1. Consider a two-period overlapping generations model without production. At each time period $t = 0, 1, 2, ...$ a new generation of measure $N_t$ is born. Population grows at a constant rate $n = N_{t+1}/N_t - 1 \geq 0$. The endowment when young and when old are $\{\omega_1 > 0, \omega_2 > 0\}$ with $\omega_1 > \omega_2$. The preference and endowment structures of successive generations are identical. The preference of the representative member of generation $t$ are given by
\[
\frac{(c_t^t)^{1-\gamma}}{1-\gamma} + \frac{(c_{t+1}^t)^{1-\gamma}}{1-\gamma}, \gamma > 0
\]
There is an initial old generation with endowment $\omega_2 > 0$ and consume $c_0^1$. (30 points)

(a) Define a Pareto Optimal allocation for this economy.

**Answer:** See lecture notes page 135.

(b) Define an Arrow-Debreu competitive equilibrium.

**Answer:** See lecture notes page 136.

(c) Obtain the offer curve, and characterize the equilibrium defined in part (b). Is it unique?

**Answer:** OC is the solution to the problem
\[
\max_{\{c_t^t, c_{t+1}^t\}} \frac{(c_t^t)^{1-\gamma}}{1-\gamma} + \frac{(c_{t+1}^t)^{1-\gamma}}{1-\gamma}
\]
\[\text{s.t.} \quad c_t^t + \frac{p_{t+1}}{p_t} c_{t+1}^t \leq \omega_1 + \frac{p_{t+1}}{p_t} \omega_2\]

The solution is
\[c_{t+1}^t = \frac{\omega_1 + \frac{p_{t+1}}{p_t} \omega_2}{(\frac{p_{t+1}}{p_t})^{1/\gamma} + \frac{p_{t+1}}{p_t}}\]
\[c_t^t = \frac{(\frac{p_{t+1}}{p_t})^{1/\gamma} (\omega_1 + \frac{p_{t+1}}{p_t} \omega_2)}{(\frac{p_{t+1}}{p_t})^{1/\gamma} + \frac{p_{t+1}}{p_t}}\]

The unique equilibrium is the autarky one
\[c_1^0 = \omega_2, c_t^t = \omega_1, c_{t+1}^t = \omega_2, \forall t \geq 1\]
\[\frac{p_{t+1}}{p_t} = \left(\frac{\omega_1}{\omega_2}\right)^{\gamma}\]
(d) Now assume that \( \gamma = 1 \). Assume that there is a pay-as-you-go (PAYG) social security system in the economy. The young pay lump-sum taxes in the amount \( \tau \) each period, and when they get old, they receive benefits in the amount \( b \). Assume that the system is self-financing each period. Compute the competitive equilibrium with social security.

**Answer**: the problem is

\[
\max \ln c_t^t + \ln c_{t+1}^t \\
\text{s.t.} \\
c_t^t + \frac{p_{t+1}}{p_t} c_{t+1}^t \leq \omega_1 - \tau + \frac{p_{t+1}}{p_t} (\omega_2 + b) \\
b = \tau (1 + n)
\]

The unique solution is

\[
c_0^t = \omega_2 + \tau (1 + n), c^t = \omega_1 - \tau, c_{t+1}^t = \omega_2 + \tau (1 + n), \forall t \geq 1
\]

\[
\frac{p_{t+1}}{p_t} = \frac{\omega_1 - \tau}{\omega_2 + \tau (1 + n)}.
\]

(e) Compute the optimal level of the PAYG social security system when \( n = 0 \); that is find the optimal tax rate that takes the economy to the Golden Rule.

**Answer**: The value function is

\[
V(\tau) = \ln(\omega_1 - \tau) + \ln(\omega_2 + \tau (1 + n))
\]

we have

\[
V'(\tau) = \frac{(1 + n)}{\omega_2 + \tau (1 + n)} - \frac{1}{\omega_1 - \tau}
\]

Notice that \( V'(0) > 0 \) iff \( n > \frac{\omega_2}{\omega_1} - 1 \). The optimal \( \tau \) is the value of \( \tau \) to make \( V'(\tau) = 0 \), i.e., \( \tau = \frac{\omega_1 (1 + n) - \omega_2}{2 (1 + n)} \). When \( n = 0 \), we have \( \tau^* = \frac{\omega_1 - \omega_2}{2} \).

