Introduction

Complete Bipartite Graphs

The Order of $S(\Gamma_{n,m})$

Identity on $S(\Gamma)$

Recurrent Configurations on $\Gamma_{n,m}$

- $n = 2, m = 1$
- $n = 2, m = 2$
- $n = 2, m = 3$
- $n = 2, m = 4$
- $n = 3, m = 2$

Fundamental Theorem of Finite Abelian Groups

Patterns on Isomorphic Finite Abelian Groups

An Isomorphism Formula

Bijection between recurrent sandpiles and spanning trees directed into $s$

Pattern of $S(\Gamma_{n,m})$ Generators

- $n = 2$
- $n = 3$
Complete Bipartite Graphs

Definition:
The complete bipartite graph $K_{n,m}$ is a graph with two sets of vertices, one with $n$ members and one with $m$ such that each vertex in one set is adjacent to every vertex in the other set and to no vertex in its own set.

Figure: $n = 9$ $m = 9$
Complete Bipartite Graphs

The Order of $S(\Gamma_{n,m})$

| $m$ | $|\Gamma_{n=2,m}|$ |
|-----|-------------------|
| 5   | 80                |
| 4   | 32                |
| 3   | 12                |
| 2   | 4                 |
| 1   | 1                 |
Complete Bipartite Graphs

The Order of $S(\Gamma_{n,m})$

| m  | $|\Gamma_{n=2,m}|$ |
|-----|-------------------|
| 5   | 80                |
| 4   | 32                |
| 3   | 12                |
| 2   | 4                 |
| 1   | 1                 |

Conjecture:
The number of recurrent configurations on $\Gamma_{n,m}$, that is, $|S(\Gamma_{n,m})| = (n^{m-1})(m^{n-1})$. 
Complete Bipartite Graphs

Tynan Lazarus
and Kendall Tada

Outline
Introduction
The Order of \( S(\Gamma_{n,m}) \)
Identity on \( S(\Gamma) \)
Recurrent Configurations on \( \Gamma_{n,m} \)
Fundamental Theorem of Finite Abelian Groups
Bijection between recurrent sandpiles and spanning trees directed into \( s \)
Pattern of \( S(\Gamma_{N,m}) \)
Generators

**The Order of \( S(\Gamma_{n,m}) \)**

| \( m \) | \( |\Gamma_{n=2,m}| \) |
|-------|------------------|
| 5     | 80               |
| 4     | 32               |
| 3     | 12               |
| 2     | 4                |
| 1     | 1                |

**Conjecture:**

The number of recurrent configurations on \( \Gamma_{n,m} \), that is,

\[ |S(\Gamma_{n,m})| = (n^{m-1})(m^{n-1}). \]

**Example:**

\[ |S(\Gamma_{n=7,m=8})| = \left(7^{(8-1)}\right)\left(8^{(7-1)}\right) = (7^7)(8^6) = 215, 886, 856, 192 \]
Conjecture:

Let $\text{sink} = n_1$. If $\sigma = e$, the identity configuration, then all $n$-vertices $= 0$ and all $m$-vertices $= n - 1$.

The identity is $n = 0, m = n - 1$ for all $S(\Gamma_{n,m})$
Conjecture:

Let \( \text{sink} = n_1 \). If \( \sigma = e \), the identity configuration, then all \( n \)-vertices = 0 and all \( m \)-vertices = \( n - 1 \).

