Co-circular Kite Central Configurations in the 4-body Problem

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Outline

Introduction

Polynomials in $r_{ij}^3$

Perfect Squares
Glossary

Definition (Variety)

A *variety* is the set of common zeros of a collection of polynomials.
Glossary

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**Definition (Ideal)**

A subset $I \subset R[x_1, \ldots, x_n]$ is an *ideal* if the following holds true:

(i) $0 \in I$

(ii) If $f, g \in I$, then $f + g \in I$

(iii) If $f \in I$ and $h \in R[x_1, \ldots, x_n]$, then $hf \in I$
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**Definition (Elimination Ideal)**

Given $I = \langle f_1, \ldots, f_s \rangle \subset R[x_1, \ldots, x_n]$ the *l-th elimination ideal* $I_l$ is the ideal of $R[x_1, \ldots, x_n]$ defined by

$$I_l = I \cap R[x_{l+1}, \ldots, x_n].$$
Albouy-Chenciner Equations

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For each pair of $1 \leq i, j \leq n$:

$$G_{ij} = \sum_{k=1}^{n} m_k S_{ki} (r_{jk}^2 - r_{ik}^2 - r_{ij}^2) = 0$$

where $S_{ki} = \begin{cases} r - 3 & \text{if } k \neq i \\ 0 & \text{if } k = i \end{cases}$. If $i = j$ for each pair $1 \leq i, j \leq n$, then we get a cancellation in the AC equations.
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where

$$S_{ki} = \begin{cases} r_{ki}^{-3} - 1 & \text{if } k \neq i \\ 0 & \text{if } k = i \end{cases}$$

Note that $r_{ij} = r_{ji}$ for all $i$ and $j$, thus $S_{ij} = S_{ji}$. If $i=j$ for each pair $1 \leq i, j \leq n$, then we get a cancellation in the AC equations.
4-Body Problem

We consider a special case of the 4-body problem, specifically the co-circular kite

**Theorem (Finiteness [Hampton, Moeckel])**

Given a collection of positive masses, there exist finitely many central configurations in the 4-body problem, up to equivalence under rotation, translation, and scaling.
Definitions

**Definition (Kite)**

A *kite* is a convex quadrilateral with two pairs of distinct congruent sides adjacent to each other.

**Definition (Co-circular Kite)**

A *co-circular kite* is a kite that is inscribed in a circle and one of the diagonals forms the diameter of the circle.
Setting Up Co-circular Kite Configurations

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\[ r_{23} = r_{12} \]
\[ r_{34} = r_{14} \]
\[ m_1 = m_3 \]
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\[ k_0 = 1 - tr_{12}r_{13}r_{24}r_{14} \]
Ptolemy

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\[ = r_{13}r_{24} - 2r_{14}r_{12} \]
Setting Up Co-circular Kite Configurations

The Cayley-Menger determinant defined by $CM$ forces the masses to be coplanar.

$$CM = \begin{vmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & r_{12}^2 & r_{13}^2 & r_{14}^2 \\
1 & r_{12}^2 & 0 & r_{23}^2 & r_{24}^2 \\
1 & r_{13}^2 & r_{23}^2 & 0 & r_{34}^2 \\
1 & r_{14}^2 & r_{24}^2 & r_{34}^2 & 0 \\
\end{vmatrix}$$
Setting Up Co-circular Kite Configurations

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0 & 1 & 1 & 1 & 1 \\
1 & 0 & r_{12}^2 & r_{13}^2 & r_{14}^2 \\
1 & r_{12}^2 & 0 & r_{23}^2 & r_{24}^2 \\
1 & r_{13}^2 & r_{23}^2 & 0 & r_{34}^2 \\
1 & r_{14}^2 & r_{24}^2 & r_{34}^2 & 0
\end{vmatrix}$$

After kite substitutions:

$$CM = (-2)r_{13}^2(r_{12}^4 + r_{13}^2r_{24}^2 - 2r_{12}^2r_{24}^2 + r_{24}^4 - 2r_{12}^2r_{14}^2 - 2r_{24}^2r_{14}^2 + r_{14}^4)$$
Example 1, $m_4 = \frac{1}{5}$

Example

$$K = [G_{12}, G_{21}, G_{13}, G_{31}, G_{14}, G_{41}, G_{34}, G_{43}, G_{24}, G_{42}, G_{23}, G_{32}, \text{nonzero, ptol, cm}]$$
Example

K = [G12, G21, G13, G31, G14, G41, G34, G43, G24, G42,
    G23, G32, nonzero, ptol, cm]

Ielim = ideal(K)
R24 = Ielim.elimination_ideal([t, r12, r14, r13, m2])
R24.gen(0).factor()
Example 1, \( m_4 = \frac{1}{5} \)

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\[ I_{elim} = \text{ideal}(K) \]
\[ R_{24} = I_{elim}.\text{elimination\_ideal}([t, r_{12}, r_{14}, r_{13}, m_{2}]) \]
\[ R_{24}.\text{gen}(0).\text{factor}() \]

Output:

\[(2230918027632\times r_{24}^{24} - 1557072438768\times r_{24}^{21} - 6057772562340\times r_{24}^{18} + 1636987206176\times r_{24}^{15} + 4385658414560\times r_{24}^{12} - 839052304212\times r_{24}^{9} - 1269888493452\times r_{24}^{6} + 9780\times r_{24}^{3} - 1)^{2}\]
Example 2, \( m_4 = \frac{1}{4} \)

**Example**

\[ K_2 = [G_{12}, G_{21}, G_{13}, G_{31}, G_{14}, G_{41}, G_{34}, G_{43}, G_{24}, G_{42}, G_{23}, G_{32}, \text{nonzero}, \text{ptol}, \text{cm}] \]
Example 2, $m_4 = \frac{1}{4}$

Example

\begin{verbatim}
K2 = [G12, G21, G13, G31, G14, G41, G34, G43, G24, G42, G23, G32, nonzero, ptol, cm]

Ielim = ideal(K2)
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R24.gen(0).factor()
\end{verbatim}
Example 2, $m_4 = \frac{1}{4}$

