Security & Economics — Part 8

Social welfare and social choice

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Outline

Outline

Introduction

Social welfare and preference aggregation

Social choice and manipulability

Social choice

Kenneth Arrow's Thesis (1948, 1951)

'In a capitalist democracy there are essentially two methods by which social choices can be made:

- voting, typically used to make "political" decisions, and
- the market mechanism, typically used to make "economic" decisions.'

Social choice

Kenneth Arrow's Thesis (1948, 1951)

'... In the emerging democracies with mixed economic systems, Great Britain, France, and Scandinavia, the same two modes of making social choices prevail, though more scope is given to the method of voting and decisions based directly or indirectly on it and less to the rule of the price mechanism. Elsewhere in the world, and even in smaller social units within the democracies, social decisions are sometimes made by single individuals or small groups.'

Problem of voting

There are 11 voters and 3 candidates: *a*, *b* and *c*. The voters need to elect one candidate. They have different preferences.

Describe a method to elect the candidate which satisfies most voters.

Voting problem

Suppose the preferences are as follows:

voters	preference
3	$a \succ b \succ c$
2	$a \succ c \succ b$
2	$b \succ c \succ a$
4	$c \succ b \succ a$

Voting problem

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- If each voter casts 1 vote, then the tally is 5:4:2 for $a \succ c \succ b$.

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- If each voter casts (1,1) votes, then the tally is 9:8:5 for $b \succ c \succ a$.

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- If each voter casts 1 vote, then the tally is 5:4:2 for $a \succ c \succ b$.
- If each voter casts (1,1) votes, then the tally is 9:8:5 for $b \succ c \succ a$.
- If each voter casts (2,1) votes, then the tally is 12:11:10 for $c \succ b \succ a$

Outline

Preference relation

Definition

A *preference* over a set $\mathcal{S}$ is a binary relation $\succ$ on $\mathcal{S}$ such that for all $X, Y, Z \in \mathcal{S}$ holds

$$X \succ Y \ \wedge \ Y \succ Z \implies X \succ Z$$

$$(X \succ Y \ \vee Y \succ X) \ \wedge X \not\succ X$$

Preference relation

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$$X \succ Y \ \wedge \ Y \succ Z \implies X \succ Z$$

$$(X \succ Y \ \vee Y \succ X) \ \wedge X \nsucc X$$

We write $x \sim y$ when $x \succ y \wedge y \succ x$ holds.

Recall: Utility function

Definition

A *utility function* corresponding to a preference preorder $\succ \subseteq \mathcal{S} \times \mathcal{S}$ is a function $u : \mathcal{S} \to \mathbb{R}$ such that

$$u(X) > u(Y) \iff X \succ Y$$

Recall: Utility function

Remark

When the preferences involve random events, then the argument X in a utility function $u(X)$ is a random variable.

Preference space

Definition

The *preference space* over a set $\mathcal{S}$ is the set $\mathbb{P}$ of all preference relations $\succ$ over $\mathcal{S}$

$$\mathbb{P} = \Big\{ \succ \subseteq \mathcal{S} \times \mathcal{S} \mid X \succ Y \succ Z \implies X \succ Z \\ \wedge (X \succ Y \vee Y \succ X) \Big\}$$

Social welfare function

Definition

For a society consisting of the players $i = 1, 2, \ldots n$, a *social welfare function (swf)* is a mapping

$$\wr - \int_w \ : \ \mathbb{P}^n \ \rightarrow \ \mathbb{P}$$
$$\succ \ \longmapsto \ \wr \succ \int_w$$

where $\succ = \langle \overset{1}{\succ}, \overset{2}{\succ}, \ldots, \overset{n}{\succ} \rangle$

Social welfare function

Definition

For a society consisting of the players $i = 1, 2, \ldots n$, a *social welfare function (swf)* is a mapping

$$\begin{array}{rcccl} \wr - \int_w & : & \mathbb{P}^n & \rightarrow & \mathbb{P} \\ & & \succ & \longmapsto & \wr \succ \int_w \end{array}$$

where $\succ = \langle \overset{1}{\succ}, \overset{2}{\succ}, \ldots, \overset{n}{\succ} \rangle$

The relation $\wr \succ \int_w$ is the *aggregate preference* (or *social welfare*) induced by the profile $\succ \in \mathbb{P}^n$.

