On competitive equilibria with common complementarities

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Abstract

We investigate the connection between equilibrium existence and its attainability through simple market mechanisms in exchange economies with indivisibilities and complementarities. The analysis suggests that attaining efficient outcomes through simple non-combinatorial auctions may be problematic even when market clearing prices exist.

JEL classification code: C62, D44, D51

Key words: competitive equilibrium; complementarities; auctions

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1 Introduction

We study the problem of allocating two heterogeneous objects among a number of agents in an environment where there exist positive complementarities in agent valuations between objects. There are two issues of interest. The first is the existence of competitive equilibrium. The second is whether the competitive equilibrium can be obtained as an outcome of a simple market mechanism.

It is well-known that competitive equilibrium may not exist in an environment with indivisible objects. Bikhchandani and Mamer (1997) present necessary and sufficient conditions for existence of equilibrium in economies with indivisibilities using linear programming techniques. Kelso and Crawford (1982) and further Gul and Stachetti (1999) show that the competitive equilibrium exists in environments with indivisibilities if the gross substitute condition is satisfied, that is, if there is a certain substitutability in agent valuations across objects. In the presence of complementarities, examples of non-existence of equilibrium are easily generated (e.g., Bykowsky et al., 2000). It is then of interest to investigate whether competitive equilibrium exists in some special classes of environments with complementarities.

The literature also shows that in environments where the gross substitute condition on agent preferences is satisfied, competitive equilibrium can be obtained as an outcome of an English-type auction (Demange, Gale and Sotomayor, 1986; Gul and Stachetti, 2000; Ausubel, 2000). This suggests that there may be a connection between equilibrium existence and its attainability though simple auction mechanisms. This connection has not been investigated in other types of environments. As stated by Bikhchandani and Mamer (1997, p. 405), the question is: "Do there exist simple market mechanisms (i.e., mechanisms that assign a price to each object) which efficiently allocate multiple indivisible objects when market clearing prices exist?"

Our anlysis suggests that, in the presence of complementarities, existence of competitive equilibrium does not necessarily imply that an equilibrium outcome may be obtained through a simple auction. We present a class of two-good environments with common additive complementarity, for which the competitive equilibrium exists. However, an example suggests that simple dynamic mechanisms are unlikely to guarantee convergence to competitive equilibrium outcomes in these environments.[1]

[1] Bykowsky et al. (2000) point out that using simultaneous English auctions in environments with complementarities may lead to "mutually destructive bidding." However, the problem they discuss may be caused by the feature of the auction that requires bidders to commit themselves to buying individual objects even when desired packages do not materialize. This feature is absent in the auction we consider.

2 Competitive equilibria with common complementarities

The setting is as follows. There are two objects, A and B, and a set N of n agents (bidders), $n < \infty$. Let a_i be bidder i's value for object A, and b_i be bidder i's value for object B, with $a_i, b_i \in [0, \bar{v}]$. Then i's value for the package AB is given by

$$u_i(AB) = a_i + b_i + k,$$

where k is the common additive complementarity term, $k \geq 0$.[2] Let W be the set of possible packages that can be sold to a bidder, $W \equiv \{\emptyset, A, B, AB\}$, and let w be an element of W. We assume that bidders have quasi-linear utilities in packages and money, and are not budget constrained. Bidder i's utility of buying a package w given prices $p = (p_a, p_b)$ is i's net value of the package, or his surplus: $S_i(w; p) = u_i(w) - \sum_{j \in w} p_j$, where j is the object index, $j \in \{a, b\}$. Specifically,

$$S_i(\emptyset; p) = 0 \tag{1}$$

$$S_i(A; p) = a_i - p_a \tag{2}$$

$$S_i(B; p) = b_i - p_b \tag{3}$$

$$S_i(AB; p) = a_i + b_i + k - p_a - p_b \tag{4}$$

For any price vector (p_a, p_b), let i's demand set be the set of packages that maximize i's surplus at this price:

$$D_i(p) = \{w \in W | S_i(w; p) = max_{v \in W} S_i(v; p)\}. \tag{5}$$

We employ standard Walrasian notion of competitive equilibrium (CE). A price $p = (p_a, p_b)$ is a competitive equilibrium price if, given p, there is an allocation of objects to bidders $\mu : \{A, B\} \to N$ such that each bidder gets a package in their demand set, i.e., there is no excess demand. Such price and allocation pair (p, μ) is called a competitive equilibrium if, in addition, the prices of all unallocated objects are zero.