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K_2 = [G_{12}, G_{21}, G_{13}, G_{31}, G_{14}, G_{41}, G_{34}, G_{43}, G_{24}, G_{42}, G_{23}, G_{32}, \text{nonzero}, \text{ptol}, \text{cm}]
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Output:

\[(106570108575\times r_{24}^{18} - 255577677530\times r_{24}^{15} + 63682445617\times r_{24}^{12} + 148432515924\times r_{24}^{9} - 85955082223\times r_{24}^{6} + 5030\times r_{24}^{3} - 1)^2\]
Example 3, Square

Let \( r_{23} = r_{34} = r_{14} = r_{12} \), all the \( m_i = 1 \), diagonals equal \( r_{12}\sqrt{2} \)
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Let $r_{23} = r_{34} = r_{14} = r_{12}$, all the $m_i = 1$, diagonals equal $r_{12}\sqrt{2}$

Example

$$K3 = [G12, G21, G13, G31, G14, G41, G34, G43, G24, G42, G23, G32, \text{nonzero, ptol, cm}]$$
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### Example

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Example

\begin{verbatim}
K3 = [G12,G21,G13,G31,G14,G41,G34,G43,G24,G42,
     G23,G32,nonzero,ptol,cm]

Ielim = ideal(K3)
R12 = Ielim.elimination_ideal([t,r13,r24])
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\end{verbatim}

Output:

$$32r_{12}^6 - 32r_{12}^3 + 7$$
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\text{Ielim} = \text{ideal}(K3)
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\]
\[
\text{R12}.\text{gen}(0).\text{factor}()
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Output:

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32r_{12}^6 - 32r_{12}^3 + 7
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The generating polynomial for the elimination ideal is not always a perfect square for all co-circular configurations.
Observation

In all our examples, we noticed that the polynomials that were generated by our elimination ideal, had exponents divisible by 3.
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However, in only examples 1 and 2, we notice that

$$(I_{AC} + \langle CM, 1 - tr_1r_4r_2r_1, r_2r_1 - 2r_1r_4\rangle) \cap \mathbb{Q}[r_{ij}] = \langle f(r_{ij})^2 \rangle$$
Observation

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However, in only examples 1 and 2, we notice that

\[(I_{AC} + \langle CM, 1 - tr_{12}r_{14}r_{24}r_{13}, r_{24}r_{13} - 2r_{12}r_{14}\rangle) \cap \mathbb{Q}[r_{ij}] = \langle f(r_{ij})^2 \rangle\]

The ideal defined above intersect the rational polynomials in \(r_{ij}\) will equal an elimination ideal generated by a perfect square polynomial, \(f(r_{ij})^2\).
Theorem (BLM)

Given a variety generated by the AC equations for a 4-body problem with fixed mass values together with a nonzero constraint, the generators of the elimination ideals for the distance variables are polynomials in $r_{ij}^3$ (i.e. have all exponents divisible by 3).
Polynomials in $r_{ij}^3$

Theorem (BLM)

Given a variety generated by the AC equations for a 4-body problem with fixed mass values together with a nonzero constraint, the generators of the elimination ideals for the distance variables are polynomials in $r_{ij}^3$ (i.e. have all exponents divisible by 3).

We will now prove this theorem!
Proof of Theorem

Proof:
Recall $k_0 = 1 - tr_{12}r_{13}r_{14}r_{23}r_{24}r_{34}$ in order to force the distance variables to be nonzero.
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Recall $k_0 = 1 - tr_{12}r_{13}r_{14}r_{23}r_{24}r_{34}$ in order to force the distance variables to be nonzero. Let $a = (r_{12}, r_{13}, r_{14}, r_{23}, r_{24}, r_{34}, t)$ be in the variety defined by $\nabla(C)$, where $C = I_{AC} + \langle k_0 \rangle$ and $I_{AC}$ is the ideal generated by the AC equations for the 4-body problem with fixed values for the masses.
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Let $\rho$ be a primitive cube root of unity.
Proof of Theorem

Proof:
Recall \( k_0 = 1 - tr_{12}r_{13}r_{14}r_{23}r_{24}r_{34} \) in order to force the distance variables to be nonzero. Let \( a = (r_{12}, r_{13}, r_{14}, r_{23}, r_{24}, r_{34}, t) \) be in the variety defined by \( \mathbb{V}(C) \), where \( C = I_{AC} + \langle k_0 \rangle \) and \( I_{AC} \) is the ideal generated by the AC equations for the 4-body problem with fixed values for the masses. This implies that \( a \) is in \( \mathbb{V}(C) \).

Let \( \rho \) be a primitive cube root of unity.

We claim that \( \rho \cdot a := (\rho r_{12}, \rho r_{13}, \rho r_{14}, \rho r_{23}, \rho r_{24}, \rho r_{34}, t) \) is in the variety of \( C \).
Proof of Theorem

The AC equations for the general 4-body problem are in the form

\[ G_{ij} = -2m_j S_{ij}(r_{ij}^2) + m_h S_{ih}(r_{jh}^2 - r_{ih}^2 - r_{ij}^2) + m_l S_{il}(r_{jl}^2 - r_{il}^2 - r_{ij}^2) = 0 \]

where \( h, l \in \{1, 2, 3, 4\} \setminus \{i\} \) such that \( h, i, l \) distinct
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\[ S_{ij} = \begin{cases} 
    r_{ij}^{-3} - 1 & \text{if } j \neq i \\
    0 & \text{if } j = i
\end{cases} \]
Proof of Theorem

