Primality in Arithmetically Generated Numerical Monoids

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Overview
Overview

- Fundamentals of $\omega$-primality
Overview

- Fundamentals of $\omega$-primality
- $\omega$-primality of generators
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- Fundamentals of $\omega$-primality
- $\omega$-primality of generators
- Bad elements and the dissonance point
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- Fundamentals of $\omega$-primality
- $\omega$-primality of generators
- Bad elements and the dissonance point
- Eventual quasilinearity of $\omega(n)$ in numerical monoids
Motivating Question

How prime are certain elements in a numerical monoid?

Numerical monoids are not unique factorization domains.

This leads to interesting properties in terms of primality and factorizations.
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Primality

Consider the monoid $\mathbb{N}^*$, which is the natural numbers under the binary operation of multiplication.
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We say an non-unit $p \in \mathbb{N}$ is prime if whenever $p \mid (a_1 \cdot a_2 \cdot \ldots \cdot a_k)$ for $a_1, a_2, \ldots, a_k \in \mathbb{N}^*$, then $p \mid a_i$ for some $i$. 
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We say an non-unit $p \in \mathbb{N}$ is prime if whenever $p \mid (a_1 \cdot a_2 \cdot \ldots \cdot a_k)$ for $a_1, a_2, \ldots, a_k \in \mathbb{N}^*$, then $p \mid a_i$ for some $i$.

In other words, whenever $p$ divides a product, it always divides at least one term in the product.
Example of primality

Consider the monoid $\mathbb{N}^*$. Notice that $3 \mid 210$. Writing 210 as all possible products of natural numbers, we have:

- $3 \mid 210 = (7 \cdot 30)$
- $3 \mid 30 = (3 \cdot 30)$
- $3 \mid 30 = (6 \cdot 45)$
- $3 \mid 45 = (5 \cdot 42)$
- $3 \mid 42 = (6 \cdot 21)$
- $3 \mid 21 = (2 \cdot 3 \cdot 5 \cdot 7)$

Notice that 3 is prime in $\mathbb{N}^*$, hence it always divides at least 1 term in the product.
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3 \mid (7 \cdot 30) \quad \Rightarrow \quad 3 \mid 30 \\
3 \mid (3 \cdot 70)
\]
Example of primality

Consider the monoid $\mathbb{N}^*$. Notice that $3 \mid 210$. Writing 210 as all possible products of natural numbers, we have:

$$3 \mid (7 \cdot 30) \implies 3 \mid 30$$

$$3 \mid (3 \cdot 70) \implies 3 \mid 3$$
Example of primality

Consider the monoid \( \mathbb{N}^* \). Notice that \( 3 \mid 210 \). Writing 210 as all possible products of natural numbers, we have:

\[
\begin{align*}
3 & \mid (7 \cdot 30) \quad \Rightarrow \quad 3 \mid 30 \\
3 & \mid (3 \cdot 70) \quad \Rightarrow \quad 3 \mid 3 \\
3 & \mid (6 \cdot 45) \quad \Rightarrow \quad 3 \mid 45 \\
3 & \mid (5 \cdot 42) \quad \Rightarrow \quad 3 \mid 42 \\
3 & \mid (2 \cdot 5 \cdot 21) \quad \Rightarrow \quad 3 \mid 21 \\
3 & \mid (2 \cdot 3 \cdot 5 \cdot 7) \quad \Rightarrow \quad 3 \mid 3
\end{align*}
\]
Example of primality

Consider the monoid $\mathbb{N}^*$. Notice that $3 \mid 210$. Writing 210 as all possible products of natural numbers, we have:

- $3 \mid (7 \cdot 30) \Rightarrow 3 \mid 30$
- $3 \mid (3 \cdot 70) \Rightarrow 3 \mid 3$
- $3 \mid (6 \cdot 45) \Rightarrow 3 \mid 45$
- $3 \mid (5 \cdot 42) \Rightarrow 3 \mid 42$
- $3 \mid (2 \cdot 5 \cdot 21) \Rightarrow 3 \mid 21$
- $3 \mid (2 \cdot 3 \cdot 5 \cdot 7) \Rightarrow 3 \mid 3$

Notice that 3 is prime in $\mathbb{N}^*$, hence it always divides at least 1 term in the product.
What about composite numbers in $\mathbb{N}^*$?

Consider $6 \in \mathbb{N}^*$, which is composite. Notice that $6|210$. 
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What about composite numbers in $\mathbb{N}^*$?

