Chapter 15

Oscillation/Periodic Motion

Modified by P. Lam 8_2_2018

Learning Goals for Chapter 15

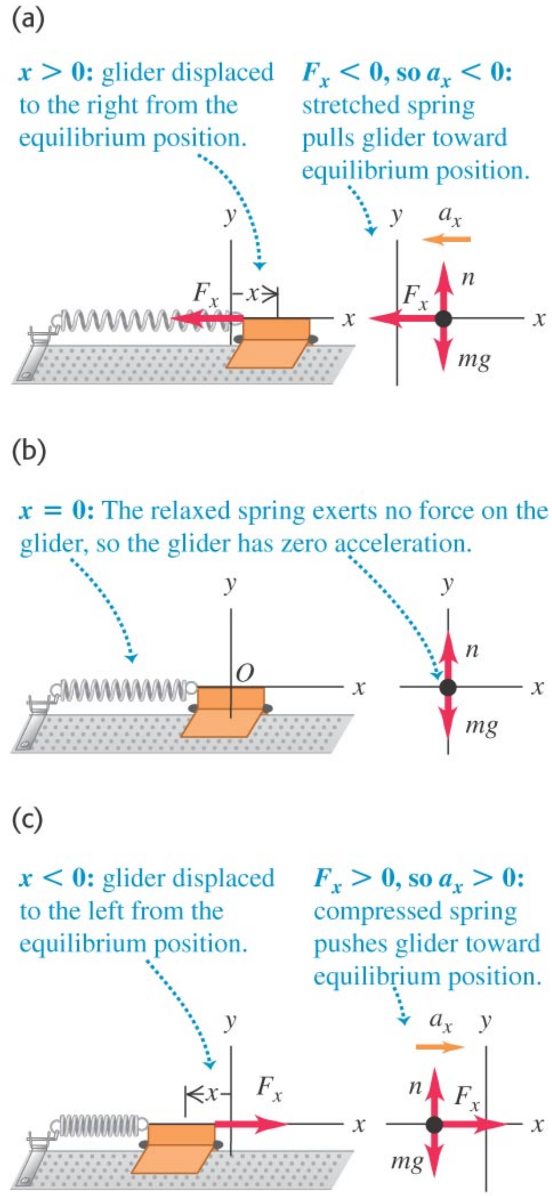
- To examine when natural oscillation occurs
- To quantify and learn the details of a special type of oscillation called simple harmonic motion
- To understand the concept of driven oscillations and resonance

An example of natural oscillation

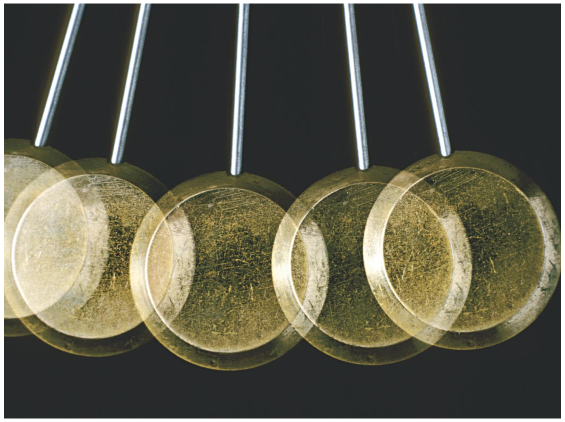
- A mass-spring system exhibits natural oscillation when the mass is displaced from its equilibrium position and then let go or give it push.
- Natural frequency

$$\omega = \sqrt{\frac{k}{m}} \ \text{rad/s}$$

- The restoring force is provided by the stretched (or compressed) spring.



Introduction - when does natural oscillation occur?