Social welfare function

There are many different ways to aggregate preferences.

- If $A = \{a, b, c\}$, then $\mathbb{P}$ has 6 elements:

$$a \succ b \succ c \qquad b \succ c \succ a \qquad c \succ b \succ a$$

$$a \succ c \succ b \qquad b \succ a \succ c \qquad c \succ a \succ b$$

Social welfare function

There are many different ways to aggregate preferences.

- If $A = \{a, b, c\}$, then $\mathbb{P}$ has 6 elements:

$$\begin{array}{ccc} a \succ b \succ c & b \succ c \succ a & c \succ b \succ a \\ a \succ c \succ b & b \succ a \succ c & c \succ a \succ b \end{array}$$

- For a society of $n = 2$ members, the number of swfs $\mathbb{P}^2 \to \mathbb{P}$ is

$$6^{36} \approx 10^{28}$$

Example 1: Utilitarianism of Jeremy Bentham

"The Greatest Pleasure Principle"

- given utilities $u_i : A \to [0,1]$ for $i = 1, 2, \ldots n$ with

$$a \overset{i}{\succ} b \iff u_i(a) > u_i(b)$$

- derive $u : A \to [0,1]$ as

$$u(x) = \sum_{i=1}^{n} \frac{u_i(x)}{n}$$

and set

$$a \wr\succ\wr_W b \iff u(a) > u(b)$$

Example 2: Borda Ranking

- Suppose that there are ℓ candidates in A.
- For each i, rename the candidates

$$A = \left\{a_0^{(i)}, a_1^{(i)}, a_2^{(i)}, \ldots a_{\ell-1}^{(i)}\right\}$$

so that

$$a_{\ell-1}^{(i)} \overset{i}{\succ} a_{\ell-2}^{(i)} \overset{i}{\succ} a_{\ell-3}^{(i)} \overset{i}{\succ} \cdots \overset{i}{\succ} a_0^{(i)}$$

and set

$$u_i\left(a_k^{(i)}\right) = k$$

Example 2: Borda Ranking

- Then derive $u : A \to \mathbb{R}$ as

$$u(x) \;=\; \sum_{i=1}^{n} u_i(x)$$

and set

$$a \wr \succ \wr_w b \iff u(a) > u(b)$$

General Ranking

Definition

A *voting vector* (or a *procedure*) for ℓ candidates is an ℓ-tuple

$$(c_{\ell-1}, c_{\ell-2}, \ldots, c_0)$$

which is descending, i.e. $c_{i+1} \geq c_i$ for all i.

General Ranking

- Suppose that there are ℓ candidates in A.
- Let $(c_{\ell-1}, c_{\ell-2}, \ldots, c_0)$ be a voting vector.
- For each i, rename the candidates

$$A = \left\{a_0^{(i)}, a_1^{(i)}, a_2^{(i)}, \ldots a_{\ell-1}^{(i)}\right\}$$

so that

$$a_{\ell-1}^{(i)} \overset{i}{\succ} a_{\ell-2}^{(i)} \overset{i}{\succ} a_{\ell-3}^{(i)} \overset{i}{\succ} \cdots \overset{i}{\succ} a_0^{(i)}$$

and set

$$u_i\left(a_k^{(i)}\right) = c_k$$

General Ranking

- Then derive $u : A \to \mathbb{R}$ as

$$u(x) \quad = \quad \sum_{i=1}^{n} u_i(x)$$

and set

$$a \wr{\succ}\textstyle\int_w b \quad \Longleftrightarrow \quad u(a) > u(b)$$

General Ranking

Instances

- **plurality vote:** $(1, 0, \ldots, 0)$
- **antiplurality vote:** $(1, 1, \ldots, 1, 0)$
- **Borda ranking:** $(\ell - 1, \ell - 2, \ldots, 0)$

Ranking problem

Exercise

Consider again the preferences

voters	preference
3	$a \succ b \succ c$
2	$a \succ c \succ b$
2	$b \succ c \succ a$
4	$c \succ b \succ a$

Compute the aggregate rankings for the voting vectors: (1,0,0), (4,1,0), (7,2,0), (7,3,0), (2,1,0), (3,2,0), (1,1,0).