The identity is \( n = 0, m = n - 1 \) for all \( S(\Gamma_{n,m}) \)

Example:

\( \Gamma_{3,4} \) Identity: \( m_1 : 2, m_2 : 2, m_3 : 2, m_4 : 2, n_2 : 0, n_3 : 0 \)

\( \Gamma_{5,3} \) Identity: \( m_1 : 4, m_2 : 4, m_3 : 4, n_2 : 0, n_3 : 0, n_4 : 0, n_5 : 0 \)
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$n = 2, m = 2$

$n = 2, m = 3$

$n = 2, m = 4$

$n = 3, m = 2$

Fundamental Theorem of Finite Abelian Groups

Bijection between recurrent sandpiles and spanning trees directed into $s$

Pattern of $S(\Gamma_{n,m})$

$n=2, m=1$:

Recurrent Configuration: $[1, 0]$
Complete Bipartite Graphs

Recurrent Configurations on $\Gamma_{n,m}$

$n = 2, m = 2$

$n=2, m=2$:

$[m_1, m_2, n_2]$

Recurrent Configurations:

$[1, 1, 1], [1, 1, 0], [0, 1, 1], [1, 0, 1]$
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$n = 3, m = 2$

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Bijection between recurrent sandpiles and spanning trees directed into $s$

Pattern of $S(\Gamma_{n,m})$

Recurrent Configurations:

$n=2, m=3$:

$[m_1, m_2, m_3, n_2]$

$[1, 1, 1, 2], [1, 1, 1, 0], [1, 1, 1, 1], [1, 1, 0, 2], [1, 0, 1, 2], [0, 1, 1, 2], [1, 1, 0, 1], [1, 0, 1, 1], [0, 1, 1, 1], [0, 0, 1, 2], [0, 1, 0, 2], [1, 0, 0, 2]$
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$n = 3, \ m = 2$
Fundamental Theorem of Finite Abelian Groups
Bijection between recurrent sandpiles and spanning trees directed into $s$

Pattern of $S(\Gamma_{n,m})$:

Recurrent Configurations:

$[m_1, m_2, m_3, m_4, n_2]$

$n=2, \ m=4$:

\[
\begin{align*}
S & \quad m_1 \\
& \quad m_2 \\
& \quad m_3 \\
& \quad m_4 \\
n_2 &
\end{align*}
\]
Complete Bipartite Graphs

Recurrent Configurations on $\Gamma_{n,m}$

- $n = 3, m = 2$

Outline

Introduction

The Order of $S(\Gamma_{n,m})$

Identity on $S(\Gamma)$

Recurrent Configurations on $\Gamma_{n,m}$

- $n = 2, m = 1$
- $n = 2, m = 2$
- $n = 2, m = 3$
- $n = 2, m = 4$
- $n = 3, m = 2$

Fundamental Theorem of Finite Abelian Groups

Bijection between recurrent sandpiles and spanning trees directed into $s$

Pattern of $S(\Gamma_{n,m})$

Recurrent Configurations:

- $[2, 1, 1, 2], [1, 1, 1, 2], [0, 1, 1, 2], [2, 0, 0, 2], [2, 1, 1, 0], [2, 0, 1, 1], [2, 1, 0, 1], [1, 1, 0, 2], [1, 0, 1, 2], [2, 1, 0, 2], [2, 0, 1, 2], [2, 1, 1, 1]$
Isomorphism to $\mathbb{Z}$ obtained from Smith Normal Form

Example $n = 4$

<table>
<thead>
<tr>
<th>m</th>
<th>SNF Diag.</th>
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<tbody>
<tr>
<td>1</td>
<td>1, 1, 1, 1</td>
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<tr>
<td>2</td>
<td>1, 1, 2, 2, 8</td>
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<tr>
<td>3</td>
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<tr>
<td>6</td>
<td>1, 1, 2, 2, 4, 4, 12, 12, 24</td>
</tr>
<tr>
<td>7</td>
<td>1, 1, 1, 1, 4, 4, 4, 28, 28, 28</td>
</tr>
<tr>
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<td>1, 1, 4, 4, 4, 4, 4, 4, 4, 8, 8, 32</td>
</tr>
<tr>
<td>9</td>
<td>1, 1, 1, 1, 4, 4, 4, 4, 4, 36, 36, 36</td>
</tr>
<tr>
<td>10</td>
<td>1, 1, 2, 2, 4, 4, 4, 4, 4, 4, 4, 20, 20, 40</td>
</tr>
</tbody>
</table>
### Rewriting SNF in terms of $n$ and $m$

