ECON607 Fall 2010
University of Hawaii
Professor Hui He
TA: Xiaodong Sun
Assignment 4 Suggested Answer

The due date for this assignment is Tuesday, November 30. (Total 100 points)

1. (*Productivity and Employment*) Consider a neoclassical growth model with leisure in the utility function

$$
\begin{aligned}
& \max \sum_{t=0}^{\infty} \beta^t u(c_t, 1-n_t) \\
& s.t. \\
c_t + i_t + g_t & \leq z f(k_t, n_t) \\
k_{t+1} & \leq (1-\delta)k_t + i_t
\end{aligned}
$$

where g_t is the government's expenditure per capita. The functions u and f are assumed to be strictly increasing in each argument, concave, and twice differentiable. In addition, we have $\lim_{k_t\to\infty} \frac{\partial f(k_t,n_t)}{\partial k_t} \to 0$. (15 points)

(a) Describe the steady state of this economy. If necessary, make additional assumptions to guarantee that it exists and is unique. If you make additional assumptions, go as far as you can giving an economic interpretation of them.

Answer: Denote $h(x) = f(x,1)$ and $g_n = \frac{u_n(an-b,1-n)}{u_c(an-b,1-n)}$ with $a > b$ and $a > 0$. We make the following assumptions:

$$
\begin{aligned}
f(\lambda k, \lambda n) &= \lambda f(k,n) \text{ (CRS)} \\
\lim_{x\to\infty} h_x(x) &= 0 \text{ (Inada condition)} \\
\lim_{x\to 0} h_x(x) &= \infty \text{ (Inada condition)} \\
\frac{\partial^2 u}{\partial c \partial n} &> 0 \\
\lim_{n\to 1} g(n) &= \infty \text{ (Inada condition)} \\
\lim_{n\to 0} g(n) &= 0 \text{ (Inada condition)}
\end{aligned}
$$

When we solve the FOCs, we should end up with following intertemporal Euler equation (inter EE) and intratemporal Euler equation (intra EE):

$$
\begin{aligned}
u_c(c_t, 1-n_t) &= \beta u_c(c_{t+1}, 1-n_{t+1})[z f_k(k_{t+1}, n_{t+1}) + 1 - \delta] \\
\frac{u_n(c_t, 1-n_t)}{u_c(c_t, 1-n_t)} &= z f_n(k_n, n_t).
\end{aligned}
$$

In steady state, we have inter EE as

$$
\begin{aligned}
1 &= \beta[z f_k(k,n) + 1 - \delta] \\
&= \beta[z n f_k(\frac{k}{n}, 1) + 1 - \delta] \\
&= \beta[z h_x(\frac{k}{n}) + 1 - \delta] \qquad (1)
\end{aligned}
$$

where $x = \frac{k}{n}$, and intra EE as

$$\begin{aligned}
\frac{u_n(c,1-n)}{u_c(c,1-n)} &= zf_n(k,n) = \frac{zf(k,n) - zkf_k(k,n)}{n} \\
&= zf(\frac{k}{n},1) - z\frac{k}{n}f_k(k,n) \\
&= zh(\frac{k}{n}) + z(-\frac{k}{n^2})nf_k(\frac{k}{n},1) \\
&= zh(\frac{k}{n}) - z\frac{k}{n}h_x(\frac{k}{n}). \qquad (2)
\end{aligned}$$

Recall the resource constraint is

$$c_t + i_t + g_t \leq zf(k_t, n_t)$$

In SS, we have

$$c + \delta k + g = zf(k,n) = nzh(\frac{k}{n})$$

Therefore, in SS we have

$$c = n(zh(x) - \delta x) - g. \qquad (3)$$

Obviously there exists a unique solution $x^* = \frac{k^*}{n^*}$ that solves equation (1). Second, substituting (3) into (2), we have

$$\frac{u_n(n(zh(x) - \delta x) - g, 1-n)}{u_c(n(zh(x) - \delta x) - g, 1-n)} = zh(x) - zxh_x(x). \qquad (4)$$

Since utility function is strictly increasing in both c and n, the RHS of equation (4) is positive and when $x = x^*$ in SS, it is a constant. Inada condition of utility function w.r.t. consumption c guarantees $c > 0$. Government expenditure $g \geq 0$. Therefore, from (3), we know $zh(x) - \delta x > 0$. Now let $a \equiv zh(x^*) - \delta x^*$ and $b \equiv g$, by assumption we made above, the LHS is an increasing function in n and goes to zero when n goes to $\frac{b}{a}$, and goes to infinity when n goes to 1. This establishes existence and uniqueness of n^*.