Ranking problem

Solution

voting vector	ranking
(1,0,0)	$a \succ c \succ b$
(4,1,0)	$a \sim c \succ b$
(7,2,0)	$c \succ a \succ b$
(7,3,0)	$c \succ a \sim b$
(2,1,0)	$c \succ b \succ a$
(3,2,0)	$b \sim c \succ a$
(1,1,0)	$b \succ c \succ a$

Dictatorship Theorem

Notation

$$a\{\succ\}b \quad = \quad \{i \mid a \overset{i}{\succ} b\}$$

Dictatorship Theorem

Theorem (K. Arrow)

Suppose that $\wr - \int_w : \mathbb{P}^n \to \mathbb{P}$ satisfies

- *Pareto* or *Unanimity Principle (UP)*:

$$\forall i. a \overset{i}{\succ} b \implies a \wr \succ \int_w b$$

- *Independence of Irrelevant Alternatives (IIA)*:

$$a\{\succ\}b = a\{\sqsupset\}b \ \wedge \ a \wr \succ \int_w b \implies a \wr \sqsupset \int_w b$$

holds for every two profiles $\succ, \sqsupset \in \mathbb{P}^n$.

Dictatorship Theorem

Theorem (K. Arrow)

Then as soon as there are more than 2 candidates in A, there must exist a dictator, i.e. a voter i such that

$$a \overset{i}{\succ} b \quad \Longleftrightarrow \quad a \wr\succ\int_W b$$

holds for every preference profile $\succ \in \mathbb{P}^n$.

Dictatorship Theorem

Remark

Note that the theorem does not say that the dictator has to actively *impose* his preferences.

Dictatorship Theorem

Remark

Note that the theorem does not say that the dictator has to actively *impose* his preferences.

The theorem says that

- for every swf satisfying UP and IIA
- there is a voter who agrees with the social welfare for every preference profile of the society.

Condorcet requirement

Definition

A swf $\wr - \int_w : \mathbb{P}^n \to \mathbb{P}$ satisfies the *Condorcet requirement* if

$$a \wr \succ \int_w b \implies \#a\{\succ\}b > \#b\{\succ\}a$$

Borda count violates Condorcet requirement

Example

Consider the preferences

voters	preference
30	$a \succ b \succ c$
1	$a \succ c \succ b$
29	$b \succ a \succ c$
10	$b \succ c \succ a$
10	$c \succ a \succ b$
1	$c \succ b \succ a$

Borda count violates Condorcet requirement

Example

Consider the preferences

voters	preference
30	$a \succ b \succ c$
1	$a \succ c \succ b$
29	$b \succ a \succ c$
10	$b \succ c \succ a$
10	$c \succ a \succ b$
1	$c \succ b \succ a$

Then $b(109) \wr\succ\int_w a(101) \wr\succ\int_w c(33)$
but $a(41) \succ b(40)$ and $a(60) \succ c(21)$.

Condorcet ranking

Definition

Given a preference profile $\succ \in \mathbb{P}^n$, the *Condorcet ranking* $\gg$ is defined by setting

$$a \gg b \iff \#a\{\succ\}b > \#b\{\succ\}a$$

Condorcet ranking allows cycles

Example

Consider the preferences

voters	preference
23	$a \succ b \succ c$
2	$b \succ a \succ c$
17	$b \succ c \succ a$
10	$c \succ a \succ b$
8	$c \succ b \succ a$

Condorcet ranking allows cycles

Example

Consider the preferences

voters	preference
23	$a \succ b \succ c$
2	$b \succ a \succ c$
17	$b \succ c \succ a$
10	$c \succ a \succ b$
8	$c \succ b \succ a$

Then

$$a(33) \gg b(27) \qquad b(42) \gg c(18) \qquad c(35) \gg a(25)$$

Condorcet ranking allows cycles

Corollary

Condorcet ranking may not be transitive.