2. Consider the economy in question 1 with \( n = 0 \). But now the initial old are also endowed with \( m \) units of unbacked fiat money. The stock of currency is constant over time. (30 points)

(a) Find a saving function of a young agent, i.e., find savings as a function of the interest rate. Characterize the saving function.

**Answer**: We focus on the case \( 0 < \gamma \leq 1 \). The case \( \gamma > 1 \) exhibits more complicated dynamics due to a non monotonic saving function.

The saving function of a young agent is:

\[
s(R) = \arg \max_s \frac{(w_1 - s)^{1-\gamma}}{1 - \gamma} + \frac{(w_2 + Rs)^{1-\gamma}}{1 - \gamma}
\]

The first order necessary and sufficient condition of this program is:

\[
(w_1 - s)^{-\gamma} = R (w_2 + Rs)^{-\gamma}
\]
which gives:

\[ s(R) = \frac{w_1 - w_2 R^{-\frac{1}{\gamma}}}{1 + R^{1 - \frac{1}{\gamma}}} \]

Note that the derivative of this function with respect to \( R \) is:

\[
\frac{R^{-\frac{1}{\gamma}}}{(1 + R^{1 - \frac{1}{\gamma}})^2} \left[ \left( \frac{1}{\gamma} - 1 \right) w_1 + w_2 R^{-\frac{1}{\gamma}} + \frac{w_2}{\gamma} R^{-1} \right]
\]

which is strictly positive for all \( 0 < \gamma \leq 1 \). This proves that the saving function is increasing, a fact that is going to be crucial to characterize unambiguously the dynamic of the system.

(b) Define an Arrow-Debreu competitive equilibrium with fiat money.

Answer; the problem faced by the young of generation \( t \) is

\[
\max_{c_t^t, c_{t+1}^t, M_t} \ u(c_t^t) + u(c_{t+1}^t)
\]

s.t.

\[
c_t^t + \frac{M_t}{p_t} \leq w_1
\]

\[
c_{t+1}^t \leq w_2 (t + 1) + \frac{M_t}{p_{t+1}}
\]

\[
[c_t^t, c_{t+1}^t, M_t] \geq 0
\]

The problem of the initial old is simply

\[
\max_{c_0^1} \ c_0^1
\]

s.t.

\[
0 \leq c_0^1 \leq \frac{M}{p_1}
\]

we can now define an equilibrium:

Definition. An equilibrium with valued fiat currency is a consumption decision for the initial old \( c_0^1 \), consumption decisions for the young born at time \( t \geq 1 \), \( \{c_t^t, c_{t+1}^t\}_{t=1}^{\infty} \), money demand \( \{M_t\}_{t=1}^{\infty} \), and a positive price sequence \( \{p_t\}_{t=1}^{\infty} \), such that the two following conditions are satisfied:

(i) Optimality: given \( p_t, c_0^1 \) solves the initial old problem. Given \( \{p_t, p_{t+1}\}, \{c_t^t, c_{t+1}^t\} \) solves agent of generation \( t \) problem, for all \( t \geq 1 \).

(ii) Feasibility: the market for good and the market for money clear for all \( t \geq 1 \):

\[
\begin{align*}
c_t^t + c_{t-1}^t &= w_1 + w_2 \\
M_1 &= M_2 \text{ and } M_{t+1} = M_t
\end{align*}
\]