- Example: A pendulum hanging straight down is in a stable equilibrium position.
- If you pull the pendulum to the side and then let go or give it a push, the pendulum will oscillate at its natural frequency,

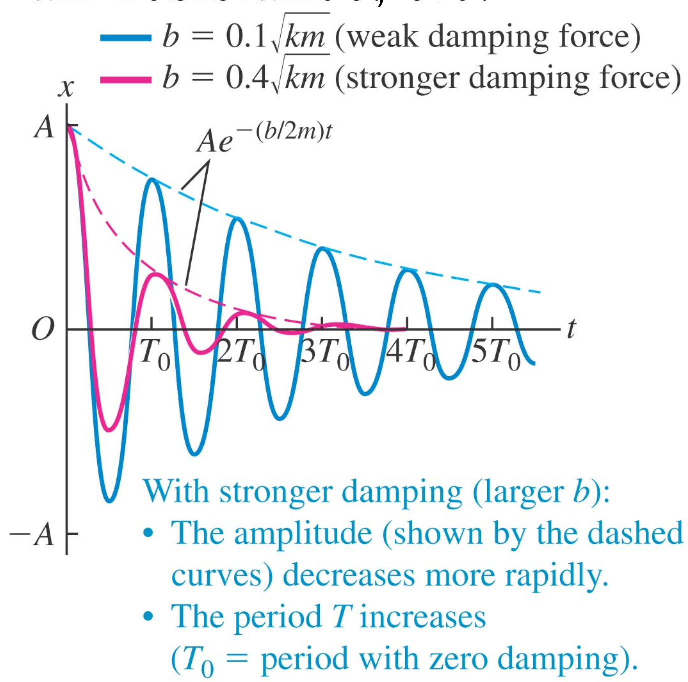
$\omega = \sqrt{\frac{g}{\ell}}$ rad/s (will derive later) if there is no damping forces.

Friction, air-resistance, and other damping forces will cause the pendulum to oscillate at a slightly different frequency and to stop eventually.

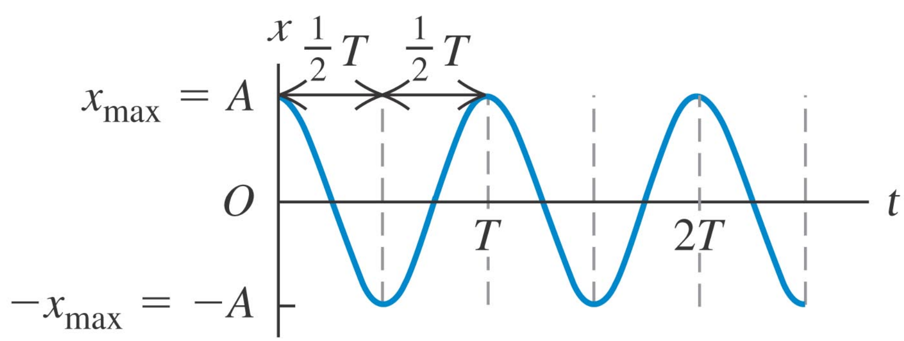
- Natural oscillation occurs whenever an object with mass (inertia) is displaced away from its ***stable equilibrium*** position. Note: the system provides a restoring force when the object is displaced from a stable equilibrium but not from an unstable equilibrium position.

Different between oscillation and periodic motion

- The text equates periodic motion with oscillation.
- Technically, a damped oscillation is not a periodic function. The motion is periodic only when there is no damping force. In real situations, all natural oscillations are damped by friction, air-resistance, etc.