(b) Assume that $u(c, 1-n) = \frac{[c^\mu(1-n)^{1-\mu}]^{1-\sigma}}{1-\sigma}$ and $f(k,n) = k^\alpha n^{1-\alpha}$. What is the effect of changes in the technology (say increases in z) on employment and output per capita?

Answer: With the specific utility function and production function, in SS, we have following FOCs:

$$\begin{aligned}
1 &= \beta[z\alpha(\frac{k}{n})^{\alpha-1} + 1 - \delta] \\
\frac{1-\mu}{\mu}\frac{c}{1-n} &= z(1-\alpha)(\frac{k}{n})^\alpha \\
c + \delta k + g &= zk^\alpha n^{1-\alpha}
\end{aligned}$$

Following the steps as in part (a), we can combine the three equations into just one equation with unknown n, and then solve n as follows

$$\begin{aligned} n &= \frac{(1-\alpha)A + \frac{1-\mu}{\mu} g z^{-1/(1-\alpha)}}{(1-\alpha)A + \frac{1-\mu}{\mu}(A - \delta A^{1/\alpha})} \\ \text{where } A &= \alpha^{\frac{\alpha}{1-\alpha}} (\frac{1}{\beta} - 1 + \delta)^{-\frac{\alpha}{1-\alpha}}. \end{aligned}$$

We then know $\frac{\partial n}{\partial z} < 0$ if $g > 0$, and $\frac{\partial n}{\partial z} = 0$ if $g = 0$. In SS, the output is determined by

$$y = z^{1/(1-\alpha)} A n$$

Therefore, we have $\frac{\partial y}{\partial z} > 0$.

(c) Consider next an increase in g. Are there conditions under which an increase in g will result in an increase in the steady-state $\frac{k}{n}$ ratio? How about an increase in the steady-state level of output per capita? Go as far as you can giving an economic interpretation of these conditions. (Try to do this for general $f(k, n)$ functions with the appropriate convexity assumptions, but if this proves too hard, use the Cobb-Douglas specification.)

Answer: For the Cobb-Douglas specification, the capital to labor ratio is a function of the parameters (β, δ, α) and the technology level z. It is independent of government spending. The partial derivative of output per capita w.r.t. the government spending is strictly positive:

$$\frac{\partial y}{\partial g} = z^{1/(1-\alpha)} A \frac{\partial n}{\partial g} = \frac{\frac{1-\mu}{\mu} A}{(1-\alpha)A + \frac{1-\mu}{\mu}(A - \delta A^{1/\alpha})} > 0.$$

2. (*H-P filter*) Choose the annual real GDP data you obtained in question 1, write a Matlab program to use the H-P filter to detrend the GDP time series and obtain the cyclical part of the GDP ($\{y_t^c\}_{t=1}^T$). Plot the original GDP series, the growth part and the cyclical part. Calculate the standard deviation of the GDP series, the growth part and the cyclical part respectively. (15 points)

Answer: Recall the H-P Filter problem is

$$\min_{\{y_t^g\}_{t=1}^T} \sum_{t=1}^{T} (y_t - y_t^g)^2$$

subject to

$$\sum_{t=2}^{T-1} [(y_{t+1}^g - y_t^g) - (y_t^g - y_{t-1}^g)]^2 \leq \mu.$$

In order to solve this problem, let's set up the Lagrangian (or loss function)

$$L = \left[\sum_{t=1}^{T} (y_t - y_t^g)^2 + \lambda \sum_{t=2}^{T-1} [(y_{t+1}^g - y_t^g) - (y_t^g - y_{t-1}^g)]^2 \right]$$

FOCs w.r.t. y_t^g for $t = 3, 4, ..., T-2$

$$\begin{aligned}\frac{\partial L}{\partial y_t^g} &= -2(y_t - y_t^g) + 2\lambda\left[(y_t^g - y_{t-1}^g) - (y_{t-1}^g - y_{t-2}^g)\right] \\ &\quad +(-2)2\lambda\left[(y_{t+1}^g - y_t^g) - (y_t^g - y_{t-1}^g)\right] + 2\lambda\left[(y_{t+2}^g - y_{t+1}^g) - (y_{t+1}^g - y_t^g)\right] \\ &= 0\end{aligned}$$

That is

$$y_t = \lambda y_{t-2}^g - 4\lambda y_{t-1}^g + (1+6\lambda)y_t^g + -4\lambda y_{t+1}^g + \lambda y_{t+2}^g.$$

