Co-Circular Central Configurations of Vortices

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In general interactions between Helmholtz vortices are quite complicated
Applications to fluid mechanics

- Helmholtz vortices were introduced in the mid 19th century as a way to model a two-dimensional slice of a columnar vortex filament.
- In the 1870’s William Thomson (Lord Kelvin) experimented with central configurations (c.c.s) of vortices using floating magnets.
- In an ideal fluid c.c.s could be used to construct periodic solutions to the $n$-vortex problem.
This sounds really familiar?
$n$-body problem

- The Newtonian $n$-body problem
Central Configurations of n-body: The acceleration vector of each body is proportional to the position vector from the center of mass, all with the same proportionality constant.
$n$-body problem, cont.

- Central Configurations of $n$-body: The acceleration vector of each body is proportional to the position vector from the center of mass, all with the same proportionality constant.
It does not make sense to talk about the “mass” of a vortex in the same way we measure the mass of a body in the $n$-body problem. Instead we measure the strength of rotation, vorticity, of a vortex. This corresponds to the curl of the velocity vector field.

Remember that the curl of a vector field can be either positive or negative, depending on whether the rotation is counter-clockwise or clockwise.
This causes more problems in finding c.c.s of vortices.

Opposite rotating vortices will attract to each other, whereas similar vortices will repel more.
The vortex case also differs from the Newtonian case in that the force scales as the inverse of the square of distance:

**Newtonian Case:**

\[
a_i = \sum_{j \neq i} \frac{m_i (q_j - q_i)}{r_{ij}^3}
\]

**Vortex Case:**

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**Vortex Case:**

\[
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\]

In our case the force scales as in the inverse of distance.
Two ways of analyzing co-circular configurations of the four vortex Problem
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Albouy-Chenciner Equations

We can form an ideal by adjoining the AC equations to a condition derived from Ptolemy’s theorem and another one derived from the Cayley-Menger determinant.
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Albouy-Chenciner Equations

We can form an ideal by adjoining the AC equations to a condition derived from Ptolemy’s theorem and another one derived from the Cayley-Menger determinant.

Cor-Roberts’ $F$ equations

We also derive an equivalent set of polynomial equations that are zero if and only if the given set of mutual distances and vorticities corresponds to a central configuration of vortices.
Similarly, the AC equations for vortices are in terms of $r_{ij}^2$ instead of $r_{ij}^3$. Newtonian:

$$G_{ij} = \sum_{k=1, k \neq i}^{n} m_k \left( \frac{1}{r_{ik}^3} - 1 \right) \left( r_{jk}^2 - r_{ik}^2 - r_{ij}^2 \right) = 0$$
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Newtonian:

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Vortex:

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G_{ij} = \sum_{k=1, k \neq i}^n m_k \left( \frac{1}{r_{ik}^2} - 1 \right) \left( r_{jk}^2 - r_{ik}^2 - r_{ij}^2 \right) = 0
\]

This form allows us to make a substitution that will make the problem more computationally manageable:

\[
G_{ij} = \sum_{k=1, k \neq i}^n m_k \left( \frac{1}{z_{ik}} - 1 \right) \left( z_{jk} - z_{ik} - z_{ij} \right) = 0
\]
Formation of an Ideal using AC Equations

Ideal is created by adjoining the following equations:

- **Albouy-Chenciner Equations**: Ensures that we have a central configuration.
- **Cayley-Menger Determinant**: Ensures that our central configurations are co-planar and geometrically realizable.
- **Equation $h$**: Ensures that each of the $z_{ij}$ (distance between vortices) is non-zero.
- **Ptolemy Equation**: Force geometric configurations to be co-circular.
Ptolemy Equation

Theorem
A convex quadrilateral is cyclic if and only if
\[ r_{13}r_{24} = r_{14}r_{23} + r_{12}r_{34}, \ r_{12}r_{34} = r_{13}r_{24} + r_{14}r_{23}, \ r_{14}r_{23} = r_{12}r_{34} + r_{13}r_{24} \]