Substituting $\rho \cdot a$, $S_{ij} = \rho^{-3} r_{ij}^{-3} - 1 = r_{ij}^{-3} - 1$ by a property of the cube root of unity, and

$$G_{ij} = -2m_j S_{ij}(\rho^2 r_{ij}^2) + m_h S_{ih}(\rho^2 r_{jh}^2 - \rho^2 r_{ih}^2 - \rho^2 r_{ij}^2) + m_l S_{il}(\rho^2 r_{jl}^2 - \rho^2 r_{il}^2 - \rho^2 r_{ij}^2)$$
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$$= -2m_j \rho^2 S_{ij}(r_{ij}^2) + m_h \rho^2 S_{ih}(r_{jh}^2 - r_{ih}^2 - r_{ij}^2) + m_l \rho^2 S_{il}(r_{jl}^2 - r_{il}^2 - r_{ij}^2)$$
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$$= \rho^2 [-2m_j S_{ij}(r_{ij}^2) + m_h S_{ih}(r_{jh}^2 - r_{ih}^2 - r_{ij}^2) + m_l S_{il}(r_{jl}^2 - r_{il}^2 - r_{ij}^2)]$$
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\[
= \rho^2 (0) = 0,
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$$G_{ij} = -2m_j S_{ij}(\rho^2 r_{ij}^2) + m_h S_{ih}(\rho^2 r_{jh}^2 - \rho^2 r_{ih}^2 - \rho^2 r_{ij}^2) +$$

$$m_l S_{il}(\rho^2 r_{jl}^2 - \rho^2 r_{il}^2 - \rho^2 r_{ij}^2)$$

$$= -2m_j \rho^2 S_{ij}(r_{ij}^2) + m_h \rho^2 S_{ih}(r_{jh}^2 - r_{ih}^2 - r_{ij}^2) + m_l \rho^2 S_{il}(r_{jl}^2 - r_{il}^2 - r_{ij}^2)$$

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$$= \rho^2 (0) = 0,$$

Thus, $\rho \cdot a$ is still a solution for the AC equations.
Proof of Theorem

Similarly, substituting $\rho \cdot a$ into $k_0$, we have

$$k_0 = 1 - t(\rho r_{12})(\rho r_{13})(\rho r_{23})(\rho r_{24})(\rho r_{14})(\rho r_{34})$$
Proof of Theorem

Similarly, substituting \( \rho \cdot a \) into \( k_0 \), we have

\[
\begin{align*}
 k_0 & = 1 - t(\rho r_{12})(\rho r_{13})(\rho r_{23})(\rho r_{24})(\rho r_{14})(\rho r_{34}) \\
 & = 1 - t(\rho^6)r_{12}r_{13}r_{23}r_{24}r_{14}r_{34}
\end{align*}
\]
Proof of Theorem

Similarly, substituting $\rho \cdot a$ into $k_0$, we have

$$k_0 = 1 - t(\rho r_{12})(\rho r_{13})(\rho r_{23})(\rho r_{24})(\rho r_{14})(\rho r_{34})$$

$$= 1 - t(\rho^6)r_{12}r_{13}r_{23}r_{24}r_{14}r_{34}$$

Since $\rho^6 = 1$, $\rho \cdot a$ is still a solution for $k_0$
Proof of Theorem

If \( r_{ij} = \alpha \) is a root of the elimination ideal generator, then \( \alpha \neq 0, \rho \cdot \alpha \neq 0, \) and \( \rho^2 \cdot \alpha \neq 0. \)
Proof of Theorem

If \( r_{ij} = \alpha \) is a root of the elimination ideal generator, then
\( \alpha \neq 0, \ \rho \cdot \alpha \neq 0, \) and \( \rho^2 \cdot \alpha \neq 0. \) Thus, by the Factor Theorem,

\[
(r_{ij} - \alpha)(r_{ij} - \rho \alpha)(r_{ij} - \rho^2 \alpha) \mid f(r_{ij})
\]
Proof of Theorem

If \( r_{ij} = \alpha \) is a root of the elimination ideal generator, then \( \alpha \neq 0, \rho \cdot \alpha \neq 0, \) and \( \rho^2 \cdot \alpha \neq 0. \) Thus, by the Factor Theorem,

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(r_{ij} - \alpha)(r_{ij} - \rho\alpha)(r_{ij} - \rho^2\alpha) \mid f(r_{ij})
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\[
r_{ij}^3 - r_{ij}^2(\rho^2\alpha + \rho\alpha + \alpha) + r_{ij}(\rho^2\alpha^2 + \rho\alpha^2 + \alpha^2) - (\alpha^3) \mid f(r_{ij})
\]
Proof of Theorem

If \( r_{ij} = \alpha \) is a root of the elimination ideal generator, then \( \alpha \neq 0 \), \( \rho \cdot \alpha \neq 0 \), and \( \rho^2 \cdot \alpha \neq 0 \). Thus, by the Factor Theorem,

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\]

Note that given a primitive cube root of unity,

\[
\rho^3 - 1 = 0
\]
Proof of Theorem

If \( r_{ij} = \alpha \) is a root of the elimination ideal generator, then \( \alpha \neq 0 \), \( \rho \cdot \alpha \neq 0 \), and \( \rho^2 \cdot \alpha \neq 0 \). Thus, by the Factor Theorem,

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\[
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Note that given a primitive cube root of unity,

\[
\rho^3 - 1 = 0
\]

\[
(\rho - 1)(\rho^2 + \rho + 1) = 0
\]
Proof of Theorem

If \( r_{ij} = \alpha \) is a root of the elimination ideal generator, then \( \alpha \neq 0, \ \rho \cdot \alpha \neq 0, \ \text{and} \ \rho^2 \cdot \alpha \neq 0. \) Thus, by the Factor Theorem,

\[
(r_{ij} - \alpha)(r_{ij} - \rho\alpha)(r_{ij} - \rho^2\alpha) \mid f(r_{ij})
\]

\[
r_{ij}^3 - r_{ij}^2(\rho^2\alpha + \rho\alpha + \alpha) + r_{ij}(\rho^2\alpha^2 + \rho\alpha^2 + \alpha^2) - (\alpha^3) \mid f(r_{ij})
\]