(c) Define an stationary Arrow-Debreu competitive equilibrium with fiat money.
In a stationary equilibrium the rate of return on currency is constant. More formally a stationary equilibrium with valued fiat currency is an equilibrium with valued fiat currency for which there is $R > 0$ such that

$$\frac{p_t}{p_{t+1}} = R$$

we simplify the list of conditions describing an equilibrium. First, Walras law allows to eliminate one market clearing condition at each $t \geq 1$. We eliminate the market clearing condition for good and keep the one for money. Second, going back to the agent problem, we note that it can be reduced to a saving problem, as described in question (a), with $R_t = \frac{p_t}{p_{t+1}}$. The optimal money holding are

$$\frac{M_t}{p_t} = s (R_t) \text{ if } s (R_t) > 0 \text{ and } 0 \text{ otherwise.}$$

The optimal consumption stream is

$$c_t = w_1 - \frac{M_t}{p_t} \text{ and } c_{t+1} = w_2 + \frac{M_t}{p_{t+1}}.$$  These remarks allow us to define an equilibrium as a positive price sequence $\{p_t\}_{t=1}^{+\infty}$ such that, for all $t$:

$$\frac{M}{p_t} = s \left( \frac{p_t}{p_{t+1}} \right)$$

Looking for a stationary equilibrium, we write $\frac{p_t}{p_{t+1}} = R$ so that

$$\frac{M}{p_t} = s (R) = \text{constant}$$

Therefore the price level is constant and $p_1$ solves

$$\frac{M}{p_1} = s (1) = \frac{w_1 - w_2}{2} \Rightarrow p_1 = \frac{2M}{w_1 - w_2}$$

Note here how the condition $w_2 < w_1$ is necessary to ensure existence of a stationary equilibrium with valued fiat currency.

(d) Describe how many equilibria with valued fiat currency there are. (You are not being asked to compute them.)

Answer: In order to describe the set of equilibria, we rewrite the equilibrium condition using the auxiliary variable $R_t = \frac{p_t}{p_{t+1}}$:

$$\frac{M}{p_1} = s (R_1)$$

$$s (R_{t+1}) = R_t s (R_t), t \geq 1$$

$$R_t = \frac{p_t}{p_{t+1}}$$

The first equation says that the saving of the initial young must equal the supply of real money balance. The second equation is found by expressing that nominal money balances stay constant over time. An equilibrium is constructed the following way:

(i) Choose a positive sequence $\{R_t\}_{t=1}^{+\infty}$ solving the difference equation $s(R_{t+1}) = R_t s (R_t)$. We will see that there are infinitely many.
(ii) Once $R_1$ is chosen, find a solution $p_1 < +\infty$ to the first equation.

(iii) Construct the sequence of price using $p_{t+1} = \frac{p_t}{R_t}$.

The assumptions made on the utility function put some structure on the set of positive sequence solving $s(R_{t+1}) = R_t s(R_t)$. We are going to show in particular that all non-stationary equilibria are associated with decreasing sequences of rate of return on currency, converging towards $R^* = \left[ \frac{w_2}{w_1} \right]^\gamma$.

Let’s first gain some intuition from a simple graphical analysis. We plot the function $R_t s(R_t)$ and $s(R_{t+1})$. A candidate interest rate sequence is constructed as follows. Choose $R_0$ on the x-axis. Go on the first curve to compute $R_0 s(R_0)$. Then go non the second curve to "solve" $s(R_1) = R_0 s(R_0)$. Go back on the x-axis. Iterate. It is clear from this exercise that $R=1$ is an unstable stationary point. Also, all sequences starting on the left of $R=1$ and on the right of $R^* = \left[ \frac{w_2}{w_1} \right]^\gamma$ converges to $R^*$. It is not possible to construct a sequence starting far on the right of $R=1$.

Lastly, sequence starting on the left on $R^*$ are not admissible as they are associated with negative savings.

Let’s make the previous arguments more formal. We go in several steps.

(1) there is no equilibrium such that $R_1 \leq \left[ \frac{w_2}{w_1} \right]^\gamma$. This follows from the fact that for all such $R_1$, $s(R_1)$ is non-positive and that equilibrium imposes that $\frac{M}{p_1} = s(R_1) > 0$.

(2) there is no equilibrium such that $R_1 > 1$.