Transforming \( r_{ij}^2 \) into \( z_{ij} \)

\[
(r_{12}r_{34}^2 - r_{14}^2r_{23}^2 - r_{13}^2r_{24}^2)^2 = (2r_{13}r_{14}r_{23}r_{24})^2
\]
\[
r_{14}^4r_{23}^4 - 2r_{13}^2r_{14}^2r_{23}^2r_{24}^2 + r_{13}^4r_{24}^4 - 2r_{12}^2r_{14}^2r_{23}^2r_{34}^2 - 2r_{12}^2r_{13}^2r_{24}^2r_{34}^2 + r_{12}^4r_{34}^4 = 0
\]
\[
(-r_{14}r_{23} - r_{13}r_{24} + r_{12}r_{34}) \cdot (-r_{14}r_{23} + r_{13}r_{24} + r_{12}r_{34}) \cdot (r_{14}r_{23} - r_{13}r_{24} + r_{12}r_{34}) \cdot (r_{14}r_{23} + r_{13}r_{24} + r_{12}r_{34})
\]
Derivation of $F$ Polynomials

Similarly to Cors and Roberts paper, we obtain the following system of equations when $\Gamma_1 = 1$ and $r_{12} = 1$, being the longest exterior side.

\[
F_1 = r_{13}^2(r_{23} + r_{14}r_{34}) - (r_{14}r_{34} + r_{14}^2r_{23} + r_{23}r_{34}^2 + r_{14}r_{23}^2r_{34})
\]

\[
F_2 = r_{24}^2(r_{14} + r_{23}r_{34}) - (r_{23}r_{34} + r_{14}r_{23}^2 + r_{14}r_{34}^2 + r_{14}^2r_{23}r_{34})
\]

\[
F_3 = (r_{13}^2 - 1)(r_{23}^2 - r_{34}^2)(r_{24}^2 - r_{14}^2) - (1 - r_{14}^2)(r_{24}^2 - r_{34}^2)(r_{23}^2 - r_{24}^2)
\]

\[
F_4 = \Gamma_2r_{13}r_{14}(r_{24}^2 - r_{23}^2) - r_{23}r_{24}(r_{13}^2 - r_{14}^2)
\]

\[
F_5 = \Gamma_3r_{12}r_{14}(r_{23}^2 - r_{34}^2) - r_{23}r_{34}(r_{12}^2 - r_{14}^2)
\]

\[
F_6 = \Gamma_4r_{12}r_{13}(r_{24}^2 - r_{34}^2) - r_{24}r_{34}(r_{13}^2 - r_{12}^2)
\]
Derivation of F-Polynomials

\[ r_{13} = \sqrt{\frac{ab}{c}} \quad \text{and} \quad r_{24} = \sqrt{\frac{ac}{b}} \quad \text{where} \]

\[ a = r_{12}r_{34} + r_{14}r_{23}, \quad b = r_{12}r_{14} + r_{23}r_{34}, \quad \text{and} \quad c = r_{12}r_{23} + r_{14}r_{34} \]

\[ F_1 = r_{13}^2(r_{23} + r_{14}r_{34}) - (r_{14}r_{34} + r_{14}^2r_{23} + r_{23}r_{34}^2 + r_{14}r_{23}^2r_{34}) \]

\[ F_2 = r_{24}^2(r_{14} + r_{23}r_{34}) - (r_{23}r_{34} + r_{14}r_{23}^2 + r_{14}r_{34}^2 + r_{14}^2r_{23}r_{34}) \]
Derivation of F-Polynomials

\[
\begin{align*}
\Gamma_2 \Gamma_1 &= r_{23}r_{24}(r_{13}^2 - r_{14}^2) \\
&\quad / r_{13}r_{14}(r_{24}^2 - r_{23}^2), \\
\Gamma_3 \Gamma_1 &= r_{23}r_{34}(r_{12}^2 - r_{14}^2) \\
&\quad / r_{12}r_{14}(r_{23}^2 - r_{34}^2), \\
\Gamma_4 \Gamma_1 &= r_{24}r_{34}(r_{13}^2 - r_{12}^2) \\
&\quad / r_{12}r_{13}(r_{24}^2 - r_{34}^2).
\end{align*}
\]

\[
\begin{align*}
F_4 &= \Gamma_2 r_{13}r_{14}(r_{24}^2 - r_{23}^2) - r_{23}r_{24}(r_{13}^2 - r_{14}^2) \\
F_5 &= \Gamma_3 r_{14}(r_{23}^2 - r_{34}^2) - r_{23}r_{34}(1 - r_{14}^2) \\
F_6 &= \Gamma_4 r_{13}(r_{24}^2 - r_{34}^2) - r_{24}r_{34}(r_{13}^2 - 1)
\end{align*}
\]
An inverse approach

- These $F$ equations will not have a solution given a general collection of vortices.
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- Similarly, not every set of four co-circular points will have a corresponding set of vorticities which will correspond to a central configuration.
An inverse approach

