For the thick cycle $C[a, a, ..., a, b, c]$ with $n \geq 2$, we have that the sandpile group is

$$S(C[a, a, ..., a, b, c]) = \mathbb{Z}_{(a, b)} \times \mathbb{Z}_{(a, c)} \times \mathbb{Z}_a^{n-2} \times \mathbb{Z}_{\frac{a(nbc+ab+ac)}{(a,b)(a,c)}}$$
Theorem 7
The sandpile group of the thick cycle \( C[a, ..., a, b, ..., b, c] \) is isomorphic to

\[
\mathbb{Z}(a,b,c) \times \mathbb{Z}(a,b) \times \mathbb{Z}(a,c) \times \mathbb{Z}_a^{n-2} \times \mathbb{Z}_b^{m-2} \times \mathbb{Z}_{ab(ab+nb+c+ma)} \cdot (a,b,c)(a,b)(a,c)
\]
Conjecture
The sandpile group of the thick cycle $C[a, ..., a, b, ..., b, c, ..., c]$ is $\mathbb{Z} \times \mathbb{Z}_{\text{lcm}(a,b)} \times \mathbb{Z}_{\text{lcm}(a,c)} \times \mathbb{Z}_{\text{lcm}(b,c)} \times \mathbb{Z}_{a}^{n-2} \times \mathbb{Z}_{b}^{m-2} \times \mathbb{Z}_{c}^{\ell-2} \times \mathbb{Z}_{\text{gcd}(abc(nbc+mac+lab), x(a,b)(a,c)(b,c))}^{\text{x}(a,b)(a,c)(b,c)}$ where $x = \text{gcd}(\text{lcm}(a, b), \text{lcm}(a, c), \text{lcm}(b, c))$, and $n, m, \ell \geq 2$
Note that,

\[ C[a, a, \ldots, a, b, b, \ldots, b] \cong P[a, a, \ldots, a] \parallel P[b, b, \ldots, b] \]

\[ n \text{ times} \quad m \text{ times} \quad n \text{ times} \quad m \text{ times} \]
Note that,

\[ C[a, a, \ldots, a, b, b, \ldots, b] \cong P[a, a, \ldots, a] \parallel P[b, b, \ldots, b] = P^a_{n+1} \parallel P^b_{m+1} \]
What about 3 - Parallel Paths?
Theorem 8

The sandpile group of $P_{n+1}^{a} \parallel P_{m+1}^{a} \parallel P_{\ell+1}^{a}$ is

$$S(P_{n+1}^{a} \parallel P_{m+1}^{a} \parallel P_{\ell+1}^{a}) \cong \mathbb{Z}^{m+n+\ell-4} \times \mathbb{Z}_{a}(m,n,\ell) \times \mathbb{Z}_{a(mn+m\ell+n\ell)} \, (m,n,\ell).$$
Theorem 9

Let \((P_{n+1}^a)^k\) denote the graph of \(k\) - Parallel Paths with each path having \(n + 1\) vertices and edge weights equal to \(a\). Then

\[ S((P_{n+1}^a)^k) \cong \mathbb{Z}_a^{k(n-2)+2} \times \mathbb{Z}_{an}^{k-2} \times \mathbb{Z}_{kan}. \]
Theorem (Schulz)

A configuration $c \in \mathcal{M}(\Gamma)$ is a minimal recurrent configuration if and only if there is a directed acyclic graph $G$ such that for any $v \in V : c(v) = \text{outdeg}_G(v)$ is true.
Theorem 10
The minimal recurrent configurations for any the thick cycle $C[a_1, a_2, a_3, \ldots, a_n]$ with a distinguished vertex $v_1$ are

\begin{align*}
(0, a_2, a_3, \ldots, a_{n-2}, a_{n-1}) \\
(a_2, 0, a_3, \ldots, a_{n-2}, a_{n-1}) \\
\vdots \\
(a_2, a_3, \ldots, a_{n-2}, 0, a_{n-1}) \\
(a_2, a_3, \ldots, a_{n-2}, a_{n-1}, 0).
\end{align*}
Minimal Recurrent Configurations

Figure: $C[a_1, a_2, a_3, a_4]$
Minimal Recurrent Configurations

Figure: The three $S - DAG's$ of $C[a_1, a_2, a_3, a_4]$
(a_2, a_3, 0)

Figure: The three $S$ - $DAG$’s of $C[a_1, a_2, a_3, a_4]$
Minimal Recurrent Configurations

\[(a_2, 0, a_3)\]

**Figure:** The three $S - DAG's$ of $C[a_1, a_2, a_3, a_4]$
Minimal Recurrent Configurations

Figure: The three $S-DAG's$ of $C[a_1, a_2, a_3, a_4]$
Theorem 11

The minimal recurrent configurations for any 3-parallel path $P_{n+1}^a \parallel P_{n+1}^b \parallel P_{n+1}^c$ are of the form