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\[6|(7 \cdot 30) \quad \Rightarrow \quad 6|30\]
\[6|(3 \cdot 70) \quad \Rightarrow \quad 6 \nmid 3 \text{ and } 6 \nmid 70\]
What about composite numbers in $\mathbb{N}^*$?

Consider $6 \in \mathbb{N}^*$, which is composite. Notice that $6|210$.

\[
\begin{align*}
6|(7 \cdot 30) & \Rightarrow 6|30 \\
6|(3 \cdot 70) & \Rightarrow 6 \nmid 3 \text{ and } 6 \nmid 70 \\
6|(6 \cdot 45) & \Rightarrow 6|6 \\
6|(5 \cdot 42) & \Rightarrow 6|42 \\
6|(2 \cdot 5 \cdot 21) & \Rightarrow 6 \nmid 2, 6 \nmid 5, \text{ and } 6 \nmid 21 \\
6|(2 \cdot 3 \cdot 5 \cdot 7) & \Rightarrow 6 \nmid 2, 6 \nmid 3, 6 \nmid 5, \text{ and } 6 \nmid 7
\end{align*}
\]
What about composite numbers in $\mathbb{N}^*$?

\[ 6 \mid (2 \cdot 5 \cdot 21) \implies 6 \nmid 2, 6 \nmid 5, \text{ and } 6 \nmid 21 \]
What about composite numbers in $\mathbb{N}^*$?

\[ 6 | (2 \cdot 5 \cdot 21) \implies 6 \nmid 2, 6 \nmid 5, \text{ and } 6 \nmid 21 \]

\[ \Rightarrow \]

Notice that 6 does not divide 2, 5 or 21, but 6 divides the product of 2 · 21.
What about composite numbers in \( \mathbb{N}^* \)?

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6 \mid (2 \cdot 5 \cdot 21) \Rightarrow 6 \nmid 2, 6 \nmid 5, \text{ and } 6 \nmid 21
\]

- Notice that 6 does not divide 2, 5 or 21, but 6 divides the product of 2 \cdot 21.
- It turns out that we can guarantee that 6 always divides some sub-product of 2 terms.
What about composite numbers in $\mathbb{N}^*$?

\[ 6 \mid (2 \cdot 5 \cdot 21) \implies 6 \nmid 2, 6 \nmid 5, \text{ and } 6 \nmid 21 \]

- Notice that 6 does not divide 2, 5 or 21, but 6 divides the product of 2 \cdot 21.
- It turns out that we can guarantee that 6 always divides some sub-product of 2 terms. While 6 isn’t prime in $\mathbb{N}^*$, it is “not very far” from being prime.
**ω-primality**

**Definition:** Let $M$ be a cancellative, commutative monoid. For any non-unit $x \in M$, $\omega(x) = m$ if the following is true: $m$ is the smallest positive integer with the property that whenever $x \mid \prod_{i=1}^{r} a_i$ for $r > m$, there exists a subset $T \subset \{1, 2, \ldots, r\}$ with $|T| \leq m$ such that $x \mid \prod_{i \in T} a_i$.
In other words... Given $x \in M$, if $\omega(x) = m$, then:
In other words... Given $x \in M$, if $\omega(x) = m$, then:

Whenever $x$ divides a product of more than $m$ terms, we know that $x$ divides some sub product of at least $m$ terms.
Irreducibility

**Definition:** Let $M$ be a monoid. We say a non-unit $x \in M$ is *irreducible* if whenever $x = a \ast b$ then $a$ is a unit or $b$ is a unit.
Prime vs. Irreducible

All prime elements are irreducible, but not all irreducible elements are prime.

In the case of $\mathbb{N}^*$, the prime and irreducible elements are the same.
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All prime elements are irreducible, but not all irreducible elements are prime.

In the case of $\mathbb{N}^*$, the prime and irreducible elements are the same.

What about numerical monoids?
Primality and irreducibility in numerical monoids
Recall that numerical monoids are additive monoids, not multiplicative.
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“Divisibility” is in terms of subtraction, not division.
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The only irreducible elements in a numerical monoid are the minimal generators.
Primality and irreducibility in numerical monoids

- Recall that numerical monoids are additive monoids, not multiplicative.
- “Divisibility” is in terms of subtraction, not division.
- The only irreducible elements in a numerical monoid are the minimal generators.
- None of the generators are prime.
Why is $\omega$-primality interesting in numerical monoids?
Why is $\omega$-primality interesting in numerical monoids?