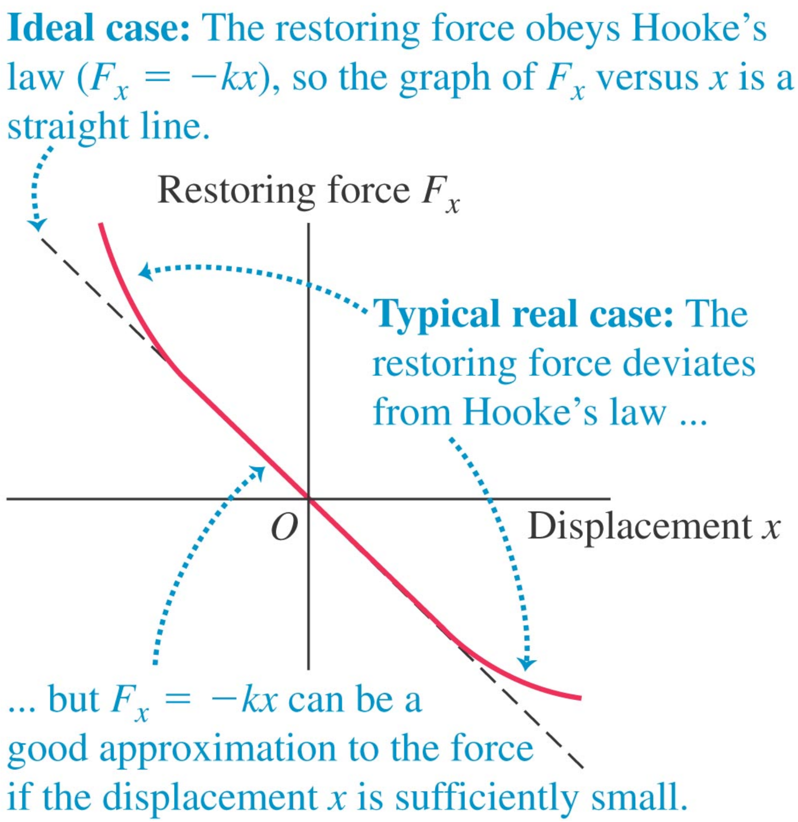
Damped oscillation



Undamped oscillation

A special type of oscillation - Simple harmonic motion

- An ideal spring responds to stretch and compression linearly, obeying Hooke' s Law.



Simple Harmonic motion occurs when there is no damping force and the restoring force is given by :

F = - kx (Hook's Law)

For real springs, Hook's Law is a good approximation only when the displacement (x) is "small".

Mathematics of simple harmonic motion (SHM)

$$\text{Newton's 2nd Law}: F = ma \Rightarrow -kx = m\frac{d^2x}{dt^2}$$

(assumed no damping force and restoring force obeys Hook's Law)

This equation is called a "differential equation" (D.E.).

That is an equation which relates the function, x(t), to

its derivatives (in this case, $\frac{\mathrm{dx}^2}{\mathrm{dt}^2}$).

A differential equation is like a puzzle.

Try to think of a function, x(t), which can

satisfy the relationship stated by the differential equation.

In this case, what function whose 2nd derivative

is proportional to the function with a negative sign?

Solution to the simple harmonic motion (SHM)

$$F = ma$$

$$-kx = m\frac{d^2x}{dt^2}$$

Solution: $x(t) = A\cos\omega t + B\sin\omega t$

Substitute:

$$-k(A\cos\omega t + B\sin\omega t) = -m\omega^2(A\cos\omega t + B\sin\omega t)$$

$\Rightarrow$ A & B can be any number

but

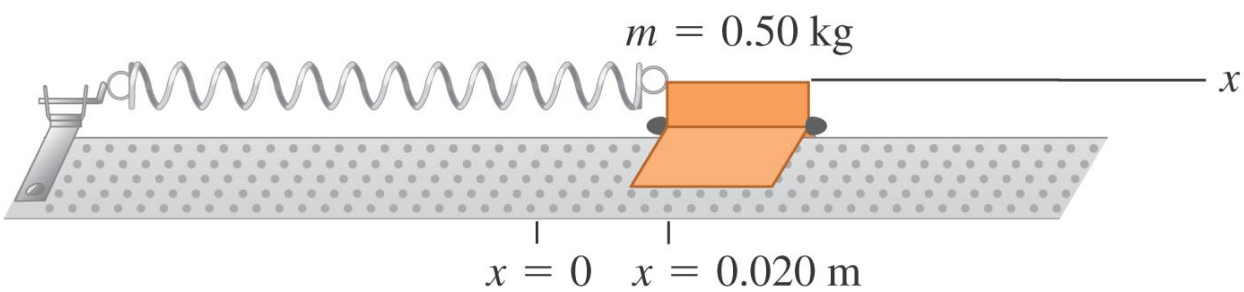
ω must be $\sqrt{\frac{k}{m}}$

ω is called the natural (angular) frequency of mass-spring system.

What are the physical meanings of A & B?

Sample problems for SHM

(b)



Given: Spring constant k=800 N/m, m=0.5kg,

initial displacment =+0.02m, initial velocity =0

a)Find: frequency of oscillation in rad/s, period of oscillation in second.

b) Find x(t) and sketch it.

c) Suppose the initial displacment = 0.02m, initial velocity = +3m/s, find x(t) and sketch it.

d) Suppose the initial displacment = 0, initial velocity = +3m/s, find x(t) and sketch it.