For $t = 1$, we have

$$\frac{\partial L}{\partial y_1^g} = -2(y_1 - y_1^g) + 2\lambda\left[(y_3^g - y_2^g) - (y_2^g - y_1^g)\right] = 0$$

which implies

$$y_1 = (1+\lambda)y_1^g - 2\lambda y_2^g + \lambda y_3^g$$

Similarly, we have for $t = 2$

$$\begin{aligned}\frac{\partial L}{\partial y_2^g} &= -2(y_2 - y_2^g) + (-2)2\lambda\left[(y_3^g - y_2^g) - (y_2^g - y_1^g)\right] \\ &\quad +2\lambda\left[(y_4^g - y_3^g) - (y_3^g - y_2^g)\right] \\ &= 0\end{aligned}$$

which implies

$$y_2 = -2\lambda y_1^g + (1+5\lambda)y_2^g - 4\lambda y_3^g + \lambda y_4^g.$$

Do the same thing to $t = T-1$ and T, we have

$$y_{T-1} = -2\lambda y_T^g + (1+5\lambda)y_{T-1}^g - 4\lambda y_{T-2}^g + \lambda y_{T-3}^g$$

and

$$y_1 = (1+\lambda)y_T^g - 2\lambda y_{T-1}^g + \lambda y_{T-2}^g.$$

We also need to check the second-order condition (SOC)

$$\frac{\partial^2 L}{\partial y_t^{g2}} = 2 + 12\lambda > 0$$

since the multiplier $\lambda > 0$. Hence we indeed obtain a minimum.

We can rewrite the solution in terms of matrix language

$$Y = AY^g$$

where

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ y_t \\ . \\ . \\ y_{T-1} \\ y_T \end{bmatrix}$$

And

$$A = \begin{bmatrix} 1+\lambda & -2\lambda & \lambda & 0 & 0 & ... & 0 & 0 & 0 \\ -2\lambda & 1+5\lambda & -4\lambda & \lambda & 0 & ... & 0 & 0 & 0 \\ \lambda & -4\lambda & 1+6\lambda & -4\lambda & \lambda & 0 & ... & 0 & 0 \\ 0 & \lambda & -4\lambda & 1+6\lambda & -4\lambda & \lambda & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & 0 & \lambda & -4\lambda & 1+6\lambda & -4\lambda & \lambda \\ 0 & 0 & ... & ... & 0 & \lambda & -4\lambda & 1+5\lambda & -2\lambda \\ 0 & 0 & ... & ... & 0 & 0 & \lambda & -2\lambda & 1+\lambda \end{bmatrix}$$

$$Y^g = \begin{bmatrix} y_1^g \\ y_2^g \\ . \\ . \\ y_t^g \\ . \\ . \\ y_{T-1}^g \\ y_T^g \end{bmatrix}.$$

Thus, the Matlab code just involves a matrix inverse operation as following

$$Y^g = A^{-1} \cdot Y.$$

And the cyclical part is calculated as

$$Y^c = Y - Y^g.$$

Using real GDP data from the first quarter of 1960 to the 4th quarter of 1998 ($T = 154$) and choosing $\lambda = 1600$, we have the following graphs:

3. (*Replicating RBC exercise*) Download Dynare Version 3.065 from the URL

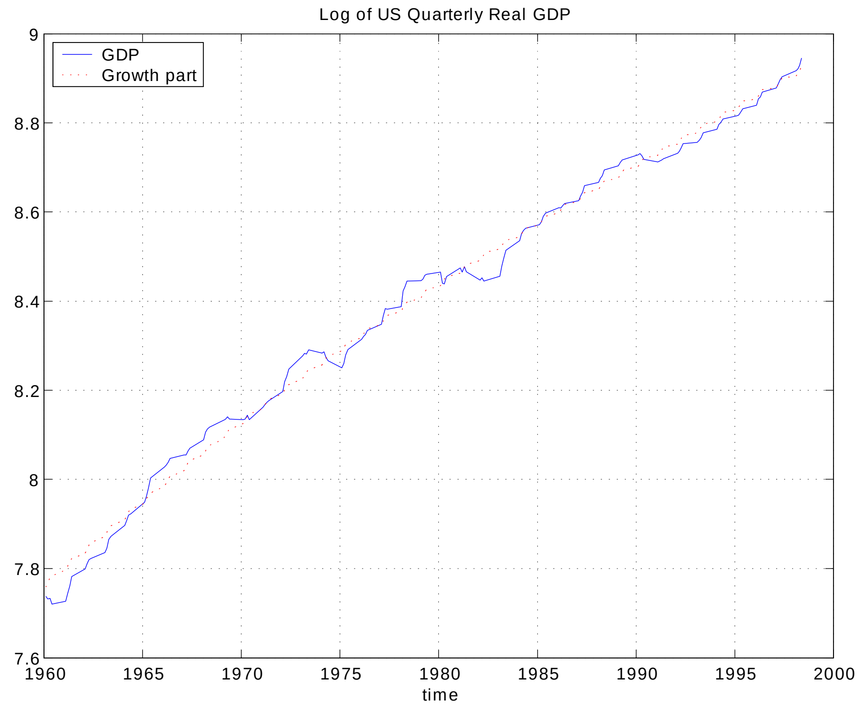
   http://www.cepremap.cnrs.fr/juillard/mambo/