- These $F$ equations will not have a solution given a general collection of vortices.
- Similarly, not every set of four co-circular points will have a corresponding set of vorticities which will correspond to a central configuration.
- Two possible inverse approaches:
  - Given what sets of vorticities will there exist a set of mutual distances representing a co-circular central configuration?
  - Given what (co-circular) sets of mutual distances will there exist vorticities corresponding to a central configuration?
The importance of $F_3$

\[
F_3 = (r_{13}^2 - 1)(r_{23}^2 - r_{34}^2)(r_{24}^2 - r_{14}^2) - (1 - r_{14}^2)(r_{24}^2 - r_{34}^2)(r_{13}^2 - r_{23}^2) 
\]

- $F_3 = 0$ is necessary and sufficient for a planar c.c. given that the six distances are geometrically realizable.
The importance of \( F_3 \)

\[
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\]

- \( F_3 = 0 \) is necessary and sufficient for a planar c.c. given that the six distances are geometrically realizable.
- Equations \( F_1 \) and \( F_2 \) allow us to solve for the diagonals \( r_{13} \) and \( r_{24} \) in terms of the three variable length external sides

\[
F_1 = r_{13}^2 (r_{23} + r_{14}r_{34}) - (r_{14}r_{34} + r_{14}^2 r_{23} + r_{23}r_{34}^2 + r_{14}r_{23}^2 r_{34})
\]
\[
F_2 = r_{24}^2 (r_{14} + r_{23}r_{34}) - (r_{23}r_{34} + r_{14}r_{23}^2 + r_{14}r_{34}^2 + r_{14}^2 r_{23}r_{34})
\]

- Doing this allows us to write \( F_3 \) as a function of \( r_{14}, r_{23}, \) and \( r_{34}: \)
The graph of $F_3$

Figure: $F_3$ as a function of $r_{23}, r_{34},$ and $r_{14}$
Figure 6. On the left is the surface $\Gamma$ of c.c.c.s in $r_{34}r_{23}r_{14}$-space. The outline of the projection onto the $r_{34}r_{23}$-plane is shown plotted in the plane $r_{14} = 0.9$. This figure was generated with Matlab [17] using a bisection algorithm. On the right is the image of $\Gamma$ in $m_2m_3m_4$-space under equations (16), (17) and (18) with $m_1 = 1$. This figure shows the full set of masses for which a co-circular c.c. exists.

Figure : Image taken from “Four-body co-circular central configurations” by Josep M Cors and Gareth E Roberts. Published in NONLINEARITY January 11, 2012
Symmetric Cases

- There are two special symmetric cases that we want to examine in the c.c.c.s cases.
Symmetric Cases

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- The Kite Cases.
Symmetric Cases

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- The Kite Cases.
- And the Trapezoid Cases.
Symmetric Cases

- There are two special symmetric cases that we want to examine in the c.c.c.s cases.
- The Kite Cases.
- And the Trapezoid Cases.
- We especially want to see how those special relations fit into our bow tie graph.
What are Kites?
What are Kites?

Definition: We call a convex quadrilateral a kite configuration if two opposite vortices lie on an axis of symmetry of the configuration and lies on the diameter of the circle.
Kite line

Special Relations of a c.c.c. Kite:

- **Adjacent sides are equal:**
  
  \[ r_{12} = r_{14} \]
  \[ r_{23} = r_{34} \]

- **Pythagorean relations hold:**
  
  \[ r_{12}^2 + r_{23}^2 = r_{13}^2 \]
  \[ r_{14}^2 + r_{34}^2 = r_{13}^2 \]
Kite line

- We can graph just the relations of the kite conditions \( r_{23} = r_{34} \).
Kite line

- The yellow plane represents all Kite configurations.
- The blue bow tie represents all c.c.c.s.
- Therefore the boundary, where the plane and the bow tie meet represents all c.c.c.s that have kite configurations.
Kite line

How do the kite configurations act along the boundary of this graph?
Kite line

How do the kite configurations act along the boundary of this graph? ??
Kite line

How do the kite configurations act along the boundary of this graph?
In order to understand the Kite Case, we can compute a Gröbner basis in terms of lexicographical order, using the A.C. equations to find our solutions.