Solution to the simple harmonic motion (SHM) -cont

What does $x(t)=A\cos\omega t + B\sin\omega t$ represent?

What are the physical meanings of A & B?

Look at the situation at t=0

$x(t=0)=A \quad \Rightarrow$ A is the initial displacement.

$$v(t)=\frac{dx}{dt}=-A\omega\sin\omega t + B\omega\cos\omega t$$

$$v(t=0)=B\omega \Rightarrow B=\frac{v_o}{\omega}$$

Note : As long as Hook's law is valid (i.e. the system exhibits simple harmonic motion),
the mass spring system will oscillate at the natural freqency no matter how large or small A or B.

Solution to the simple harmonic motion (SHM) -cont

Most general solution to $\mathrm{m}\frac{d^2x}{dt^2} = -kx$ is:

$$\mathrm{x(t)} = \mathrm{A}\cos\omega\mathrm{t} + \mathrm{B}\sin\omega\mathrm{t}$$

$$= \mathrm{x_o}\cos\omega\mathrm{t} + \frac{\mathrm{v_o}}{\omega}\sin\omega\mathrm{t}$$

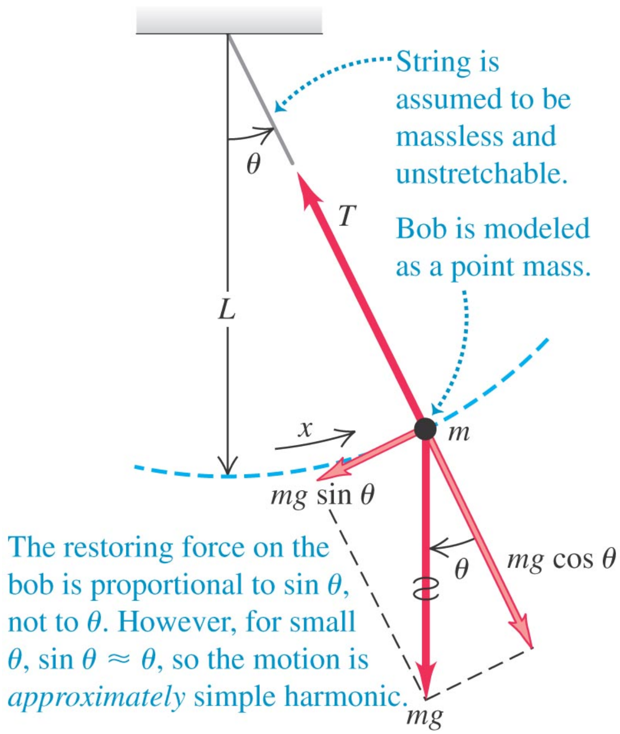
$$= \sqrt{\mathrm{x_o}^2 + \frac{\mathrm{v_o}^2}{\omega^2}}\cos(\omega t - \phi) \quad \text{(another way of writing the solution)}$$

$$\tan\phi = \frac{\mathrm{v_o}}{\omega\mathrm{x_o}}$$

$$[Note: \cos(\omega t - \phi) = \cos\omega t\cos\phi + \sin\omega t\sin\phi]$$

Simple Pendulum

(b) An idealized simple pendulum



$$\tau = I\alpha$$

$$-L(mg\sin\theta) = (mL^2)\frac{d^2\theta}{dt^2}$$

$$-\frac{g}{L}\sin\theta = \frac{d^2\theta}{dt^2}$$

For "small" θ, $\sin\theta \approx \theta \Rightarrow -\frac{g}{L}\theta = \frac{d^2\theta}{dt^2}$

compare with SHM: $-\frac{k}{m}x = \frac{d^2x}{dt^2}$

$\Rightarrow$ pendulum exhibits a SHM when θ is small

(e.g. less than 1 radian)