Proof

If Condorcet ranking were transitive, then $a \gg b$ and $b \gg c$ and $c \gg a$ would imply $a \gg a$.

But by the definition of Condorcet ranking, this would mean that $\#a\{\succ\}a \ > \ \#a\{\succ\}a$, which is impossible.

Outline

Social choice

It is often

- not necessary to aggregate the individual preferences $\overset{i}{\succ}$ into a full social preference relation $\wr\succ\int_w$, but it is
- sufficient to elect the best candidate c, i.e. such that $c \wr\succ\int_w x$ for all $x \in A$.

Social choice function and relation

Definition

A *social choice function (scf)* is a mapping $\wr - \int_f \; : \mathbb{P}^n \to A$.

A *social choice relation (scr)* is a mapping $\wr - \int_r \; : \mathbb{P}^n \to \wp A$.

Social choice function and relation

Example 1

A swf $\wr\!-\!\int_w$ always induces a scr

$$c \in \wr\!\succ\!\int_r \quad\Longleftrightarrow\quad \forall x.\ c \wr\!\succ\!\int_w x$$

It induces a scf if the aggregate preferences have top elements.

Social choice function and relation

Example 2

If the space of alternative choices A can be presented in the form

$$A = \prod_{i=1}^{n} A_i$$

where each A_i is controlled by the player i, then the scr can be defined to be

$$\wr\succ\int_r = \{\sigma \in A \mid \sigma\; BR\; \sigma\}$$

i.e. the social choices are the equilibria of the game.

Manipulability

Definition

A social function $\wr - \smallint_f : \mathbb{P}^n \to A$ is *manipulable* if there is a voter i and a preference profile $\succ \in \mathbb{P}^n$ such that

$$\wr \underset{i}{\sqsupset} \smallint_f \quad \overset{i}{\succ} \quad \wr \succ \smallint_f$$

where

$$\underset{i}{\sqsupset} \quad = \quad \left\langle \overset{1}{\succ}, \overset{2}{\succ}, \ldots, \overset{i}{\sqsupset}, \ldots \overset{n}{\succ} \right\rangle$$

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where

$$\underset{i}{\sqsupset} \quad = \quad \left\langle \overset{1}{\succ}, \overset{2}{\succ}, \ldots, \overset{i}{\sqsupset}, \ldots \overset{n}{\succ} \right\rangle$$

i.e., i can induce an $\overset{i}{\succ}$-preferred social choice if she does not vote honestly, according to $\overset{i}{\succ}$, but dishonestly, according to some $\overset{i}{\sqsupset}$.

Manipulability

Terminology

A social choice function that is not manipulable is said to be *incentive compatible*.

Manipulability

Theorem (Gibbard-Satterthwaite)

A surjective scf $\wr : \int_f \mathbb{P}^n \to A$ between more than 2 candidates in A is either manipulable, or a dictatorship, or both.

Manipulability

Exercise 1 (easy)

Prove that a scf is incentive compatible if and only if it is monotone, i.e. satisfies

$$a_0 \overset{i}{\underset{0}{\succ}} a_1 \quad \text{and} \quad a_1 \overset{i}{\underset{1}{\succ}} a_0$$

whenever

$$\begin{aligned} a_0 &= \wr \overset{1}{\succ}, \dots, \overset{i}{\underset{0}{\succ}}, \dots, \overset{n}{\succ} \int_f \\ a_1 &= \wr \overset{1}{\succ}, \dots, \overset{i}{\underset{1}{\succ}}, \dots, \overset{n}{\succ} \int_f \end{aligned}$$

Manipulability

Comment

The monotonicity of a scf $\wr - \int_f$ means that

- if changing only $\underset{0}{\overset{i}{\succ}}$ to $\underset{1}{\overset{i}{\succ}}$ causes the social choice to change from a_0 to a_1
- then the change must have been from $a_0 \underset{0}{\overset{i}{\succ}} a_1$ to $a_1 \underset{1}{\overset{i}{\succ}} a_0$.

Manipulability

Exercise 2 (hard)

Derive the Gibbard-Satterthwaite Theorem from Arrow's Theorem.