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(\rho^2\alpha + \rho\alpha + \alpha) = \alpha(\rho^2 + \rho + 1)
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\]

\[
(\rho^2 \alpha^2 + \rho \alpha^2 + \alpha^2) = \alpha^2(\rho^2 + \rho + 1)
\]
\[ r_{ij}^3 - r_{ij}^2(\alpha(\rho^2 + \rho + 1)) + r_{ij}(\alpha^2(\rho^2 + \rho + 1)) - (\alpha^3) \mid f(r_{ij}) \]
\[ r_{ij}^3 - r_{ij}^2(\alpha(\rho^2 + \rho + 1)) + r_{ij}(\alpha^2(\rho^2 + \rho + 1)) - (\alpha^3) \mid f(r_{ij}) \]

Hence \((r_{ij}^3 - \alpha^3) \mid f(r_{ij}).\)
\[ r_{ij}^3 - r_{ij}^2(\alpha(\rho^2 + \rho + 1)) + r_{ij}(\alpha^2(\rho^2 + \rho + 1)) - (\alpha^3) \mid f(r_{ij}) \]

Hence \( (r_{ij}^3 - \alpha^3) \mid f(r_{ij}) \). Therefore, \( f(r_{ij}) \) will be a polynomial in \( r_{ij}^3 \). Q.E.D.
Corollary

Given a variety generated by $\overline{I_{AC}} + \langle p, k_0, r_{14} - r_{34}, r_{12} - r_{23} \rangle$, the generators of the elimination ideals for the distance variables are polynomials in $r_{ij}^3$ (i.e. have all exponents divisible by 3).
Corollary

*Given a variety generated by $\overline{I_{AC}} + \langle p, k_0, r_{14} - r_{34}, r_{12} - r_{23} \rangle$, the generators of the elimination ideals for the distance variables are polynomials in $r_{ij}^3$ (i.e. have all exponents divisible by 3).*

Now we wish to set up the conditions necessary to have a co-circular kite configuration. Recall
Corollary

Given a variety generated by $I_{AC} + \langle p, k_0, r_{14} - r_{34}, r_{12} - r_{23} \rangle$, the generators of the elimination ideals for the distance variables are polynomials in $r_{ij}^3$ (i.e. have all exponents divisible by 3).

Now we wish to set up the conditions necessary to have a co-circular kite configuration. Recall

$$p = r_{13}r_{24} - r_{14}r_{23} - r_{12}r_{34}$$

in order to satisfy the co-circular condition and include
Corollary

Given a variety generated by \( \overline{I_{AC}} + \langle p, k_0, r_{14} - r_{34}, r_{12} - r_{23} \rangle \), the generators of the elimination ideals for the distance variables are polynomials in \( r_{ij}^3 \) (i.e. have all exponents divisible by 3).

Now we wish to set up the conditions necessary to have a co-circular kite configuration. Recall

\[
p = r_{13}r_{24} - r_{14}r_{23} - r_{12}r_{34}
\]

in order to satisfy the co-circular condition and include

\[
\begin{align*}
r_{14} - r_{34} \\
r_{12} - r_{23}
\end{align*}
\]

in order to set up the geometry of a kite.
Proof of Corollary

Proof:
Proof of Corollary

Proof:
Using a similar argument as Theorem BLM, let
\( a = (r_{12}, r_{13}, r_{23}, r_{24}, r_{14}, r_{34}, t) \) be in the variety defined by \( \mathbb{V}(D) \),
where \( D = I_{AC} + \langle k_0 \rangle + \langle p \rangle + \langle r_{14} - r_{34} \rangle + \langle r_{12} - r_{23} \rangle \).
Proof of Corollary

Proof:
Using a similar argument as Theorem BLM, let
\( a = (r_{12}, r_{13}, r_{23}, r_{24}, r_{14}, r_{34}, t) \) be in the variety defined by \( \mathbb{V}(D) \),
where \( D = I_{AC} + \langle k_0 \rangle + \langle p \rangle + \langle r_{14} - r_{34} \rangle + \langle r_{12} - r_{23} \rangle \). This implies
\( a \) is in the variety of \( D \).
Proof of Corollary

Proof:
Using a similar argument as Theorem BLM, let $a = (r_{12}, r_{13}, r_{23}, r_{24}, r_{14}, r_{34}, t)$ be in the variety defined by $\mathbb{V}(D)$, where $D = I_{AC} + \langle k_0 \rangle + \langle p \rangle + \langle r_{14} - r_{34} \rangle + \langle r_{12} - r_{23} \rangle$. This implies $a$ is in the variety of $D$.
Now we will prove that $\rho \cdot a := (\rho r_{12}, \rho r_{13}, \ldots, \rho r_{34}, t)$ where $\rho$ is a cube root of unity is also in the variety of $D$.
Substituting $\rho \cdot a$ into $p$, we have
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$$p = (\rho r_{13})(\rho r_{24}) - (\rho r_{14})(\rho r_{23}) - (\rho r_{12})(\rho r_{34})$$

$$= \rho^2 (r_{13}r_{24} - r_{14}r_{23} - r_{12}r_{34})$$
Substituting $\rho \cdot a$ into $p$, we have

$$p = (\rho r_{13})(\rho r_{24}) - (\rho r_{14})(\rho r_{23}) - (\rho r_{12})(\rho r_{34})$$

$$= \rho^2(r_{13}r_{24} - r_{14}r_{23} - r_{12}r_{34})$$

$$= \rho^2(0) = 0$$
Substituting $\rho \cdot a$ into $p$, we have

\begin{align*}
p &= (\rho r_{13})(\rho r_{24}) - (\rho r_{14})(\rho r_{23}) - (\rho r_{12})(\rho r_{34}) \\
&= \rho^2(r_{13}r_{24} - r_{14}r_{23} - r_{12}r_{34}) \\
&= \rho^2(0) = 0
\end{align*}

Thus $\rho \cdot a$ substituted into $p$ is still a root.
Also substituting $\rho \cdot a$ into the kite conditions, we have
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$$\rho r_{14} - \rho r_{34}$$
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$$\rho r_{14} - \rho r_{34} = \rho (r_{14} - r_{34})$$
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$$= \rho (0) = 0$$

with a similar result for $r_{12} - r_{23}$. 
Also substituting $\rho \cdot a$ into the kite conditions, we have

$$\rho r_{14} - \rho r_{34} = \rho(r_{14} - r_{34})$$

$$= \rho(0) = 0$$

with a similar result for $r_{12} - r_{23}$.