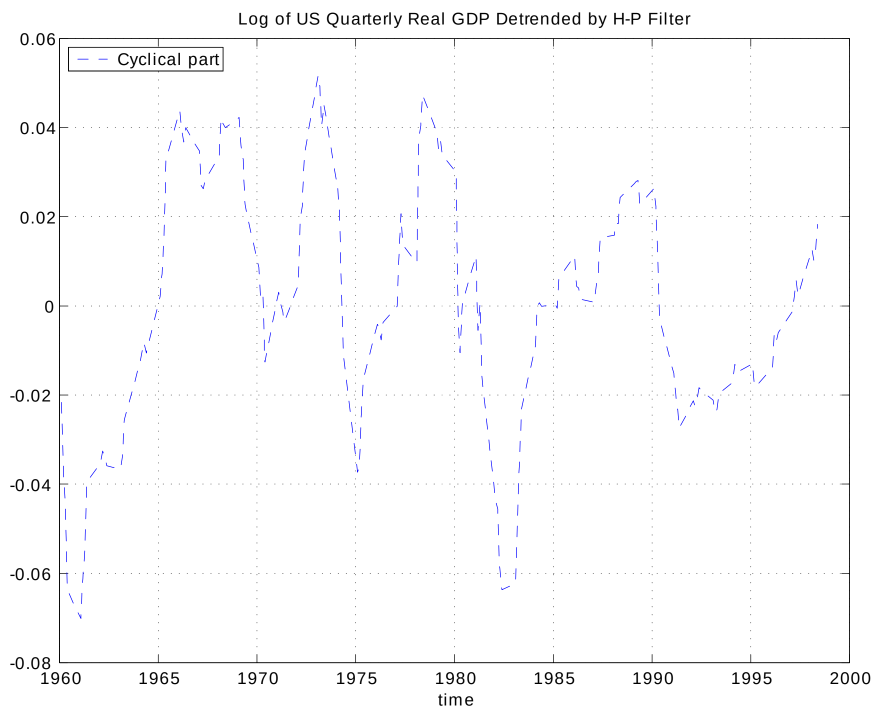
   Spending some time with Dynare and read the user's guide. Then use it to replicate the results shown in Table 8.6 on the lecture notes. Plot the impulse response of the shock to consumption c, capital stock k, working hours h, and output y. (15 points)

   Answer: Using Dynare to compute the standard RBC model as the one in Cooley and Prescott (1995) with H-P filter and linearlized decision rules, we will end up with Table 1 for the quarterly moments

   where here the capital stock data is for the nonresidential structures. Overall, we would like to say that we sort of replicate the main features of Table 1.1 in Cooley and Prescott (1995). The major discrepancy is that our computation generates much less volatile investment.

   Dynare also generates the following impulse response functions for us. Impulse response functions are the expected future path of the endogenous