<table>
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<tr>
<th>$m$</th>
<th>SNF Diag. without 1’s</th>
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## Rewriting SNF in terms of $n$ and $m$

<table>
<thead>
<tr>
<th>$m$</th>
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<th>#n</th>
<th>#m</th>
<th>#mn</th>
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<td>1</td>
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<td>2</td>
<td>2m</td>
<td>mn</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3, 12, 12</td>
<td>3</td>
<td>m</td>
<td>2m</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>4, 4, 4, 4, 16</td>
<td>4</td>
<td>4n</td>
<td>mn</td>
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<tr>
<td>5</td>
<td>4, 20, 20, 20</td>
<td>5</td>
<td>n</td>
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<td>4n</td>
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<td>8n</td>
<td>2m</td>
<td>mn</td>
</tr>
</tbody>
</table>

**Question:**

What is the pattern?
Recall that if $p$ and $q$ are relatively prime, then

$$\mathbb{Z}_{p,q} \cong \mathbb{Z}_p \times \mathbb{Z}_q$$

**Example $n = 4$**

<table>
<thead>
<tr>
<th>m</th>
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<th>#m</th>
<th>#mn</th>
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<td>n</td>
<td></td>
<td>?</td>
</tr>
<tr>
<td>2</td>
<td>n</td>
<td>2m</td>
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<tr>
<td>3</td>
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<td>2mn</td>
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<tr>
<td>4</td>
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<tr>
<td>5</td>
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</table>
Recall that if $p$ and $q$ are relatively prime, then
\[ \mathbb{Z}_{p,q} \cong \mathbb{Z}_p \times \mathbb{Z}_q \]

### Example $n = 4$

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<th>$#m$</th>
<th>$#mn$</th>
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<th>$#mn$</th>
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<td>2m</td>
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<td></td>
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<td>2mn</td>
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<td></td>
<td>2n</td>
<td>2m</td>
</tr>
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<td>4n</td>
<td></td>
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<td>2m</td>
<td>mn</td>
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</tbody>
</table>
Conjecture:

In $K_{n,m}$, the sandpile group $S(\Gamma)_{n,m} \cong \mathbb{Z}^{m-2}_n \times \mathbb{Z}^{n-2}_m \times \mathbb{Z}_{n\cdot m}$.
Conjecture:

In $K_{n,m}$, the sandpile group $S(\Gamma)_{n,m} \cong \mathbb{Z}_{n-2}^m \times \mathbb{Z}_{m-2}^n \times \mathbb{Z}_{n\cdot m}$.

Example:

$n = 4, \ m = 9$

4, 4, 4, 4, 4, 36, 36, 36

Since 4 and 9 are relatively prime, $\mathbb{Z}_{36}$ can be rewritten as $\mathbb{Z}_4 \times \mathbb{Z}_9$. 
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Conjecture:

In $K_{n,m}$, the sandpile group $S(\Gamma)_{n,m} \cong \mathbb{Z}_n^{m-2} \times \mathbb{Z}_m^{n-2} \times \mathbb{Z}_{n,m}$.

Example:

$n = 4, m = 9$

4, 4, 4, 4, 4, 36, 36, 36

Since 4 and 9 are relatively prime, $\mathbb{Z}_{36}$ can be rewritten as $\mathbb{Z}_4 \times \mathbb{Z}_9$.

Thus $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_{36} \times \mathbb{Z}_{36} \times \mathbb{Z}_{36}$

$\cong \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_9 \times \mathbb{Z}_{36}$. 
Spanning tree to recurrent configuration

Case 1: For each $m$, there exists an edge from $m$ to the sink $(n_1)$.

**Example 1**

![Diagram showing a spanning tree to recurrent configuration](image)
Spanning tree to recurrent configuration

Case 1: For each $m$, there exists an edge from $m$ to the sink $(n_1)$.