$\Rightarrow$ natural angular frequency for a simple pendulum is

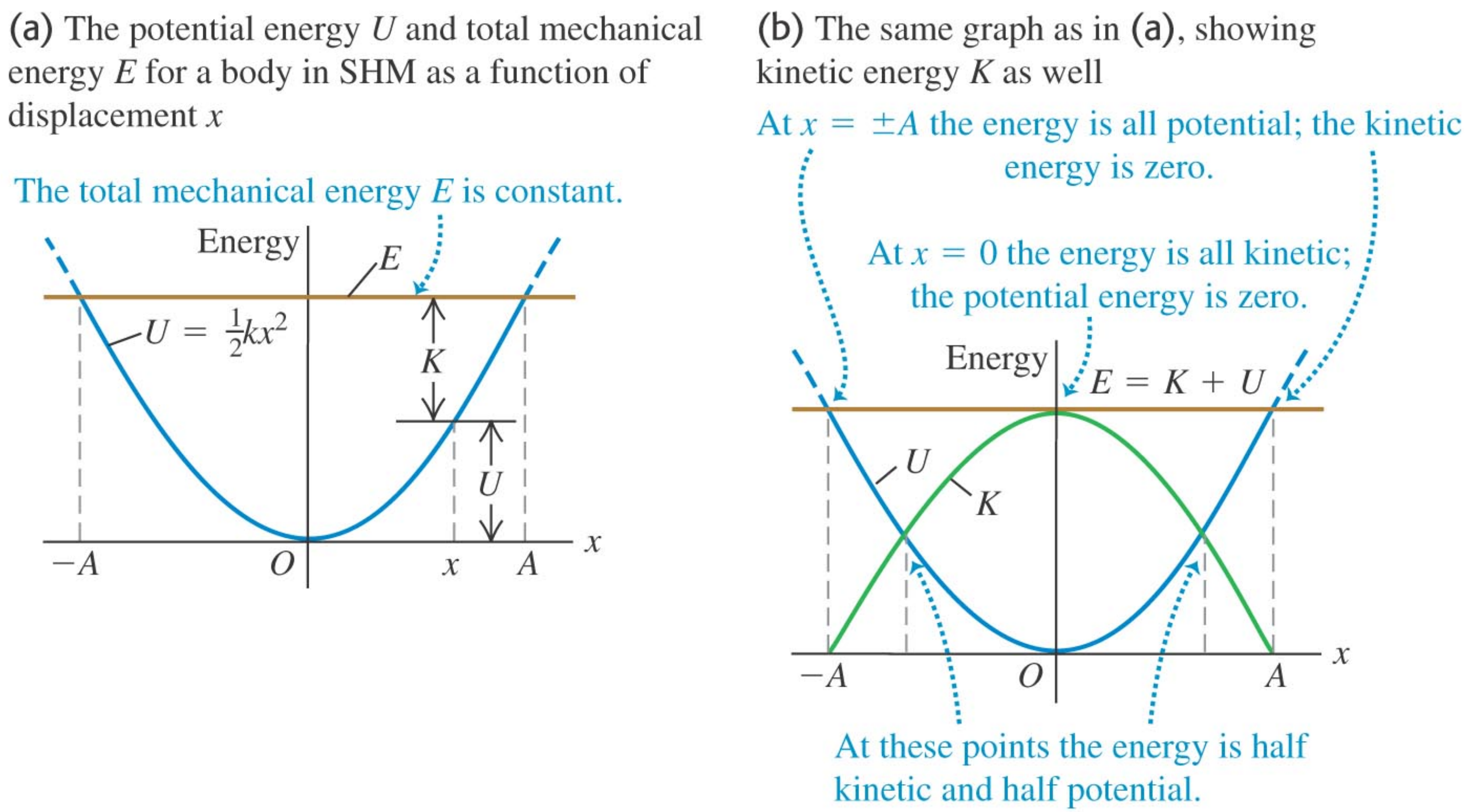
$\omega = \sqrt{\frac{g}{L}}$ $\quad$ (*Note*: does not depend on m)

$\Rightarrow \theta(t) = A\cos\omega t + B\sin\omega t$

Q. Find θ(t) for a pendulum with mass=3kg, L= 2.5m, released from rest with an initial displacement is 0.5 rad.

Conservation of energy in SHM – graphical view

- Figure 13.15 shows the interconversion of kinetic and potential energy with an energy versus position graphic.