Assume that there is one. First note that $s(R_{t+1}) - s(R_t) = (R_t - 1) s(R_t)$. Therefore, $R_t > 1$, then $s(R_{t+1}) > s(R_t)$. And since $s(.)$ is increasing, this implies that $R_{t+1} > R_t$. Thus, if $R_0 > 1$, the sequence $\{R_t\}_{t=1}^\infty$ is increasing. As any increasing sequence, it has a limit, finite or infinite. If it has a finite limit $R^*$, it must satisfy, by continuity of $s(.)$

$$s(R^*) = R^* s(R^*)$$
It is easy to see that this equation has only two solutions $R^* = 1$ and $R^* = \left[ \frac{\omega_2}{\omega_1} \right]^\gamma$, both less than 1. Since $R_t > R_0 > 1$, the sequence $R_t$ cannot have such a finite limit. Therefore $R_t$ goes to infinity. This also implies that $R_t s(R_t) > R_t s(1)$ goes to infinity. But $s(R_{t+1})$ is bounded above by $w_1$ since an agent cannot save more than her young period endowment. This means that, for $t$ large enough, this sequence violates the equality

$$s(R_{t+1}) = R_t s(R_t)$$

(3) All equilibria are such that $\left[ \frac{\omega_2}{\omega_1} \right]^\gamma < R_1 \leq 1$.

We already know that $R_1 = 1$ is associated with the stationary equilibrium. For $\left[ \frac{\omega_2}{\omega_1} \right]^\gamma < R_1 < 1$, write as before that $s(R_{t+1}) = s(R_t)$. Since $s(.)$ is increasing, this implies that $R_{t+1} < R_t$, Furthermore, $R_t > \left[ \frac{\omega_2}{\omega_1} \right]^\gamma$ implies that $s(R_t) > 0$ so that $s(R_{t+1}) > 0$ and $R_{t+1} > \left[ \frac{\omega_2}{\omega_1} \right]^\gamma$. Thus the sequence $\{R_t\}_{t=1}^{\infty}$ is decreasing and bounded below. It thus converges towards a finite limit $R^*$ such that

$$s(R^*) = R^* s(R^*)$$

and

$$\left[ \frac{\omega_2}{\omega_1} \right]^\gamma \leq R^* < 1$$

We already know the solutions of this equation. It must be that

$$R^* = \left[ \frac{\omega_2}{\omega_1} \right]^\gamma$$

3. Consider an economy with overlapping generations of a constant population of an even number $N$ of two-period-lived agents. New young agents are born at each date $t \geq 1$. Half of the young agents are endowed with $\omega_1 > 0$ when young and 0 when old. The other half are endowed with 0 when young and $\omega_2 > 0$ when old. Assume that $\omega_1 > \omega_2$. Preferences of all young agents are as in question 1, with $\gamma = 1$. Half of the $N$ initial old are endowed with $\omega_2$ units of the consumption good and half are endowed with nothing. Each old person at $t = 1$ is endowed with $m$ units of unbacked fiat currency. No other generation is endowed with fiat currency. The stock of fiat currency is fixed over time. (40 points)

(a) Find the saving function of each of the two types of young person for $t \geq 1$

Answer: Let $(w^y, w^0)$ be the endowment of the agent. The saving function of a young agent is

$$s(R) = \arg \max_s \log (w^y - s) + \log (w^0 + Rs)$$

$$\frac{1}{w^y - s} = \frac{R}{w^0 + Rs}$$
Which gives:

\[ s(R) = \frac{w^0}{2} - \frac{w^y}{2R} \]

Thus, a consumer of type 1 has saving function

\[ s_1(R) = \frac{w_1}{2} \]

And a consumer of type 2 has saving function

\[ s_2(R) = -\frac{w_2}{2R} \]

(b) Define an equilibrium without valued fiat currency. Compute all such equilibria. Answer: The problem faced by a young of generation of type \( h = 1, 2 \) is

\[
\max_{c^h_t, c^h_{t+1}, b_t} u(c^h_t) + u(c^h_{t+1}) \]

s.t.