The rest of this proof follows from the proof of Theorem BLM. Q.E.D.
Reminders

Co-circular Kite Constraints:
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\[ r_{23} = r_{12} \]
\[ r_{34} = r_{14} \]
\[ m_1 = m_3 \]
Reminders

Co-circular Kite Constraints:

\[ r_{23} = r_{12} \]
\[ r_{34} = r_{14} \]
\[ m_1 = m_3 \]
\[ k_0 = 1 - tr_{12}r_{13}r_{24}r_{14} \]
Reminders

Co-circular Kite Constraints:

\[ r_{23} = r_{12} \]
\[ r_{34} = r_{14} \]
\[ m_1 = m_3 \]
\[ k_0 = 1 - tr_{12}r_{13}r_{24}r_{14} \]
\[ \text{Ptol} = r_{13}r_{24} - r_{14}r_{23} - r_{12}r_{34} \]
Reminders

Co-circular Kite Constraints:

\[
\begin{align*}
    r_{23} & = r_{12} \\
    r_{34} & = r_{14} \\
    m_1 & = m_3 \\
    k_0 & = 1 - tr_{12}r_{13}r_{24}r_{14} \\
    \text{Ptol} & = r_{13}r_{24} - r_{14}r_{23} - r_{12}r_{34} \\
    & = r_{13}r_{24} - 2r_{14}r_{12}
\end{align*}
\]
Reminders

Co-circular Kite Constraints:

\[
\begin{align*}
    r_{23} &= r_{12} \\
    r_{34} &= r_{14} \\
    m_1 &= m_3 \\
    k_0 &= 1 - tr_{12}r_{13}r_{24}r_{14} \\
    \text{Ptol} &= r_{13}r_{24} - r_{14}r_{23} - r_{12}r_{34} \\
    &= r_{13}r_{24} - 2r_{14}r_{12} \\
    \overline{CM} &= (-2)r_{13}^2(r_{12}^4 - 2r_{12}^2r_{14}^2 + r_{14}^4 - 2r_{12}^2r_{24}^2 + r_{13}^2r_{24}^2 - 2r_{14}^2r_{24}^2 + r_{24}^4)
\end{align*}
\]
Perfect Squares

Observation:

\[
(I_{AC} + \langle CM, k_0, r_{13}r_{24} - 2r_{12}r_{14} \rangle) \cap \mathbb{Q}[r_{ij}] = \langle f(r_{ij})^2 \rangle
\]
Perfect Squares

Observation:

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(I_{AC} + \langle CM, k_0, r_{13}r_{24} - 2r_{12}r_{14}\rangle) \cap \mathbb{Q}[r_{ij}] = \langle f(r_{ij})^2 \rangle
\]

\[
r_{24}^2 = r_{12}^2 + r_{14}^2
\]
Observation:

\[(I_{AC} + \langle CM, k_0, r_{13}r_{24} - 2r_{12}r_{14} \rangle) \cap \mathbb{Q}[r_{ij}] = \langle f(r_{ij})^2 \rangle\]

\[r_{24}^2 = r_{12}^2 + r_{14}^2\]

When we add \(r_{24}^2 = r_{12}^2 + r_{14}^2\), we get

\[(I_{AC} + \langle CM, k_0, r_{13}r_{24} - 2r_{12}r_{14}, r_{14}^2 + r_{12}^2 - r_{24}^2 \rangle) \cap \mathbb{Q}[r_{ij}] = \langle f(r_{ij}) \rangle\]
**Example 1, \( m_2 = \frac{1}{5} \)**

**Example**

\[
K = \{G12, G21, G13, G31, G14, G41, G34, G43, G24, G42, G23, G32, \text{nonzero, ptol, cm}\}
\]

\[
\text{Ielim} = \text{ideal}(K)
\]

\[
\text{R24} = \text{Ielim.eliminationIdeal}([t, r12, r14, r13, m4])
\]

\[
\text{R24.gen(0).factor()}
\]

**Output:**

\[
(2230918027632*\text{r24}^24 - 1557072438768*\text{r24}^21 - 6057772562340*\text{r24}^18 + 1636987206176*\text{r24}^15 + 4385658414560*\text{r24}^12 - 839052304212*\text{r24}^9 - 1269888493452*\text{r24}^6 + 9780*\text{r24}^3 - 1)^2
\]
Example 1, $m_2 = \frac{1}{5}$

Example

\[
K = [G_{12}, G_{21}, G_{13}, G_{31}, G_{14}, G_{41}, G_{34}, G_{43}, G_{24}, G_{42}, G_{23}, G_{32}, \text{nonzero}, \text{ptol}, \text{cm}, \text{pythag}]
\]

\[
\text{Ielim} = \text{ideal}(K)
\]

\[
\text{R}_{24} = \text{Ielim} . \text{elimination\_ideal}([t, r_{12}, r_{14}, r_{13}, m_4])
\]

\[
\text{R}_{24} . \text{gen}(0) . \text{factor}()
\]

Output:

\[
2230918027632 \times r_{24}^{24} - 1557072438768 \times r_{24}^{21} - 6057772562340 \times r_{24}^{18} + 1636987206176 \times r_{24}^{15} + 4385658414560 \times r_{24}^{12} - 839052304212 \times r_{24}^{9} - 1269888493452 \times r_{24}^{6} + 9780 \times r_{24}^{3} - 1
\]
Example 2, $m_4 = \frac{1}{4}$

Example

\[
K3 = [G_{12}, G_{21}, G_{13}, G_{31}, G_{14}, G_{41}, G_{34}, G_{43}, G_{24}, G_{42}, G_{23}, G_{32}, \text{nonzero}, \text{ptol}, \text{cm}]
\]

Ielim = ideal(K3)

j = Ielim.elimination_ideal([t, r_{13}, r_{14}, r_{12}, m2])

j.gen(0).factor()