$\Gamma = \langle 6, 7 \rangle$
Recall: The irreducible elements in a numerical monoid are the minimal generators.
Primality of generators in numerical monoids

**Recall:** The irreducible elements in a numerical monoid are the minimal generators.

Although none of these elements are prime, previous work has investigated the $\omega$-primality of the generators to see “how prime” they are.
Previously known results

Chapman et al. provide a closed form for the $\omega$-primality of generators in $\Gamma = \langle m, m + 1, m + 2 \rangle$. 
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Chapman et al. provide a closed form for the $\omega$-primality of generators in $\Gamma = \langle m, m + 1, m + 2 \rangle$.

We sought to generalize these results for $\Gamma = \langle m, m + d, m + 2d \rangle$ for step size $d$. 
Example

\[
\begin{array}{|c|c|}
\hline
\text{Generator} & \omega\text{-primality} \\
\hline
3 & 2 \\
5 & 4 \\
7 & 4 \\
\hline
\end{array}
\]

\(\omega(n)\) for the generators of the monoid \(\langle 3, 5, 7 \rangle\).
Numerical monoids with step size 2

<table>
<thead>
<tr>
<th>Gen</th>
<th>ω</th>
<th>Gen</th>
<th>ω</th>
<th>Gen</th>
<th>ω</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>5</td>
<td>3</td>
<td>7</td>
<td>4</td>
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<tr>
<td>5</td>
<td>4</td>
<td>7</td>
<td>5</td>
<td>9</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>4</td>
<td>9</td>
<td>5</td>
<td>11</td>
<td>6</td>
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<tr>
<td>Gen</td>
<td>ω</td>
<td>Gen</td>
<td>ω</td>
<td>Gen</td>
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<tr>
<td>9</td>
<td>5</td>
<td>11</td>
<td>6</td>
<td>13</td>
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<tr>
<td>11</td>
<td>7</td>
<td>13</td>
<td>8</td>
<td>15</td>
<td>9</td>
</tr>
<tr>
<td>13</td>
<td>7</td>
<td>15</td>
<td>8</td>
<td>17</td>
<td>9</td>
</tr>
</tbody>
</table>

ω(n) of the generators for \( \Gamma = \langle m, m + 2, m + 4 \rangle \) where \( m \) is odd.
Example

This table displays $\omega(n)$ for the generators of the monoid $\langle 5, 9, 13 \rangle$.

<table>
<thead>
<tr>
<th>Generator</th>
<th>$\omega$-primality</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>9</td>
<td>7</td>
</tr>
<tr>
<td>13</td>
<td>7</td>
</tr>
</tbody>
</table>
Numerical monoids with step size 4

\[
\begin{array}{c|c}
\text{Gen} & \omega \\
5 & 3 \\
9 & 7 \\
13 & 7 \\
\end{array}
\begin{array}{c|c}
\text{Gen} & \omega \\
7 & 4 \\
11 & 8 \\
15 & 8 \\
\end{array}
\begin{array}{c|c}
\text{Gen} & \omega \\
9 & 5 \\
13 & 9 \\
17 & 9 \\
\end{array}
\begin{array}{c|c}
\text{Gen} & \omega \\
11 & 6 \\
15 & 10 \\
19 & 10 \\
\end{array}
\begin{array}{c|c}
\text{Gen} & \omega \\
13 & 7 \\
17 & 11 \\
21 & 11 \\
\end{array}
\]

\(\omega(n)\) of the generators for \(\Gamma = \langle m, m + 4, m + 8 \rangle\) where \(m\) is odd.
Conjecture: Let $\Gamma = \langle m, m + d, m + 2d \rangle$ be a numerical monoid.
Conjecture: Let $\Gamma = \langle m, m + d, m + 2d \rangle$ be a numerical monoid. Then,

$$\omega(m) = \left\lceil \frac{m}{2} \right\rceil$$
$$\omega(m + d) = \left\lceil \frac{m}{2} \right\rceil + d + 1$$
$$\omega(m + 2d) = \left\lceil \frac{m}{2} \right\rceil + d.$$
“Bad elements” in $\Gamma$
“Bad elements” in $\Gamma$

$\Gamma = \langle 4, 7, 10 \rangle$
Bad elements

A bad element $b \in \Gamma$ is such that $\omega(b)$ does not follow the eventual quasilinear pattern. The set of bad elements in $\Gamma$ is denoted $\text{Bad}(\Gamma)$. 
Bad elements

A **bad element** $b \in \Gamma$ is such that $\omega(b)$ does not follow the eventual quasilinear pattern.