\[
c^h_t + b^h_t \leq w^h_t
\]
\[
c^h_{t+1} \leq w^h_{t+1} + R_t b^h_t
\]
\[
[c^h_t, c^h_{t+1}] \geq 0
\]

The problem of the initial old is simply

\[
\max_{c^o_1} \]

s.t.

\[
0 < c^o_1 < w^o_0
\]

We can now define an equilibrium:

An equilibrium without value fiat currency is a consumption decision for the initial old \( c^o_1, h = 1, 2 \), consumption decisions for the young born at time \( t > 1 \), \( (c^h_t, c^h_{t+1}) \) for all \( t = 1, \infty \), \( h = 1, 2 \), lending/borrowing decisions \( \{b^h_t\}_{t=1}^{\infty} \), and a positive return sequence \( \{R_t\}_{t=1}^{\infty} \), such that the two following conditions are satisfied:

(i) Optimality: Given \( p_1, c^o_1 \) solves the initial old problem. \( h = 1, 2 \) given \( R_t, (c^h_t, c^h_{t+1}) \) solves agent of generation \( t \) problem for all \( t \geq 1, h = 1, 2 \).

(ii) Reasibility the market for good, the market for private lending clear for all \( t \) clear for all \( t > 1 \):

\[
\frac{N}{2} \sum_{h=1,2} c^h_t + \frac{N}{2} \sum_{h=1,2} c^h_{t-1} = \frac{N}{2} w_1 + \frac{N}{2} w_2
\]
\[
\frac{N}{2} \sum_{h=1,2} b^h_t = 0
\]

To characterize such an equilibrium we first notice that walrus law allows to restrict attention to the market for private lending. Using the saving function derived above, market clearing can be written:

\[ s_1(R_t) + s_2(R_t) = 0 \]
And the only solution of this equation is \( R_t = \frac{w_2}{w_1} \). Thus there is only one equilibrium without valued fiat currency, it is stationary and the interest rate is lower than the inverse of the discount factor \( R_t = \frac{w_2}{w_1} < 1 \).

(c) Define an equilibrium with valued fiat currency.

The problem faced by a young of generation \( t \) of type \( h=1, 2 \) is

\[
\max_{c_t^h, c_{t+1}^h, b_t, M_t^h} u(c_{t+1}^h) + u(c_t^h) \\
\text{st} \\
c_t^h + b_t^h + \frac{M_t^h}{p_t} \leq w_t^h \\
\frac{c_{t+1}^h}{M_t^h} \leq w_{t+1}^h + R_t b_t^h + \frac{M_t^h}{p_{t+1}} \\
[c^h_t, c^h_{t+1}, M^h_t] \geq 0
\]

An equilibrium without value fiat currency is a consumption decision for the initial old \( c_{h=1, 2}^h, h = 1, 2 \), consumption decisions for the young born at time \( t>1 \), \( (c^h_t, c^h_{t+1})_{t=1}^{\infty}, h = 1, 2 \), lending/borrowing decisions \( \{b^h_t\}_{t=1}^{\infty} \), money holding decisions \( \{M^h_t\}_{t=1}^{\infty} \), a positive price sequence \( \{p_t\}_{t=1}^{\infty} \) and a positive return sequence \( \{R_t\}_{t=1}^{\infty} \), such that the two following conditions are satisfied:

(i) Optimality: Given \( p_1, c_{h=1}^h \) solves the initial old problem. \( h=1, 2 \), given \( R_t, \{p_t, p_{t+1}\} \) \( (c^h_t, c^h_{t+1}) \) solves agent of generation \( t \) problem for all \( t \geq 1, h = 1, 2 \).

(ii) Reasibility the market for good, the market for private lending and the market for lending clear for all \( t > 1 \):

\[
\frac{N}{2} \sum_{h=1, 2} c^h_t + \frac{N}{2} \sum_{h=1, 2} c^{h-1}_t = \frac{N}{2} w_1 + \frac{N}{2} w_2 \\
\frac{N}{2} \sum_{h=1, 2} b^h_t = 0 \\
\frac{N}{2} \sum_{h=1, 2} M^h_t = NM
\]

(d) Compute all the (nonstochastic) equilibria with valued fiat currency.