Output:

\[
(106570108575 \times r_{24}^{18} - 255577677530 \times r_{24}^{15} + 63682445617 \times r_{24}^{12} + 148432515924 \times r_{24}^{9} - 85955082223 \times r_{24}^{6} + 5030 \times r_{24}^{3} - 1)^2
\]
Example 2, \( m_4 = \frac{1}{4} \)

Example

\[
K_3 = [G_{12}, G_{21}, G_{13}, G_{31}, G_{14}, G_{41}, G_{34}, G_{43}, G_{24}, G_{42}, G_{23}, G_{32}, \text{nonzero}, \text{ptol}, \text{cm}, \text{pythag}]
\]

\[
I_{\text{elim}} = \text{ideal}(K_3)
\]

\[
j = I_{\text{elim}}.\text{elimination}\_\text{ideal}([t, r_{13}, r_{14}, r_{12}, m_2])
\]

\[
j\.gen(0).\text{factor}()
\]

Output:

\[
106570108575r_{24}^{18} - 255577677530r_{24}^{15} + 63682445617r_{24}^{12} + 148432515924r_{24}^{9} - 85955082223r_{24}^{6} + 5030r_{24}^{3} - 1
\]
Observations

More observations:

\[(r_{14}^2 + r_{12}^2 - r_{24}^2)^2 \in \langle CM, k_0, r_{13}r_{24} - r_{14}r_{23} - r_{12}r_{34}, r_{12} - r_{23}, r_{14} - r_{34} \rangle\]
Observations

More observations:

\[(r_{14}^2 + r_{12}^2 - r_{24}^2)^2 \in \langle CM, k_0, r_{13}r_{24} - r_{14}r_{23} - r_{12}r_{34}, r_{12} - r_{23},
\]
\[r_{14} - r_{34}\rangle\]

\[
\overline{CM} = (-2)r_{13}^2(r_{12}^4 - 2r_{12}^2r_{14}^2 + r_{14}^4 - 2r_{12}^2r_{24}^2 + r_{13}^2r_{24}^2
\]
\[-2r_{14}^2r_{24}^2 + r_{24}^4)\]
Observations

More observations:

\[(r^2_{14} + r^2_{12} - r^2_{24})^2 \in \langle CM, k_0, r_{13}r_{24} - r_{14}r_{23} - r_{12}r_{34}, r_{12} - r_{23}, r_{14} - r_{34} \rangle \]

\[
CM = \left(-2\right)r^2_{13}\left(r^4_{12} - 2r^2_{12}r^2_{14} + r^4_{14} - 2r^2_{12}r^2_{24} + r^2_{13}r^2_{24} - 2r^2_{14}r^2_{24} + r^4_{24}\right)
\]

\[
CM_0 = r^4_{12} - 2r^2_{12}r^2_{14} + r^4_{14} - 2r^2_{12}r^2_{24} + r^2_{13}r^2_{24} - 2r^2_{14}r^2_{24} + r^4_{24}
\]
More observations:

\[(r_{14}^2 + r_{12}^2 - r_{24}^2)^2 \quad \in \quad \langle CM, k_0, r_{13}r_{24} - r_{14}r_{23} - r_{12}r_{34}, r_{12} - r_{23}, r_{14} - r_{34} \rangle \]

\[CM = (-2)r_{13}^2(r_{12}^4 - 2r_{12}^2r_{14}^2 + r_{14}^4 - 2r_{12}^2r_{24}^2 + r_{13}^2r_{24}^2 - 2r_{14}^2r_{24}^2 + r_{24}^4) \]

\[CM_0 = r_{12}^4 - 2r_{12}^2r_{14}^2 + r_{14}^4 - 2r_{12}^2r_{24}^2 + r_{13}^2r_{24}^2 - 2r_{14}^2r_{24}^2 + r_{24}^4 \]

\[(r_{14}^2 + r_{12}^2 - r_{24}^2)^2 = CM_0 - (r_{13}r_{24} - 2r_{12}r_{14})(r_{13}r_{24} + 2r_{12}r_{14}) \]
Observations

More observations:

\[(r_{14}^2 + r_{12}^2 - r_{24}^2)^2 \in \langle CM, k_0, r_{13}r_{24} - r_{14}r_{23} - r_{12}r_{34}, r_{12} - r_{23},
                      r_{14} - r_{34} \rangle\]

\[\overline{CM} = (-2)r_{13}^2(r_{12}^4 - 2r_{12}^2r_{14}^2 + r_{14}^4 - 2r_{12}^2r_{24}^2 + r_{13}^2r_{24}^2 - 2r_{14}^2r_{24}^2 + r_{24}^4)\]

\[CM_0 = r_{12}^4 - 2r_{12}^2r_{14}^2 + r_{14}^4 - 2r_{12}^2r_{24}^2 + r_{13}^2r_{24}^2 - 2r_{14}^2r_{24}^2 + r_{24}^4\]

\[(r_{14}^2 + r_{12}^2 - r_{24}^2)^2 = CM_0 - (r_{13}r_{24} - 2r_{12}r_{14})(r_{13}r_{24} + 2r_{12}r_{14})\]

\[r_{14}^2 + r_{12}^2 - r_{24}^2 \not\in \langle CM, k_0, r_{13}r_{24} - r_{14}r_{23} - r_{12}r_{34}, r_{12} - r_{23},
                      r_{14} - r_{34} \rangle\]
Theorem

Definition (Radical Ideal)

The radical of an ideal \( I \) is the ideal
\[
\sqrt{I} = \{ f \mid f^n \in I \text{ for some } n \geq 1 \}
\]
and we call \( I \) a radical ideal if and only if \( I = \sqrt{I} \).
Theorem

Definition (Radical Ideal)

The radical of an ideal $I$ is the ideal
\[ \sqrt{I} = \{ f | f^n \in I \text{ for some } n \geq 1 \} \] and we call $I$ a radical ideal if and only if $I = \sqrt{I}$.