The set of bad elements in $\Gamma$ is denoted $Bad(\Gamma)$. 
Quasilinearity and the dissonance point

Given a sufficiently large \( n \in \Gamma \), \( \omega(n) \) is eventually quasilinear.

The dissonance point, denoted \( \text{dis}(\Gamma) \), is the largest \( n \in \Gamma \) such that \( \omega(n) \) does not follow the eventual quasilinear pattern. That is, \( \text{dis}(\Gamma) = \max(\text{Bad}(\Gamma)) \).
Quasilinearity and the dissonance point

Given a sufficiently large $n \in \Gamma$, $\omega(n)$ is *eventually* quasilinear,
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The *dissonance point*, denoted $\text{dis}(\Gamma)$, is the largest $n \in \Gamma$ such that $\omega(n)$ does not follow the eventual quasilinear pattern.

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Bullets

Let $\Gamma = \langle n_1, n_2, \ldots, n_k \rangle$ be a numerical monoid.
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Let $\Gamma = \langle n_1, n_2, \ldots, n_k \rangle$ be a numerical monoid.

A factorization $[a_1, \ldots, a_k] \in \mathbb{N}^k$ is a bullet for $n \in \Gamma$ if the following hold:

1. $\left( \sum_{i=1}^{k} a_i n_i \right) - n \in \Gamma$
2. $\left( \sum_{i=1}^{k} a_i n_i - n_j \right) - n_j \in \Gamma$ for $a_j > 0$
Bullets

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Bullets

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A factorization $[a_1, \ldots, a_k] \in \mathbb{N}^k$ is a **bullet** for $n \in \Gamma$ if the following hold:

\[
\begin{align*}
\sum_{i=1}^{k} a_i n_i - n &\in \Gamma \\
\sum_{i=1}^{k} a_i n_i - n_j - n &\notin \Gamma \text{ for } a_j > 0
\end{align*}
\]

The set of bullets for $n \in \Gamma$ is denoted $\text{bul}(n)$. 
Let $\Gamma = \langle n_1, n_2, \ldots, n_k \rangle$ be a numerical monoid.

A factorization $[a_1, \ldots, a_k] \in \mathbb{N}^k$ is a **bullet** for $n \in \Gamma$ if the following hold:

\[
\left( \sum_{i=1}^{k} a_i n_i \right) - n \in \Gamma
\]

\[
\left( \sum_{i=1}^{k} a_i n_i - n_j \right) - n \notin \Gamma \text{ for } a_j > 0
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The set of bullets for $n \in \Gamma$ is denoted $\text{bul}(n)$.

The $\omega$-primality for an element is equal to the length of its maximum length bullet.
Motivation

Let $\Gamma = \langle 3, 4 \rangle$
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For $n = 15$, if $\omega(n)$ was completely quasilinear, we would have $\omega(15) = 5$
Motivation

Let \( \Gamma = \langle 3, 4 \rangle \)

For \( n = 15 \), if \( \omega(n) \) was completely quasilinear, we would have \( \omega(15) = 5 \)

However, notice that \( \omega(15) = 6 \).
Conjecture: Let $\Gamma = \langle m, m + d, m + 2d \rangle$. Then,
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- $Bad(\Gamma) = \{0, \text{dis}(\Gamma)\}$
Conjecture: Let $\Gamma = \langle m, m + d, m + 2d \rangle$. Then,

- $Bad(\Gamma) = \{0, \text{dis}(\Gamma)\}$
- For every $b \in Bad(\Gamma)$, $b$ has a unique maximal length bullet.
**Conjecture:** Let $\Gamma = \langle m, m + d, m + 2d \rangle$. Then,

- $Bad(\Gamma) = \{0, \text{dis}(\Gamma)\}$
- For every $b \in Bad(\Gamma)$, $b$ has a unique maximal length bullet.
- The *only* maximal length bullet of dis$(\Gamma)$ is of the form $[0, \omega, 0]$. 
Dissonance points for $\Gamma$

Let $\Gamma = \langle m, m + d, m + 2d \rangle$. 
Dissonance points for $\Gamma$

Let $\Gamma = \langle m, m + d, m + 2d \rangle$.

- Examples suggest that arithmetically generated numerical monoids of embedding dimension 3 only have a non-trivial dissonance point if the multiplicity is even.
Dissonance points for $\Gamma$

Let $\Gamma = \langle m, m + d, m + 2d \rangle$.