An allocation is efficient if the total value of allocation is maximized. In the presence of complementarity, equilibrium and efficiency conditions will differ depending on whether the objects are allocated to the same or to different bidders. We will say that "packaging" is efficient if it is efficient to allocate both items to the same bidder $i \in N$. "Splitting" is efficient if it is efficient to allocate the items to two different bidders.

[2] The framework is similar to Brusco and Lopomo (2002), who consider two-object environments with either no complementarities or large additive complementarities in their study of bidder collusion in multi-unit ascending price auctions. Assuming that the object values are drawn independently across bidders from the same probability distribution, and the objects are allocated using a simultaneous ascending bid auction, Brusco and Lopomo show that with either no complementarity or with large complementarities there exists a "competitive" Perfect Bayesian Equilibrium of this auction that leads to an efficient allocation. The case of moderate complementarities is not considered.

In this framework, we establish the following.

Proposition 1 *For any finite number of bidders, $n < \infty$, and any common complementarity term, $k \geq 0$, the set of CE prices and allocations is non-empty, and any CE allocation is efficient.*[3] *The set of CE prices is characterized as follows:*

- *Suppose that allocating both items to one bidder, or packaging, is efficient:*

$$a_i + b_i + k \geq \max\{\max_{j \in N} a_j + \max_{j \in N} b_j,\ \max_{j \neq i}(a_j + b_j) + k\} \tag{6}$$

for some $i \in N$. Then the set of CE prices is given by (p_a, p_b) such that

$$\max_{j \neq i}(a_j + b_j) + k \leq p_a + p_b \leq a_i + b_i + k; \tag{7}$$

$$\max_{j \neq i} a_j \leq p_a \leq a_i + k; \tag{8}$$

$$\max_{j \neq i} b_j \leq p_b \leq b_i + k. \tag{9}$$

- *Suppose that splitting of items between bidders is efficient:*

$$a_i + b_j \geq \max\{\max_{l \in N} a_l + \max_{l \in N} b_l,\ \max_{l \in N}(a_l + b_l) + k\}, \tag{10}$$

for some $i, j \in N$, $i \neq j$. Then the set of CE prices is given by (p_a, p_b) such that

$$\max_{l \neq i \neq j}(a_l + b_l) + k \leq p_a + p_b; \tag{11}$$

$$\max\{a_j + k, \max_{l \neq i \neq j} a_l\} \leq p_a \leq a_i; \tag{12}$$

$$\max\{b_i + k, \max_{l \neq i \neq j} b_l\} \leq p_b \leq b_j. \tag{13}$$

The proof is straightforward and is given in the appendix.

3 Failure of Simple Auctions

We now consider whether competitive equilibrium outcomes may be achieved by honest (non-strategic) bidders under a simple English-type auction. We will consider a variant of the progressive auction mechanism of Demange, Gale and Sotomayor, that has been shown to lead to competitive equilibrium outcomes when goods are substitutes (Demange, Gale and Sotomayor, 1986; Gul and Stachetti, 2000). Assume that all values are discrete; specifically, all prices are integers, and all bidder valuations are even integers. The auction starts with an initial price vector $(p_a^0, p_b^0) = (0, 0)$ announced by the auctioneer. Each

[3] Bikhchandani and Mamer (1997) show that if market clearing prices exist in an exchange economy with indivisibilities, then the corresponding allocations must be efficient. We re-establish equilibrium efficiency here for the sake of completeness.

bidder announces which packages $w \in W$ are in their demand set at this price. It is required that all bidders report all packages in their demand sets. If it is possible to assign items $\{A, B\}$ to bidders so that each bidder gets a package in their demand set, then the prices must be at a CE, and the auction stops. If no such assignment exists, then the auctioneer raises prices by one unit on items in $\{A, B\}$ which are overdemanded. An item is overdemanded at price p if it is necessary to increase the supply of this item (and, possibly, some other items) to find an assignment so that each bidder gets a package in their demand set. Let $O(p)$ denote the set of overdemanded items at price p. After the prices are raised, the bidders report their new demand sets, and the procedure continues until a price vector is reached at which no excess demand exists.