We notice that when we have a fixed value for $m_2$ or $m_4$,

\[ (I_{AC} + \langle CM, k_0, r_{13}r_{24} - 2r_{12}r_{14} \rangle) \cap \mathbb{Q}[r_{ij}] = \langle f(r_{ij})^2 \rangle \quad (1) \]
\[ (I_{AC} + \langle CM, k_0, r_{13}r_{24} - 2r_{12}r_{14}, r_{14}^2 + r_{12}^2 - r_{24}^2 \rangle) \cap \mathbb{Q}[r_{ij}] = \langle f(r_{ij}) \rangle \quad (2) \]
Definition (Radical Ideal)

The radical of an ideal $I$ is the ideal
\[ \sqrt{I} = \{ f \mid f^n \in I \text{ for some } n \geq 1 \} \]
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We notice that when we have a fixed value for $m_2$ or $m_4$,

\begin{align*}
(\overline{I_{AC}} + \langle CM, k_0, r_{13}r_{24} - 2r_{12}r_{14} \rangle) \cap \mathbb{Q}[r_{ij}] &= \langle f(r_{ij})^2 \rangle \quad (1) \\
(\overline{I_{AC}} + \langle CM, k_0, r_{13}r_{24} - 2r_{12}r_{14}, r_{14}^2 + r_{12}^2 - r_{24}^2 \rangle) \cap \mathbb{Q}[r_{ij}] &= \langle f(r_{ij}) \rangle \quad (2)
\end{align*}

Theorem (Co-circular Kite Radical Ideals)

Except for a finite number of special cases, (2) is the radical ideal of (1)
Proof of Theorem

Theorem (Conditions for a Radical Ideal [Cox, Little, O’Shea])

Let $I \subset \mathbb{C}[x_1, \ldots, x_n]$ be a zero-dimensional ideal. For each $i = 1, \ldots, n$, let $p_i$ be the unique monic generator of $I \cap \mathbb{C}[x_i]$, and let $p_{i,\text{red}}$ be the square-free part of $p_i$. Then

$$\sqrt{I} = I + \langle p_{1,\text{red}}, \ldots, p_{n,\text{red}} \rangle.$$
Proof of Theorem

Theorem (Conditions for a Radical Ideal [Cox, Little, O’Shea])

Let $I \subset \mathbb{C}[x_1, \ldots, x_n]$ be a zero-dimensional ideal. For each $i = 1, \ldots, n$, let $p_i$ be the unique monic generator of $I \cap \mathbb{C}[x_i]$, and let $p_{i,\text{red}}$ be the square-free part of $p_i$. Then

$$\sqrt{I} = I + \langle p_{1,\text{red}}, \ldots, p_{n,\text{red}} \rangle.$$ 

Note that if $p_i = p_{i,\text{red}}$ for all $i$, then $I = \sqrt{I}$. 
Mass Relation

\[
\left( \overline{I_{AC}} + \langle CM, k_0, r_{13}r_{24} - 2r_{12}r_{14}, r_{14}^2 + r_{12}^2 - r_{24}^2 \rangle \right) \cap \mathbb{Q}[m_2, m_4] = \langle M(m_2, m_4) \rangle
\]
Mass Relation

\[
(I_{AC} + \langle CM, k_0, r_{13}r_{24} - 2r_{12}r_{14}, r_{14}^2 + r_{12}^2 - r_{24}^2 \rangle) \cap \mathbb{Q}[m_2, m_4] = \langle M(m_2, m_4) \rangle
\]
Mass Relation

\[
\left( \overline{I_{AC}} + \langle \overline{CM}, k_0, r_{13}r_{24} - 2r_{12}r_{14}, r_{14}^2 + r_{12}^2 - r_{24}^2 \rangle \right) \cap \mathbb{Q}[m_2, m_4] = \langle M(m_2, m_4) \rangle
\]
**Discriminant**

**Definition (Resultant)**

The resultant $R(P, Q, x)$ of two univariate polynomials $P$ and $Q$ in $x$ is a polynomial function of their coefficients that is zero if and only if the two polynomials have common roots in an algebraically closed field containing the coefficients.
## Discriminant

### Definition (Resultant)

The resultant $R(P, Q, x)$ of two univariate polynomials $P$ and $Q$ in $x$ is a polynomial function of their coefficients that is zero if and only if the two polynomials have common roots in an algebraically closed field containing the coefficients.

- Recall that the discriminant is defined as $D(h) = R(h, h', x)$. Our notation for the discriminant will be $D(h)$. 
Discriminant

Definition (Resultant)

The resultant $R(P, Q, x)$ of two univariate polynomials $P$ and $Q$ in $x$ is a polynomial function of their coefficients that is zero if and only if the two polynomials have common roots in an algebraically closed field containing the coefficients.

- Recall that the discriminant is defined as $D(h) = R(h, h', x)$. Our notation for the discriminant will be $D(h)$.
- A polynomial has a root with multiplicity greater than one if and only if its discriminant is zero.
Checking for Multiplicity

Now we compute the elimination ideal

$$(I_{AC} + \langle CM, k_0, r_{13}r_{24} - 2r_{12}r_{14}, r_{14}^2 + r_{12}^2 - r_{24}^2 \rangle) \cap \mathbb{Q}[r_{ij}, m_2, m_4] = J$$
Checking for Multiplicity

Now we compute the elimination ideal

$$(I_{AC} + \langle CM, k_0, r_{13}r_{24} - 2r_{12}r_{14}, r_{14}^2 + r_{12}^2 - r_{24}^2 \rangle) \cap \mathbb{Q}[r_{ij}, m_2, m_4] = J$$

- $M$ and $f(m_4, r_{ij})$ are generators of $J$
Checking for Multiplicity

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- \(M\) and \(f(m_4, r_{ij})\) are generators of \(J\)
- If \(f\) is square-free for all of the \(r_{ij}\) variables, then by the Conditions for a Radical Ideal theorem,

\[(I_{AC} + \langle CM, k_0, r_{13}r_{24} - 2r_{12}r_{14}, r_{14}^2 + r_{12}^2 - r_{24}^2 \rangle) \cap \mathbb{Q}[r_{ij}] = \langle f(r_{ij}) \rangle\]

is a radical ideal.
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is a radical ideal.