- Examples suggest that arithmetically generated numerical monoids of embedding dimension 3 only have a non-trivial dissonance point if the multiplicity is even.

- When the multiplicity is odd, $\Gamma$ has a trivial dissonance point.
Conjecture: Let $\Gamma = \langle m, m + d, m + 2d \rangle$.

$$dis(\Gamma) = \begin{cases} \frac{m^2}{2} & \text{if } m \text{ is even.} \\ 0 & \text{if } m \text{ is odd.} \end{cases}$$
Example

\[ \omega(n) \] for Generators

Dissonance points in \( \Gamma \)

Quasilinearity in \( \Gamma \)

Bad elements

Dissonance points

\[ \Gamma_1 = \langle 3, 8, 13 \rangle \]

\[ \Gamma_2 = \langle 4, 9, 14 \rangle \]

Notice that we have \( \text{dis}(\Gamma_1) = 0 \) and \( \text{dis}(\Gamma_2) = 8 \).
Example

\[ \Gamma_1 = \langle 3, 8, 13 \rangle \quad \text{and} \quad \Gamma_2 = \langle 4, 9, 14 \rangle \]
Example

\[ \Gamma_1 = \langle 3, 8, 13 \rangle \quad \Gamma_2 = \langle 4, 9, 14 \rangle \]

Notice that we have \( \text{dis}(\Gamma_1) = 0 \) and \( \text{dis}(\Gamma_2) = 8 \).
\[ \Gamma = \langle 4, 7, 10 \rangle \]
Example: Consider $\Gamma = \langle 4, 7, 10 \rangle$ and the element $8 \in \Gamma$.
Example: Consider $\Gamma = \langle 4, 7, 10 \rangle$ and the element $8 \in \Gamma$.

- $\text{dis}(\Gamma) = \frac{4^2}{2} = 8$
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- $\text{dis}(\Gamma) = \frac{4^2}{2} = 8$
- The set of bullets is
  $$\text{bul}(8) = \{ [2, 0, 0], [0, 2, 1], [1, 2, 0], [0, 4, 0] \}.$$
Example: Consider \( \Gamma = \langle 4, 7, 10 \rangle \) and the element \( 8 \in \Gamma \).

- \( \text{dis}(\Gamma) = \frac{4^2}{2} = 8 \)
- The set of bullets is
  \[
  \text{bul}(8) = \{ [2, 0, 0], [0, 2, 1], [1, 2, 0], [0, 4, 0] \}.
  \]
- The maximal length bullet of 8 is \([0, 4, 0]\).
Eventual Quasilinearity in $\omega(n)$

**Recap:** We have explored the $\omega$-primality for the following cases:
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- Generators
Eventual Quasilinearity in $\omega(n)$

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- The early, “noisy” period
Eventual Quasilinearity in $\omega(n)$

**Recap:** We have explored the $\omega$-primality for the following cases:

- Generators
- The early, “noisy” period
- What’s next?
Fundamentals
\( \omega(n) \) for Generators
Dissonance points in \( \Gamma \)
Quasilinearity in \( \Gamma \)

What do we know?
Closed form
Implications

\[ \Gamma = \langle 11, 12, 13 \rangle \]
\[ \Gamma = \langle 4, 5, 6 \rangle \]
ω(n) for past the dissonance point

What do we know about eventual quasilinearity?

O’Neill and Pelayo showed that in a numerical monoid:
ω(n) for past the dissonance point

What do we know about eventual quasilinearity?

O’Neill and Pelayo showed that in a numerical monoid:

» ω(n) is well-behaved for all n greater than the dissonance point;
What do we know about eventual quasilinearity?

O’Neill and Pelayo showed that in a numerical monoid:

- $\omega(n)$ is well-behaved for all $n$ greater than the dissonance point;
- $\omega(n)$ becomes *eventually* quasilinear with a period $n_1$ and a common slope $\frac{1}{n_1}$;
ω(n) for past the dissonance point

What do we know about eventual quasilinearity?

O’Neill and Pelayo showed that in a numerical monoid:

- ω(n) is well-behaved for all n greater than the dissonance point;
- ω(n) becomes eventually quasilinear with a period $n_1$ and a common slope $\frac{1}{n_1}$;
- The lines are usually equidistant from one another.
<table>
<thead>
<tr>
<th>Fundamentals</th>
<th>What do we know?</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega(n) ) for Generators</td>
<td>Closed form</td>
</tr>
<tr>
<td>Dissonance points in ( \Gamma )</td>
<td>Implications</td>
</tr>
<tr>
<td>Quasilinearity in ( \Gamma )</td>
<td></td>
</tr>
</tbody>
</table>

**\( \omega(n) \) for past the dissonance point**

What should the equation look like?