Similar to Group 1’s presentation, we will create an ideal in terms of the \( z_{ij} \)’s using...
Kite Co-Circular Central Configurations

- The A.C. equations: \( AC = [G_{12}, \ldots, G_{34}] \)
- Ptolemy’s Condition: \( Pt = z_{13}^2 z_{24}^2 - 2z_{13}z_{24}z_{12}z_{34} + z_{12}^2 z_{34}^2 - 2z_{13}z_{24}z_{14}z_{23} + 2z_{12}z_{34}z_{14}z_{23} + z_{14}^2 z_{23}^2 - 4z_{14}z_{12}z_{34}z_{23} \)
- Zero condition: \( h = 1 - t z_{12}z_{13}z_{14}z_{23}z_{24}z_{34} \)
- And the Kite Conditions:
  - \( K_1 = z_{12} + z_{23} - z_{13} \)
  - \( K_2 = z_{14} + z_{34} - z_{13} \)
  - \( K_3 = z_{12} - z_{14} \)
  - \( K_4 = z_{23} - z_{34} \)
- Thus our Ideal will look something like this: \( \text{Ideal}([AC, Pt, h, K_1, K_2, K_3, K_4]) \).
Because of the Lexicographical ordering, the last two elements of our Gröbner basis will be left only in terms of vorticities. Therefore we can find solutions for all vorticities of kites that are c.c.c.s.

- $GB[10] = \Gamma_2 - \Gamma_4$
- $GB[11] = \Gamma_3\Gamma_4 + 2\Gamma_3 - 4\Gamma_4^2 + \Gamma_4$
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- $GB[11] = \Gamma_3 \Gamma_4 + 2\Gamma_3 - 4\Gamma_4^2 + \Gamma_4$

By setting $\Gamma_1 = 1$, we can see from the Gröbner basis that $\Gamma_2 = \Gamma_4$ and $\Gamma_3 = \frac{4\Gamma_4^2 - \Gamma_4}{\Gamma_4 + 2}$. 
Kite Co-Circular Central Configurations

- This is a graph of all the possible vorticities we get for kite c.c.c.s, with an asymptote at $\Gamma_4 = -2$. Where the horizontal axis represents $\Gamma_3$ and the vertical is $\Gamma_4$.

- Vorticities:
  $\Gamma_1 = 1,$
  $\Gamma_2 = \Gamma_4,$
  $\Gamma_3 = \frac{4\Gamma_4^2 - \Gamma_4}{\Gamma_4 + 2}.$
We then need to find all real solutions where the distance \( r_{ij} > 0 \) or \( z_{ij} > 0 \). After solving inequalities, we get

\[
\begin{align*}
z_{34} &= \frac{6\Gamma_4^2 + 3\Gamma_4}{6\Gamma_4^2 + 4\Gamma_4 + 2} \\
z_{34} &= \frac{\Gamma_3 + 2\Gamma_4}{\Gamma_3 + 2\Gamma_4 + 1} \\
z_{24} &= \frac{3\Gamma_4(\Gamma_4 + 2)}{\Gamma_4(3\Gamma_4 + 2) + 1} \\
z_{14} &= \frac{2\Gamma_4^2 + 5\Gamma_4 + 2}{6\Gamma_4^2 + 4\Gamma_4 + 2} \\
z_{13} &= \frac{(2\Gamma_4 + 1)^2}{\Gamma_4(3\Gamma_4 + 2) + 1} \\
z_{12} &= (\Gamma_4 + 2)(2\Gamma_4 + 1)
\end{align*}
\]
After solving for all possible real solutions we get this graph.

The white represents vorticities that give you negative distances.

The yellow represents vorticities that give you positive distances.
Kite Co-Circular Central Configurations
Degenerate Case

- By using the findings of Cors Roberts for the n-body problem and examining the domain of our graph, we find that the boundary of the other part of the bow tie graph is what we call a degenerate case, illustrated by
  \[ r_{23}^2 + r_{34}^2 + r_{34}r_{23} - 1 = 0 \]
Degenerate Case

- Since we want to look at the intersection of the degenerate plain and the bow tie graph, we can look at the Gröbner basis as well.
- This time we will use the F equations instead of the AC equations.
Degenerate Case

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- This time we will use the F equations instead of the AC equations.

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F_1 = r_{13}^2(r_{23} + r_{14}r_{34}) - (r_{14}r_{34} + r_{14}^2r_{23} + r_{23}r_{34}^2 + r_{14}r_{23}^2r_{34}) \\
F_2 = r_{24}^2(r_{14} + r_{23}r_{34}) - (r_{23}r_{34} + r_{14}r_{23}^2 + r_{14}r_{34}^2 + r_{14}^2r_{23}r_{34}) \\
F_3 = (r_{13}^2 - 1)(r_{23}^2 - r_{34}^2)(r_{24}^2 - r_{14}^2) - (1 - r_{14}^2)(r_{24}^2 - r_{34}^2)(r_{13}^2 - r_{23}^2)
\]