- For each of the $r_{ij}$s, we compute $J$ and $D(f, r_{ij}) = g(m_4)$
Checking for Multiplicity

We check each factor of \( D(f, r_{ij}) = g(m_4) \) that could have a positive real root \( a \) that would make \( g(a) = 0 \)
Checking for Multiplicity

We check each factor of \( D(f, r_{ij}) = g(m_4) \) that could have a positive real root \( a \) that would make \( g(a) = 0 \)

Example \((D(f, r_{12}))\)

\[
(4 \cdot m^4 - 1)^9 \ast (4 \cdot m^4 + 1)^9 \ast (m^4 + 1)^{12} \ast m^{120} \ast (m^4 + 10 \cdot m^4 + 39 \cdot m^4 + 72 \cdot m^4 + 153 \cdot m^6 + 582 \cdot m^5 + 1373 \cdot m^4 + 1676 \cdot m^3 + 2979 \cdot m^2 + 4158 \cdot m + 3969)^3 \ast (413343 \cdot m^4 + 648324 \cdot m^3 + 844720 \cdot m^2 - 2064960 \cdot m + 1804032 \cdot m^2 - 104976)^3 \ast (63700992 \cdot m^4 - 445906944 \cdot m^3 + 3327229804571210891151)^6
\]
Example 2, $m_4 = \frac{1}{4}$

Example

\[ K_2 = [G_{12}, G_{21}, G_{13}, G_{31}, G_{14}, G_{41}, G_{34}, G_{43}, G_{24}, G_{42}, G_{23}, G_{32}, \text{nonzero}, \text{ptol}, \text{cm}, \text{pythag}] \]

\[ \text{Ielim} = \text{ideal}(K_2) \]
\[ \text{R}24 = \text{Ielim}.\text{elimination\_ideal}([t, r_{13}, r_{14}, r_{12}, m_2]) \]
\[ \text{R}24.\text{gen}(0).\text{factor}() \]

Output:

\[ 106570108575 \ast r_{24}^{18} - 255577677530 \ast r_{24}^{15} + 63682445617 \ast r_{24}^{12} + 148432515924 \ast r_{24}^9 - 85955082223 \ast r_{24}^6 + 5030 \ast r_{24}^3 - 1 \]
Conclusion

We were able to use Sturm’s theorem in Sage to determine that there are only a finite number of positive real roots (one of which is $\frac{1}{4}$) for each factor of $D(f, r_{ij}) = g(m_4)$ for each $r_{ij}$. 
We were able to use Sturm’s theorem in Sage to determine that there are only a finite number of positive real roots (one of which is $\frac{1}{4}$) for each factor of $D(f, r_{ij}) = g(m_4)$ for each $r_{ij}$.

Therefore except for a finite number of special cases, $(\overline{I_{AC}} + \langle CM, k_0, r_{13}r_{24} - 2r_{12}r_{14}, r_{14}^2 + r_{12}^2 - r_{24}^2 \rangle) \cap \mathbb{Q}[r_{ij}]$ is a radical ideal. Q.E.D.
Future Work

\[(\overline{I_{AC}} + \langle CM, k_0, r_{13}r_{24} - 2r_{12}r_{14} \rangle) \cap \mathbb{Q}[r_{ij}] = \langle f(r_{ij})^2 \rangle\]

\[(\overline{I_{AC}} + \langle CM, k_0, r_{13}r_{24} - 2r_{12}r_{14}, r_{14}^2 + r_{12}^2 - r_{24}^2 \rangle) \cap \mathbb{Q}[r_{ij}] = \langle f(r_{ij}) \rangle\]
Future Work

\[
\left( I_{AC} + \langle CM, k_0, r_{13}r_{24} - 2r_{12}r_{14} \rangle \right) \cap \mathbb{Q}[r_{ij}] = \langle f(r_{ij})^2 \rangle \\
\left( I_{AC} + \langle CM, k_0, r_{13}r_{24} - 2r_{12}r_{14}, r_{14}^2 + r_{12}^2 - r_{24}^2 \rangle \right) \cap \mathbb{Q}[r_{ij}] = \langle f(r_{ij}) \rangle
\]

Work that will help prove the above:
Future Work

\[
(\overline{I_{AC}} + \langle CM, k_0, r_{13}r_{24} - 2r_{12}r_{14}\rangle) \cap \mathbb{Q}[r_{ij}] = \langle f(r_{ij})^2\rangle \\
(\overline{I_{AC}} + \langle CM, k_0, r_{13}r_{24} - 2r_{12}r_{14}, r_{14}^2 + r_{12}^2 - r_{24}^2\rangle) \cap \mathbb{Q}[r_{ij}] = \langle f(r_{ij})\rangle
\]

Work that will help prove the above:

\[
(\overline{I_{AC}} + \langle CM, k_0, r_{13}r_{24} - 2r_{12}r_{14}, r_{14}^2 + r_{12}^2 - r_{24}^2\rangle) \cap \mathbb{Q}[r_{ij}]
\]

is a radical ideal
Future Work

\[(I_{AC} + \langle CM, k_0, r_{13}r_{24} - 2r_{12}r_{14} \rangle) \cap \mathbb{Q}[r_{ij}] = \langle f(r_{ij})^2 \rangle\]
\[(I_{AC} + \langle CM, k_0, r_{13}r_{24} - 2r_{12}r_{14}, r_{14}^2 + r_{12}^2 - r_{24}^2 \rangle) \cap \mathbb{Q}[r_{ij}] = \langle f(r_{ij}) \rangle\]

Work that will help prove the above:

\[(I_{AC} + \langle CM, k_0, r_{13}r_{24} - 2r_{12}r_{14}, r_{14}^2 + r_{12}^2 - r_{24}^2 \rangle) \cap \mathbb{Q}[r_{ij}]\]

is a radical ideal, and

\[(r_{14}^2 + r_{12}^2 - r_{24}^2)^2 = CM_0 - (r_{13}r_{24} - 2r_{12}r_{14})(r_{13}r_{24} + 2r_{12}r_{14})\]

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