Conservation of energy in SHM

$Given: x(t) = x_o \cos\omega t + \frac{v_o}{\omega}\sin\omega t$ for SHM

Calculate the kinetic energy (K=$\frac{1}{2}mv^2$) and potential energy (U=$\frac{1}{2}kx^2$)

and show that K+U does not change with time.

Try it yourselves first before looking at the answer below.

$$v(t) = \frac{dx}{dt} = -x_o\omega\sin\omega t + v_o\cos\omega t$$

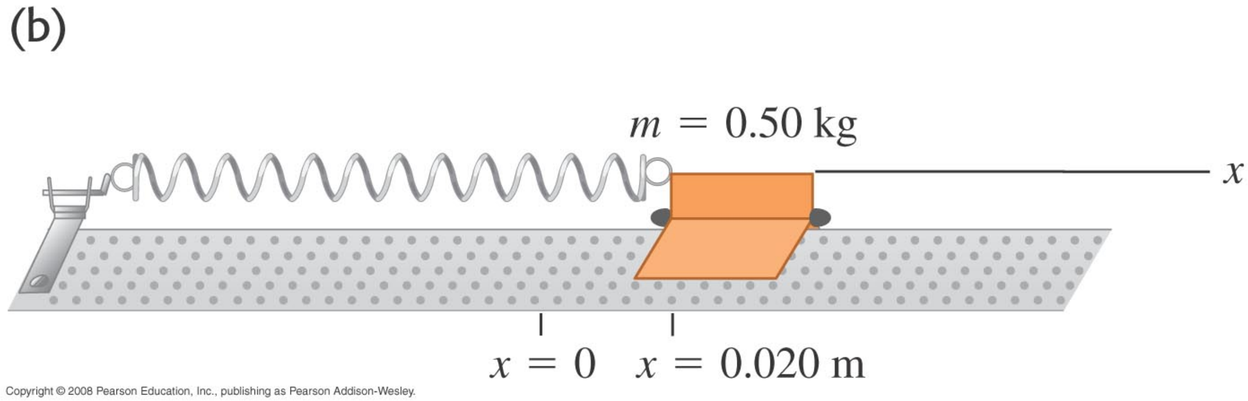
$$K(t) = \frac{1}{2}m[v(t)]^2 = \frac{1}{2}m\left[(x_o\omega)^2\sin^2\omega t - 2x_o\omega v_o\sin\omega t\cos\omega t + (v_o)^2\cos^2\omega t\right]$$

$$U(t) = \frac{1}{2}k[x(t)]^2 = \frac{1}{2}k\left[(x_o)^2\cos^2\omega t + 2x_o\frac{v_o}{\omega}\sin\omega t\cos\omega t + (\frac{v_o}{\omega})^2\sin^2\omega t\right]$$

$$Note: \quad k = m\omega^2$$

$$\Rightarrow K(t) + U(t) = \frac{1}{2}mv_o^2 + \frac{1}{2}kx_o^2 = K_o + U_o = \text{ constant}$$

Energy Expression for SHM



Given: Spring constant k=800 N/m, m=0.5kg,

initial displacment =+0.03m, initial velocity =2 m/s

Find: The maximum displacement.

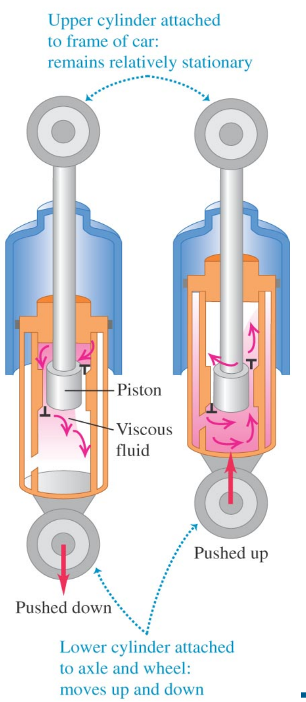
Find: The maximum velocity of the mass.