- The closed form for the \( \omega \)-primality past the dissonance point should be a set of lines.
- Each line has a common slope of \( \frac{1}{n} \).
- The intercept between each line should vary.
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ω(n) for past the dissonance point

What should the equation look like?

- The closed form for the ω-primality past the dissonance point should be a set of lines.
- Each line has a common slope of \( \frac{1}{n_1} \).
- The intercept between each line should vary.
\[ \omega(n) \text{ in arithmetically generated } \Gamma \]

**Conjecture:** Let \( \Gamma = \langle m, m + d, \ldots, m + (e - 1)d \rangle \) be a numerical monoid.
**Conjecture:** Let $\Gamma = \langle m, m + d, \ldots, m + (e - 1)d \rangle$ be a numerical monoid.

Then for all $n > \text{dis}(\Gamma)$, we have:

$$w_i(n) = \begin{cases} 
\frac{1}{m}n + \left\lceil \frac{m + e - 4}{e - 1} \right\rceil + \frac{d \cdot i}{m}, & \text{if } i < m - 1 \\
\frac{1}{m}n + \left\lceil \frac{m + e - 3}{e - 1} \right\rceil + \frac{d \cdot i}{m}, & \text{if } i = m - 1 
\end{cases}$$

$$n \equiv -d \cdot i \mod m.$$
ω(n) for Generators
Dissonance points in Γ
Quasilinearity in Γ

ω(n) in arithmetically generated Γ

In “most” cases, all lines in the quasilinear patterns are equidistant from one another.

Sometimes the top line (i = m − 1) is “shifted” up by 1.
\( \omega(n) \) in arithmetically generated \( \Gamma \)

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\[ \Gamma = \langle 4, 5, 6 \rangle \]
ω(n) in arithmetically generated Γ

**Conjecture:**

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Ashley Mailloux, Meghan Malachi, John Spaw

Primality in Arithmetically Generated Numerical Monoids
Conjecture: Let $\Gamma = \langle m, m + d, m + 2d, \ldots, m + (e - 1)d \rangle$ be a numerical monoid.
ω(n) in arithmetically generated Γ

Conjecture: Let $\Gamma = \langle m, m+d, m+2d, \ldots, m+(e-1)d \rangle$ be a numerical monoid.

$$m_{\text{shift}}(e) = \{ e + \ell(e - 1) + 1 \mid \ell \in \mathbb{N} \}. $$
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\[
= \{ 4, 6, 8, 10, \ldots \}
\]
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Fundamentals

ω(n) for Generators

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Closed form

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- We have an upward shift in the top line of the quasilinear pattern.
- There exist non-trivial bad elements.
- There exists a non-trivial dissonance point.
\[ \Gamma = \langle 10, 11, 12, 13, 14 \rangle \]
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\[ m_{\text{shift}}(5) = \{ 6 + 4\ell \mid \ell \in \mathbb{N} \} \]
\[ \Gamma = \langle 10, 11, 12, 13, 14 \rangle \]

- \( m_{\text{shift}} (5) = \{6 + 4\ell \mid \ell \in \mathbb{N}\} \)
- Shifts will occur when \( m = 6, 10, 14, 18, 22, \ldots \).
\( \Gamma = \langle 10, 11, 12, 13, 14 \rangle \)

- \( m_{\text{shift}}(5) = \{ 6 + 4\ell \mid \ell \in \mathbb{N} \} \)
- Shifts will occur when \( m = 6, 10, 14, 18, 22, \ldots \).
- Notice: A top line shift will not occur for \( m = 11 \).
**Figure:** Plots of $\omega(n)$ for $\langle 10, 11, 12, 13, 14 \rangle$ and $\langle 11, 12, 13, 14, 15 \rangle$
**Fundamentals**

$\omega(n)$ for Generators

Dissonance points in $\Gamma$

Quasilinearity in $\Gamma$

What do we know?

Closed form

Implications

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**Figure**: Plots of $\omega(n)$ for $\langle 10, 11, 12, 13, 14 \rangle$ and $\langle 11, 12, 13, 14, 15 \rangle$

Notice that the top line is shifted in the left plot and the lines are all equidistant in the right plot.
## Acknowledgements

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Questions?