Example 1
Example 2

Bijection between recurrent sandpiles and spanning trees directed into $s$

Complete Bipartite Graphs
Example 2

n₁ → m₁

n₂ → n₁ → m₁

n₃ → n₂ → n₁ → m₁

m₂ → n₃ → n₂ → n₁ → m₁

<table>
<thead>
<tr>
<th>t</th>
<th>Queue</th>
</tr>
</thead>
</table>

Parking Function: 

n₂ = 0, n₃ = 1, m₁ = 1, m₂ = 0

Parking Function = Recurrent Configuration
Example 2

<table>
<thead>
<tr>
<th>t</th>
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</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$n_1$</td>
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Complete Bipartite Graphs

Bijection between recurrent sandpiles and spanning trees directed into $s$
Complete Bipartite Graphs

Bijection between recurrent sandpiles and spanning trees directed into $s$

---

Example 2

<table>
<thead>
<tr>
<th>$t$</th>
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<tbody>
<tr>
<td>0</td>
<td>$n_1$</td>
</tr>
<tr>
<td>1</td>
<td>$m_2$</td>
</tr>
</tbody>
</table>

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Bijection between recurrent sandpiles and spanning trees directed into $s$

Pattern of $S(\Gamma_{N,m})$

Generators
Example 2

<table>
<thead>
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<tbody>
<tr>
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<td>(n_1)</td>
</tr>
<tr>
<td>1</td>
<td>(m_2)</td>
</tr>
<tr>
<td>2</td>
<td>(n_2)</td>
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</table>

Parking Function: \(n_2 = 0\), \(n_3 = 1\), \(m_1 = 1\), \(m_2 = 0\)

\[\sigma_{\max} - \text{Parking Function} = \text{Recurrence Configuration}\]
Example 2

\begin{tabular}{c|c}
  t & Queue \\
  \hline
  0 & \(n_1\) \\
  1 & \(m_2\) \\
  2 & \(n_2\) \\
  3 & \(m_1\) \\
\end{tabular}
Example 2

<table>
<thead>
<tr>
<th>t</th>
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<tbody>
<tr>
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<td>$n_1$</td>
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Example 2

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Example 2

Parking Function:

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Example 2

Parking Function: $n_2 = 0$, 

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<tr>
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</tr>
<tr>
<td>5</td>
<td>-</td>
</tr>
</tbody>
</table>
Example 2

Parking Function: $n_2 = 0$, $n_3 = 1$,

<table>
<thead>
<tr>
<th>t</th>
<th>Queue</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$n_1$</td>
</tr>
<tr>
<td>1</td>
<td>$m_2$</td>
</tr>
<tr>
<td>2</td>
<td>$n_2$</td>
</tr>
<tr>
<td>3</td>
<td>$m_1$</td>
</tr>
<tr>
<td>4</td>
<td>$n_3$</td>
</tr>
<tr>
<td>5</td>
<td>-</td>
</tr>
</tbody>
</table>
Example 2

Parking Function: \( n_2 = 0, n_3 = 1, m_1 = 1 \),

\[
\begin{array}{c|c}
\mathbf{t} & \text{Queue} \\
0 & n_1 \\
1 & m_2 \\
2 & n_2 \\
3 & m_1 \\
4 & n_3 \\
5 & - \\
\end{array}
\]
Complete Bipartite Graphs

Bijection between recurrent sandpiles and spanning trees directed into $s$

Example 2

Parking Function: $n_2 = 0$, $n_3 = 1$, $m_1 = 1$, $m_2 = 0$

$= [0, 1, 1, 0]$
Example 2

Parking Function: \( n_2 = 0, \ n_3 = 1, \ m_1 = 1, \ m_2 = 0 \)
\[ = [0, 1, 1, 0] \]
\( \sigma_{max} \) — Parking Function = Recurrent Configuration
Example 2