1. $(a, \ldots, a, b, \ldots, b, c, \ldots, c, 0)$
2. $(a, \ldots, a, b, \ldots, b, 0, b, \ldots, b, c, \ldots, c, b)$
3. $(a, \ldots, a, b, \ldots, b, c, \ldots, c, 0, c \ldots, c, c)$
4. $(a, \ldots, a, b, \ldots, b, 0, b, \ldots, b, b, c, \ldots, c, 0, c, \ldots, c, b + c)$. 
Figure: The four S-DAGs of...
Example

What is the sandpile group of $\Gamma$, the edge join of the book graphs $B(3, 3)$ and $B(4, 2)$?
Example

The dual graph of $\Gamma$, formed by assigning a vertex to each region partitioned by the edges of $\Gamma$, with vertices adjacent if and only if they share an edge, is isomorphic to $C[2, 2, 2, 3, 3, 1]$. Thus,

$$S(\Gamma) \cong (C[2, 2, 2, 3, 3, 1])$$
Example

The dual graph of $\Gamma$, formed by assigning a vertex to each region partitioned by the edges of $\Gamma$, with vertices adjacent if and only if they share an edge, is isomorphic to $C[2, 2, 2, 3, 3, 1]$. Thus,

$$S(\Gamma) \cong (C[2, 2, 2, 3, 3, 1])$$
Recall:
The sandpile group of the thick cycle $C[a, \ldots, a, b, \ldots, b, c]$ is isomorphic to
\[
\mathbb{Z}(a,b,c) \times \mathbb{Z}(a,b) \times \mathbb{Z}(a,c) \times \mathbb{Z}_a^{n-2} \times \mathbb{Z}_b^{m-2} \times \mathbb{Z}_{\frac{ab(ab+nb+c+ma)}{(a,b,c)(a,b)(a,c)}}.
\]
Recall:
The sandpile group of the thick cycle $C[a, \ldots, a, b, \ldots, b, c]$, $n$ times, $m$ times,
is isomorphic to

$$\mathbb{Z}(a,b,c) \times \mathbb{Z}(a,b) \times \mathbb{Z}(a,c) \times \mathbb{Z}_a^{n-2} \times \mathbb{Z}_b^{m-2} \times \mathbb{Z}_{ab(ab+nb+c+mc)}.$$

Therefore, since $S(\Gamma) \cong S(C[2,2,2,3,3,1])$,.
Recall:
The sandpile group of the thick cycle $C[a, \ldots, a, b, \ldots, b, c]$ is isomorphic to

$$\mathbb{Z}(a,b,c) \times \mathbb{Z}(a,b) \times \mathbb{Z}(a,c) \times \mathbb{Z}_a^{n-2} \times \mathbb{Z}_b^{m-2} \times \mathbb{Z}_{\frac{ab(ab+nb+mc)}{(a,b,c)(a,b)(a,c)}}.$$ 

Therefore, since $S(\Gamma) \cong S(C[2,2,2,3,3,1])$,

$$S(\Gamma) \cong \mathbb{Z}_2 \times \mathbb{Z}_{6\cdot19}$$
Theorem 12
The sandpile group of the edge join of two book graphs $B(n, k)$ and $B(m, \ell)$ is isomorphic to

$$\mathbb{Z}_{(n,m)} \times \mathbb{Z}_{n}^{k-2} \times \mathbb{Z}_{m}^{\ell-2} \times \mathbb{Z}_{\frac{mn(mn+km+\ell n)}{(m,n)}}.$$
We categorized the sandpile groups of thick cycles with varying edge weights.
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We categorized the sandpile group of different types of k-parallel paths.
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We categorized the sandpile group of different types of $k$-parallel paths.
We described the recurrences of both thick cycles and $k$-parallel paths.
We categorized the sandpile groups of thick cycles with varying edge weights. We categorized the sandpile group of different types of k-parallel paths. We described the recurrants of both thick cycles and k-parallel paths. We used the sandpile groups of thick cycles to describe the sandpile groups of the edge join of book graphs.
Prove the sandpile group of $C[a, \ldots, a, b, \ldots, b, c, \ldots, c]$.

$n$ times $m$ times $\ell$ times
Future Questions

Prove the sandpile group of \( C[a, ..., a, b, ..., b, c, ..., c] \).

Generalize for \( C[a_1, ..., a_1, a_2, ..., a_2, \ldots a_m, ..., a_m] \).
Prove the sandpile group of \( C[a, ..., a, b, ..., b, c, ..., c] \).

Generalize for \( C[a_1, ..., a_1, a_2, ..., a_2, \ldots, a_m, ..., a_m] \).

What can we say about the sandpile group of the parallel composition of two graphs?
Thank you!

Special Thanks to the following individuals in our PURE Math Program Matrix,

\[
\begin{bmatrix}
Luis & Rebecca & Lauren \\
Brian & Lisa & NSA \\
NSF & PURE & Math
\end{bmatrix}
\]