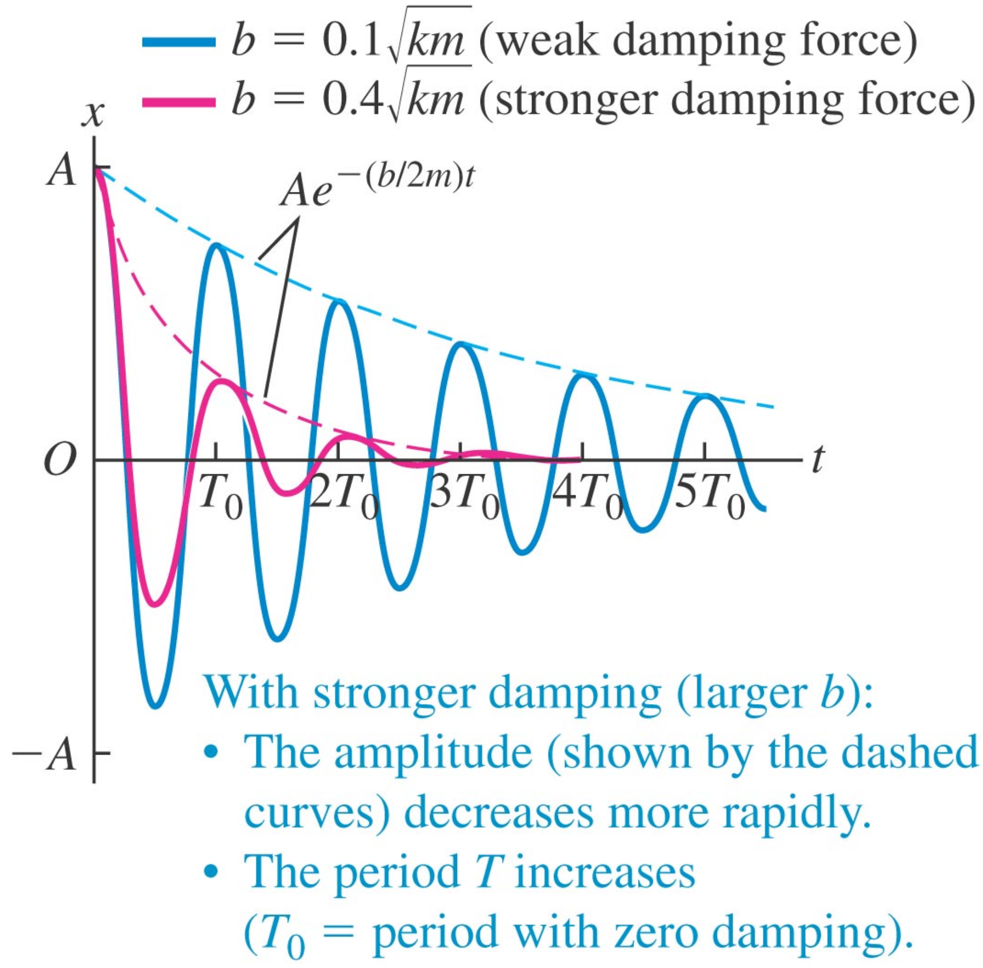
Find: The kinetic energy, potential energy and total energy at t=2 s.

Damped oscillations

- Consider shock absorbers on your automobile. Without damping, hitting a pothole would set your car into SHM on the springs that support it.