$\sigma_{\text{max}} - \text{Parking Function} = \text{Recurrent Configuration}$

$[1, 1, 2, 2] - [0, 1, 1, 0] = [1, 0, 1, 2]$  

$[n_2, n_3, m_1, m_2]$  

$\text{Recurrent Configurations:}$

$[1, 1, 2, 2], [1, 1, 2, 1], [1, 1, 2, 0], [0, 0, 2, 2], [1, 1, 0, 2], [0, 1, 1, 2], [1, 0, 1, 2], [1, 0, 2, 1], [0, 1, 2, 1], [1, 0, 2, 2], [0, 1, 2, 2], [1, 1, 1, 2]$
Recurrent configuration to spanning tree

Example 2

\[ \sigma_{\text{max}} - \text{Recurrent configuration} = \text{Parking Function} \]

\[ [1, 1, 2, 2] - [1, 0, 1, 2] = [0, 1, 1, 0] \]
Recursent configuration to spanning tree

Example 2

\[ \sigma_{max} - \text{Recurent configuration} = \text{Parking Function} \]
\[ [1, 1, 2, 2] - [1, 0, 1, 2] = [0, 1, 1, 0] \]
Recurrent configuration to spanning tree

Example 2

$\sigma_{max} - \text{Recurrent configuration} = \text{Parking Function}$

$[1, 1, 2, 2] - [1, 0, 1, 2] = [0, 1, 1, 0]$

Diagram:

- Complete Bipartite Graphs
- Bijection between recurrent sandpiles and spanning trees directed into $s$
- Recurrent configuration to spanning tree
- Example 2
- $\sigma_{max}$ - Recurrent configuration = Parking Function
- $[1, 1, 2, 2] - [1, 0, 1, 2] = [0, 1, 1, 0]$

Diagram shows a network with nodes labeled $S$, $m_1$, $n_2$, $n_3$, $m_2$, $0$, and $-1$, and a queue with entries $0$, $n_1$, $1$, $m_2$, and $2$, $n_2$.
Complete Bipartite Graphs

Recurrence configuration to spanning tree

Example 2

\[ \sigma_{max} \rightarrow \text{Recurrence configuration} = \text{Parking Function} \]

\[ [1, 1, 2, 2] - [1, 0, 1, 2] = [0, 1, 1, 0] \]
Complete Bipartite Graphs

Bijection between recurrent sandpiles and spanning trees directed into $s$

Recurrent configuration to spanning tree

Example 2

$\sigma_{\text{max}}$ — Recurrent configuration $=$ Parking Function

$[1, 1, 2, 2] - [1, 0, 1, 2] = [0, 1, 1, 0]$
Recurrent configuration to spanning tree

Example 2

\[ \sigma_{\text{max}} - \text{Recurrent configuration} = \text{Parking Function} \]

\[ [1, 1, 2, 2] - [1, 0, 1, 2] = [0, 1, 1, 0] \]
Recurrent configuration to spanning tree

Example 2

$$\sigma_{max} - \text{Recurrent configuration} = \text{Parking Function}$$

$$[1, 1, 2, 2] - [1, 0, 1, 2] = [0, 1, 1, 0]$$

$$
\begin{array}{c|c}
 t & \text{Queue} \\
 0 & n_1 \\
 1 & m_2 \\
 2 & n_2 \\
 3 & m_1 \\
 4 & n_3 \\
 5 & - \\
\end{array}
$$