We focus on this particular auction mechanism because it appears the most promising within the family of simple non-combinatorial auctions. Honest bidding under this mechanism can never lead to bidder losses. Analogous mechanisms guarantee convergence to competitive equilibrium in environments with substitutes. Further, with common complementarities, in the special cases of either two bidders and an arbitrary complementarity term, or any number of bidders and a large complementarity term, $k > \bar{v}$, this auction does lead to an efficient allocation at minimal competitive equilibrium prices (Sherstyuk, 2002).

Unfortunately, the following example demonstrates that with more than two bidders, the mechanism may result in inefficient allocations and prices out of equilibrium range.

Example 1 Let there be three bidders, $n = 3$, and let $a_1 = b_1 = 20$, $a_2 = 36$, $b_2 = 0$, $a_3 = b_3 = 16$, and $k = 20$. Hence it is efficient to allocate both items to bidder 1; from 7-9, the set of CE prices is given by:

$$56 \leq p_a + p_b \leq 60 \quad (14)$$

$$36 \leq p_a \leq 40 \quad (15)$$

$$16 \leq p_b \leq 40 \quad (16)$$

Consider the bidding dynamics as illustrated in table 1. All three bidders will initially demand package AB only, and therefore the prices will rise on both items simultaneously. At $p_a = p_b = 20$, bidder 2 switches his demand from AB to A: $S_2(AB; p) = S_2(A; p) = 16$. However, bidders 1 and 3 keep demanding AB only, and hence the prices rise on both items until they reach $p_a = p_b = 26$. At this point bidder 3 reports $\emptyset \in D_3(p)$, given $S_3(AB; p) = S_3(\emptyset; p) = 0$, and the price of B stops rising. Now bidder 1 demands AB, and bidder 2 demands A, hence the price of A keeps rising until the prices reach the level of $p_a = 34$, $p_b = 26$. At this point $S_1(AB; p) = S_1(\emptyset; p) = 0$, and bidder 1 reports $\emptyset \in D_1(p)$; bidder 2 still demands A, with $S_2(A; p) = 2$. Hence the auction stops with item

A allocated to bidder 2, and item B not allocated; the resulting prices, $(p_a, p_b) = (34, 26)$, are out of the equilibrium range: $p_a < 36$.

TABLE 1 AROUND HERE

This example suggests that in environments with complementarities, attaining efficient outcomes through simple auction mechanisms may be problematic even when market clearing prices exist. Hence in situations where achieving an efficient equilibrium outcome is critical, mechanism designers should turn to more complex combinatorial auctions that would allow for package bidding.

Appendix: Proof of Proposition 1

Before turning to the proof, it is useful to write out explicitly conditions under which a package $w \in W$ is demanded by a bidder $i \in N$. Let $p = (p_a, p_b)$ be a price vector. Applying definitions 1-4 and 5, we obtain:

- $AB \in D_i(p)$ if and only if:

$$a_i + b_i + k \geq p_a + p_b \tag{17}$$

$$b_i + k \geq p_b \tag{18}$$

$$a_i + k \geq p_a. \tag{19}$$

- $A \in D_i(p)$ if and only if:

$$a_i \geq p_a \tag{20}$$

$$b_i + k \leq p_b. \tag{21}$$

- $B \in D_i(p)$ if and only if:

$$b_i \geq p_b \tag{22}$$

$$a_i + k \leq p_a. \tag{23}$$

- $\emptyset \in D_i(p)$ if and only if:

$$a_i + b_i + k \leq p_a + p_b \tag{24}$$

$$a_i \leq p_a \tag{25}$$

$$b_i \leq p_b. \tag{26}$$

The following efficiency conditions will be also useful:

- Efficiency condition 6 holds, i.e., it is efficient to allocate the package AB to bidder $i \in N$, if and only if:

$$\begin{aligned} a_i + b_i &\geq a_j + b_j \quad \text{for all } j \in N & (27)\\ a_i + b_i + k &\geq a_j + b_l \quad \text{for all } j, l \neq i & (28)\\ b_i + k &\geq b_j \quad \text{for all } j \neq i & (29)\\ a_i + k &\geq a_j \quad \text{for all } j \neq i. & (30) \end{aligned}$$

- Efficiency condition 10 holds, i.e., it is efficient to allocate item A to bidder $i \in N$, and item B to bidder $j \in N$, $i \neq j$, if and only if:

$$\begin{aligned} a_i &\geq a_l \quad \text{for all } l \in N & (31)\\ b_j &\geq b_l \quad \text{for all } l \in N & (32)\\ b_j &\geq b_i + k & (33)\\ a_i &\geq a_j + k. & (34)\\ a_i + b_j &\geq \max_{l \neq i \neq j}(a_l + b_l) + k. & (35) \end{aligned}$$

Proof of proposition 1 The sets of CE prices are derived by solving for the no excess demand equilibrium conditions. Let (μ, p) be a CE price and allocation pair. Suppose under allocation μ each bidder $i \in N$ is assigned a package $w_i \in W$, so that $\cup_i w_i = \{A, B\}$, $w_i \cap w_j = \emptyset$ for all $i, j \in N$, $i \neq j$. The no excess demand conditions are:

$$S_i(w_i; p) \geq S_i(v; p) \quad \text{for any } v \in W. \tag{36}$$

There may be only two types of equilibrium allocations: either both items in $\{A, B\}$ are given to one of the bidders, or the items are split between the bidders. Consider equilibrium conditions for each of the two cases in turn.

CASE 1: Suppose that, in equilibrium, the package AB is assigned to bidder $i \in N$. The no excess demand conditions are conditions 17-19 for bidder i, and conditions 24-26 for all other bidders $j \neq i$. Combining the inequalities, we obtain the characterization of the set of CE prices as given in 7-9. Note that a price vector satisfying the ineqalities 7-9 exists if and only if conditions 27-30 hold: Obviously, if conditions 27-30 are satisfied, we can find prices (p_a, p_b) that satisfy 7-9. Conversely, suppose there exists a price vector (p_a, p_b) satisfying 7-9. Then 7 implies 27, 8 implies 30, 9 implies 29; finally, adding 8 and 9, we obtain $\max_{j \neq i} a_j + \max_{l \neq i} b_l \leq p_a + p_b$, which, together with 7, implies 28. Hence we obtain that a set of CE prices supporting the allocation of the package AB to bidder i is non-empty if and only if such allocation is efficient.

Case 2: Now suppose that, in equilibrium, item A is assigned to bidder i, and item B is assigned to bidder j, for some $i, j \in N$, $j \neq i$. Hence $A \in D_i(p)$, $B \in D_j(p)$, and $\emptyset \in D_l(p)$ for all $l \neq i \neq j$; that is, inequalities 20-21 hold for i, inequalities 22-23 hold for j, and inequalities 24-26 hold for all other bidders $l \neq i \neq j$. Combining these inequalities, we obtain the characterization of the set of equilibrium prices as given by 11-13. As in the previous case, it is straightforward to show that a price vector satisfying ineqalities 11-13 exists if and only if efficiency conditions 31-35 hold. $\square$

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p_a	p_b	$\max S_1(p)$	$D_1(p)$	$\max S_2(p)$	$D_2(p)$	$\max S_3(p)$	$D_3(p)$	$O(p)$
0	0	60	AB	56	AB	52	AB	A,B
...								
20	20	20	AB	16	A,AB	12	AB	A,B
21	21	18	AB	15	A	10	AB	A,B
...								
26	26	8	AB	10	A	0	AB,$\emptyset$	A
27	26	7	AB	9	A	0	$\emptyset$	A
...								
34	26	0	AB,$\emptyset$	2	A	0	$\emptyset$	$\emptyset$

Table 1: An example of failure of a simple progressive auction to reach a competitive equilibrium outcome.