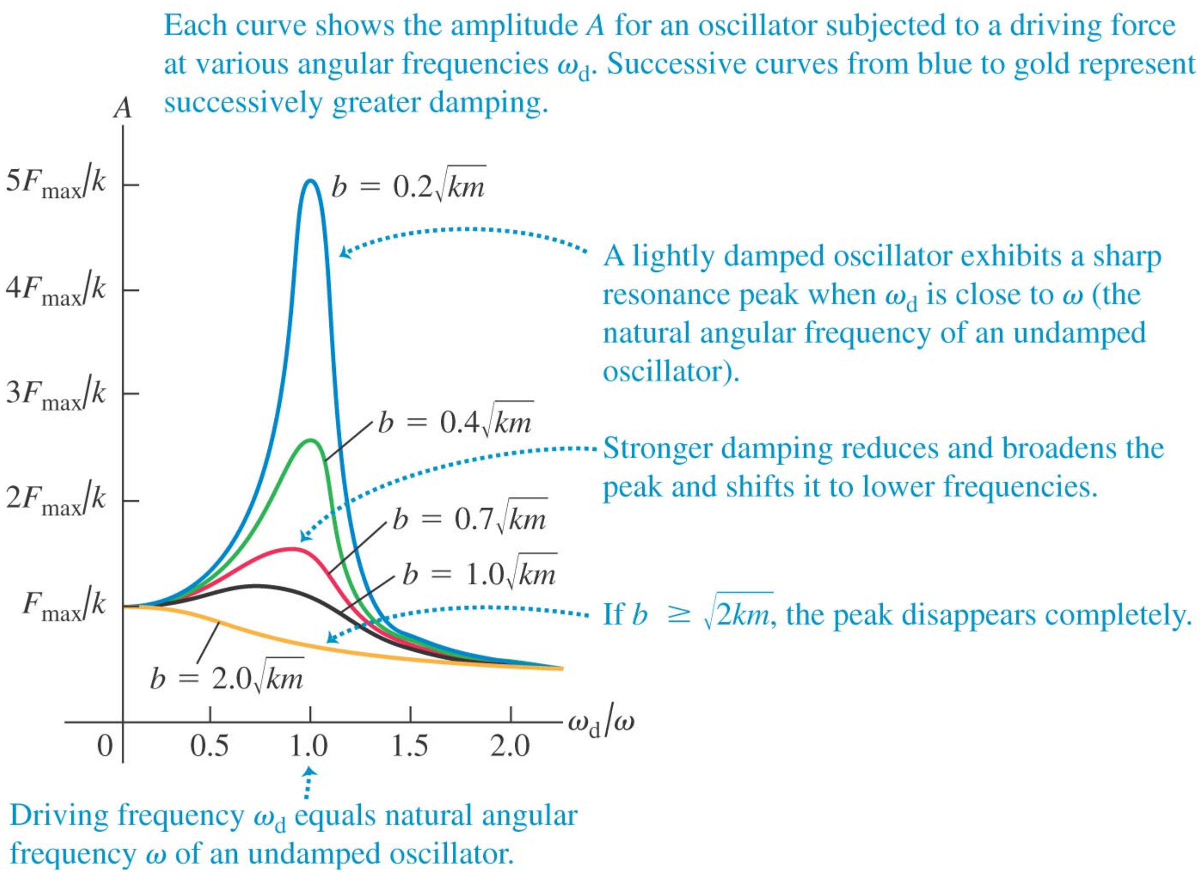
$$m\ddot{x} = -kx - b\dot{x}; \quad b = \text{damping coefficient}$$





Forced (driven) oscillations and resonance

- A force applied "in synch" with a motion already in progress will resonate and add energy to the oscillation (refer to Figure 13.28).
- ***Resonance occurs when driven frequency=natural frequency***



The system has the largest amplitude of oscillation at resonance, see Figure to the left. Damping decreases the amplitude.

Mathematics of driven oscillator

$$m\frac{d^2x}{dt^2} + kx = 0 \quad \text{(natural oscillation, natural freq.} = \omega_o = \sqrt{\frac{k}{m}}\text{)}$$

$$m\frac{d^2x}{dt^2} + kx = F_o \cos\omega t \text{ (driven ocillation, driven freq.} = \omega\text{)}$$

(assumed no damping force)

Solution: Let $x(t) = x_o \cos\omega t$ (system is forced to oscillate at driven freq.)

Substittue:

$$m\left(-\omega^2\right)x_o \cos\omega t + kx_o \cos\omega t = F_o \cos\omega t$$

$$\Rightarrow x_o = \frac{F_o}{k - m\omega^2} = \frac{F_o/m}{k/m - \omega^2} = \frac{F_o/m}{\omega_o{}^2 - \omega^2}$$

$\Rightarrow \infty$ amplitude when $\omega = \omega_o$

with damping force, amplitude is largest when $\omega = \omega_o$,

but not infinite.

Forced (driven) oscillations and resonance II

- The Tacoma Narrows Bridge suffered spectacular structural failure after absorbing too much resonant energy (refer to Figure 13.29).