- And we will substitute \( r_{13} = \sqrt{\frac{(r_{14}^2r_{23} + r_{14}r_{34}(r_{23}^2 + 1) + r_{23}r_{34}^2)}{(r_{23} + r_{14}r_{34})}} \)

\[
r_{24} = \sqrt{\frac{(r_{14}^2r_{23}r_{34} + r_{14}(r_{34}^2 + r_{23}^2) + r_{23}r_{34})}{(r_{14} + r_{23}r_{34})}}
\]
Degenerate Case

- Our Ideal will then look something like this
- \[\text{Ideal}([F_1, F_2, F_3, h, r_{23}^2 + r_{34}^2 + r_{34}r_{23} - 1])\]
Degenerate Case

- After computing a Gröbner basis and solving for distances and masses, we are able to interpret the border of our bow tie graph.
Degenerate Case

\[ \Gamma_3 = 0, \text{ and } r_{12} = r_{14} = r_{24} = 1 \]
Degenerate Case

\[ \Gamma_3 = 0, \text{ and } r_{12} = r_{14} = r_{24} = 1 \]
Degenerate Meets Kite Case

\[ \Gamma_3 = 0, \text{ and } r_{12} = r_{14} = r_{24} = 1 \]
Co-Circular Trapezoid

Ideal also consist of the following

Trapezoid Conditions:

- Diagonals are equal:
  \[ r_{13} = r_{24} \]

- Non-parallel sides are equal
  \[ r_{14} = r_{23} \]

- Special Condition
  \[ r_{12}r_{34} - r_{13}^2 + r_{14}^2 \]
Trapezoid

Vorticities
\[ \Gamma_1 = 1 = \Gamma_2, \]
\[ \Gamma_3 = \Gamma_4, \]

Special Case:
\[ z_{34} = \frac{\Gamma_4^2 + 2\Gamma_4}{\Gamma_4^2 + 2\Gamma_4 + 1}. \]
Interpretation of the Trapezoid along the bow-tie graph.
Distribution of Vorticities

Figure: The distribution of masses on $F_3$, where green represents a central configuration with three positive vorticities and one negative.
Conjecture

This distribution of vorticities leads to a natural conjecture:
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Co-Circular Central Configuration Conjecture

For \( r_{14} \geq r_{23} \) for every \( r_{23} > \frac{\sqrt{3}}{3} \) each point on the graph of \( F_3 \) corresponds to a central configuration where all four vorticities are positive. For every \( r_{23} < \frac{\sqrt{3}}{3} \) each point on the graph of \( F_3 \) corresponds to a central configuration where \( \Gamma_3 < 0 \) and \( \Gamma_1 > \Gamma_2, \Gamma_3 > 0 \).
Evidence to support conjecture: Kite case

As we examine the boundary cases, we can see evidence of our Conjecture being valid. By substituting \( r_{13} \) and \( r_{24} \) from our previous equations in terms of \( r_{14}, r_{23}, r_{34} \), then using the fact that \( r_{12} = r_{14}, r_{23} = r_{34}, \) and \( \Gamma_2 = \Gamma_4 \) for all kites, we can then re-write the vorticities in terms of one variable:

\[
\Gamma_3 = \frac{-r_{34}^3 (3r_{34}^2 - 1)}{r_{34}^3 - 3r_{34}} \quad (1)
\]

\[
\Gamma_4 = \frac{-2r_{34}^3}{r_{34}^3 - 3r_{34}} \quad (2)
\]

Solving for positive and negative vorticities yields:
Evidence to support conjecture: Kite case

\[ \Gamma_3 \begin{cases} 
> 0 & \text{if } r_{34} \in \left( \frac{1}{\sqrt{3}}, \sqrt{3} \right) \\
< 0 & \text{if } r_{34} \in (0, \frac{1}{\sqrt{3}}) \text{ or } r_{34} \in (3, \infty) \\
eq 0 & \text{if } r_{34} = \frac{1}{\sqrt{3}}, \sqrt{3} 
\end{cases} \]

\[ \Gamma_4 \begin{cases} 
> 0 & \text{if } r_{34} \in (0, \sqrt{3}) \\
< 0 & \text{if } r_{34} \in (\sqrt{3}, \infty) \\
\text{undefined} & \text{if } r_{34} = 0 \text{ or } \sqrt{3} 
\end{cases} \]

Therefore the relation of masses to the right of the center of the bowtie yield all positive masses, and the masses to the left of the center yield one negative mass, for all c.c.c. of kites.
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