Walras law allows to restrict attention to the last two market clearing conditions. Note also that, since agents can borrow at rate \( R_t \), no arbitrage imposes that the return on money, \( \frac{p_t}{p_{t+1}} \), is lower than \( R_t \), furthermore, in an equilibrium with valued fiat currency, agents are holding positive money balance. Thus \( \frac{p_t}{p_{t+1}} = R_t \). Given
this equality, savers agents of type 1) are indifferent between holding money and lending:
\[ b_1^t + \frac{M_1^t}{p_t} = s_1 (R_t) \]

This implies that the last two market clearing conditions can be replaced by their sum:
\[ \frac{N}{2} s_1 (R_t) + \frac{N}{2} s_2 (R_t) = \frac{NM}{p_t} \]

This equation says that all the money demand must equal the aggregate saving of type 1 agents minus the borrowing of the type 2 agents. Using the expressions derived in part (a) and \( R_t = \frac{w_2}{w_1} \), this equations can be reduced to the following linear difference equation in \( p_t \)
\[ p_{t+1} = -\frac{4M}{w_2} + \frac{w_1}{w_2} p_t \]

we first solve for a stationary point
\[ p^* = -\frac{4M}{w_2} + \frac{w_1}{w_2} p^* \]

We find \( p^* = \frac{4M}{w_1 - w_2} \). we subtract the equation defining \( p^* \) to the difference equation for \( p_t \)
\[ p_{t+1} - p^* = \frac{w_1}{w_2} (p_t - p^*) \]

we then iterate on this equation to find
\[ p_{t+1} = \left[ \frac{w_1}{w_2} \right]^t (p_1 - p^*) \]

Note that \( \frac{w_1}{w_2} > 1 \) and \( p_t > 0 \) imposes that \( p_1 \geq p^* \) otherwise the price level would be negative for \( t \) large enough. It is apparent from this formula that there are a continuum of equilibria with valued fiat currency indexed by \( p_1 \geq p^* \).

(e) Argue that there is a unique stationary equilibrium with valued fiat currency.

Answer: stationary equilibria are such that the rate of return on currency is constant. We can compute this rate of return explicitly using the solution we derived in part (d):
\[ \frac{p_t}{p_{t+1}} = R_t = \frac{w_1}{w_2} + \left( 1 - \frac{w_1}{w_2} \right) \frac{p^*}{p^* + \left[ \frac{w_1}{w_2} \right]^t (p_1 - p^*)} \]

This is constant if and only if \( \left[ \frac{w_1}{w_2} \right]^t (p_1 - p^*) + p^* \) is constant, that is if and only if \( p_1 = p^* \)

there is an unique stationary equilibria, associated with the lowest price level \( p_1 \) and rate of return on currency equal to the discount factor, \( R = 1 \).

(f) How are the various equilibria with valued fiat currency ranked by the Pareto criterion?
Answer: Along an equilibrium path, the utility of an agent of type 1 is:

\[ \log \left[ \frac{w_1}{2} \right] + \log \left[ \frac{R_t w_1}{2} \right] \]

and is increasing in \( R_t \). Similarly the utility of an agent of type 2 is

\[ \log \left[ \frac{w_2}{2R_t} \right] + \log \left[ \frac{w_2}{2} \right] \]

And is decreasing in \( R_t \). Now, from the formula that we derived in part(e), it is clear that \( R_t \) is a decreasing function of \( p_1 \). Note that this property is quite strong: higher \( p_1 \) will correspond to uniformly lower rate of return on currency, formally,

\[ p'_1 > p_1 \Rightarrow \forall t, R_t (p'_1) < R_t (p_1) \]

Therefore, type 1 agents will be worse off in a higher \( p_1 \) economy— their saving will earn lower interest. Conversely type 2 agents will be better off - they will borrow at a lower rate.

Considering the initial old, there are clearly worse off for higher \( p_1 \) which correspond to lower real money balance.