A unique bijection!
## Generators on $n = 2$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$m$</th>
<th>SNF diag.</th>
<th>Generators $(m_1, m_2, \ldots, n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>$1 \times 1 \times 4$</td>
<td>$(0, 0, 1) = (1, 0, 1)$</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>$1 \times 1 \times 2 \times 3$</td>
<td>$(0, 0, 1, 0) = (0, 1, 1, 2)$, $(0, 0, 0, 1) = (1, 0, 1, 1)$</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>$1 \times 1 \times 2 \times 2 \times 8$</td>
<td>$(0, 0, 1, 0, 0) = (0, 0, 1, 1, 3)$, $(0, 0, 0, 1, 0) = (0, 1, 0, 1, 3)$, $(0, 0, 0, 0, 1) = (0, 1, 1, 1, 3)$</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>$1 \times 2 \times 2 \times 2 \times 10$</td>
<td>$(0, 0, 1, 0, 0, 0) = (0, 0, 0, 1, 1, 4)$, $(0, 0, 0, 1, 0, 0) = (0, 0, 1, 0, 1, 4)$, $(0, 0, 0, 0, 1, 0) = (0, 1, 0, 0, 1, 4)$, $(0, 0, 0, 0, 0, 1) = (0, 1, 1, 1, 0, 4)$</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>$1 \times 1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 12$</td>
<td>$(0, 0, 1, 0, 0, 0, 0, 0) = (0, 0, 0, 0, 1, 1, 5)$, $(0, 0, 0, 1, 0, 0, 0, 0) = (0, 0, 0, 0, 1, 0, 1, 5)$, $(0, 0, 0, 0, 1, 0, 0, 0) = (0, 0, 1, 0, 0, 1, 5)$, $(0, 0, 0, 0, 0, 1, 0, 0) = (0, 1, 0, 0, 0, 1, 5)$, $(0, 0, 0, 0, 0, 0, 1, 0) = (1, 0, 0, 0, 0, 1, 5)$</td>
</tr>
</tbody>
</table>
### Generators on $n = 3$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$m$</th>
<th>Generators $(m_1, m_2, \ldots, n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>(0, 0, 1, 0) = (2, 2, 1, 1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0, 0, 0, 1) = (2, 1, 0, 1)</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>(0, 0, 1, 0, 0) = (2, 2, 2, 0, 1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0, 0, 0, 1, 0) = (0, 1, 2, 2, 2)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0, 0, 0, 0, 1) = (2, 1, 2, 0, 1)</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>(0, 0, 0, 1, 0, 0) = (0, 1, 0, 2, 3, 3)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0, 0, 0, 0, 1, 0) = (1, 1, 0, 2, 3, 1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0, 0, 0, 0, 0, 1) = (0, 2, 2, 1, 2, 3)</td>
</tr>
</tbody>
</table>
Generators on \( n = 3 \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( m )</th>
<th>( (m_1, m_2, \ldots, n) )</th>
</tr>
</thead>
</table>
| 3 | 2 | \((0, 0, 1, 0) = (2, 2, 1, 1)\)  
\((0, 0, 0, 1) = (2, 1, 0, 1)\) |
| 3 | 3 | \((0, 0, 1, 0, 0) = (2, 2, 2, 0, 1)\)  
\((0, 0, 0, 1, 0) = (0, 1, 2, 2, 2)\)  
\((0, 0, 0, 0, 1) = (2, 1, 2, 0, 1)\) |
| 3 | 4 | \((0, 0, 0, 1, 0, 0) = (0, 1, 0, 2, 3, 3)\)  
\((0, 0, 0, 0, 1, 0) = (1, 1, 0, 2, 3, 1)\)  
\((0, 0, 0, 0, 0, 1) = (0, 2, 2, 1, 2, 3)\) |

Inconclusive evidence!
Complete Bipartite Graphs

Tynan Lazarus and Kendall Tada

Outline

Introduction

The Order of $S(\Gamma_{n,m})$

Identity on $S(\Gamma)$

Recurrent Configurations on $\Gamma_{n,m}$

Fundamental Theorem of Finite Abelian Groups

Bijection between recurrent sandpiles and spanning trees directed into $s$

Pattern of $S(\Gamma_{N,m})$

Generators $n=2$

Pattern of $S(\Gamma_{N,m})$ Generators $n=3$

Thank you PURE Math!!!