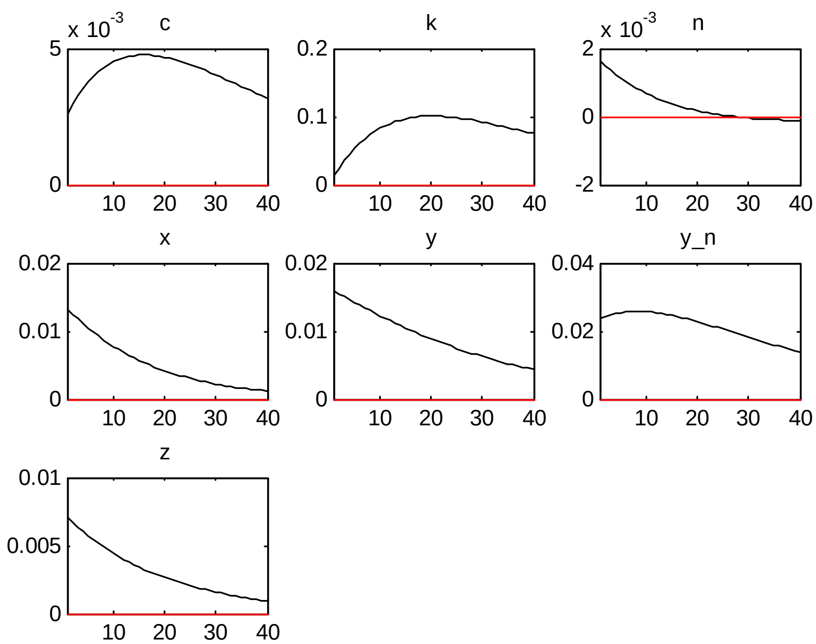

Log of US Quarterly Real GDP
GDP
Growth part
9
8.8
8.6
8.4
8.2
8
7.8
7.6
1960
1965
1970
1975
1980
1985
1990
1995
2000
time


Log of US Quarterly Real GDP Detrended by H-P Filter
Cyclical part
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
1960
1965
1970
1975
1980
1985
1990
1995
2000
time

Variable	SD (%)	ρ(variable, y)	Data: SD (%)	Data: ρ(variable, y)
y	1.278	1.0	1.72	1.0
c	0.410	0.886	0.86	0.77
i	2.735	0.994	8.24	0.91
n	0.602	0.979	1.69	0.92
y/n	0.670	0.985	0.73	0.34
k	0.374	0.379	0.40	-0.03

Table 1: Real Business Cycles: Model vs. Data

variables conditional on a shock in period 1 of one standard deviation (i.e., $\varepsilon_1 = 0.007$ in our case).



From this graph, we see clearly that the shock on TFP z gradually dies out following the initial shock on the innovation ε_1. In response to the positive shock, labor input initially jumps and then gradually decreases from its new level to the end it is even lower than the pre-shock level. Keep other things equal, increase in z will raise the real wage, the substitution effect jumps in at the first place, opportunity cost for enjoying leisure increases suddenly, people thus substitute leisure for working, n increases. But as the initial positive shock on TFP gradually dies out, the income effect gradually takes over, the labor supply thus decreases.

TFP shock will also affect the rental rate of capital, therefore, initially capital input also increases, which makes the output goes up. Consumption then catches up with the output. But as n gradually decreases, output y also decreases from its initial high level. Both c and k thus exhibit the hump shape.

4. (*Calibrating a RBC model with a labor indivisibility*) Consider a stochastic growth

model with preferences

$$E\sum_{t=0}^{\infty}\beta^t\{\log c_t+\alpha\log(1-n_t)\}$$

where $n_t\in\{0,\frac{1}{2}\}$. In other words, labor supply is subject to a indivisibility constraint. A person either does not work or work to a certain amount of time. The production function is

$$y_t=k_t^{\theta}n_t^{1-\theta}.$$

The resource constraint is given by

$$c_t+i_t=y_t.$$

The law of motion for the capital stock is

$$k_{t+1}=(1-\delta)k_t+i_t.$$

Calibrating this model economy (i.e., choose the parameter values for $\beta,\delta,\alpha,\theta$) so that its competitive equilibrium allocation matches the long-run observations in the US economy $c/y=0.85$, $k/y=3.00$, labor income share in GDP is 70%, and the fraction of working time is 80%. Due to the labor indivisibility, the model has to be convexified. Richard Rogerson (1988) shows that introducing a lottery technology can solve this issue. In particular, let π_t be the probability of working $n_t=\frac{1}{2}$, and $1-\pi_t$ be the probability of working $n_t=0$. Then the period utility function becomes

$$\log c_t+\alpha\pi_t\log(1-\frac{1}{2})+\alpha(1-\pi_t)\log(1-0).$$

The HHs now maximize utility function over the sequences of $\{c_t,i_t,\pi_t\}$. (15 points)

Answer: The HH's problem is

$$\begin{aligned}
&\max_{\{c_t,i_t,\pi_t\}}\sum_{t=0}^{\infty}\beta^t\{\log c_t-\alpha\pi_t\log 2\}\\
&s.t.\\
c_t+i_t &= r_tk_t+w_t[\pi_t\cdot\frac{1}{2}+(1-\pi_t)\cdot 0]\\
k_{t+1} &= (1-\delta)k_t+i_t
\end{aligned}$$

FOCs are

$$\begin{aligned}
c_t &: \beta^t/c_t=\lambda_t\\
k_{t+1} &: \lambda_t=\lambda_{t+1}(r_t+1-\delta)\\
\pi_t &: \beta^t\alpha\log 2=\lambda_t\frac{w_t}{2}
\end{aligned}$$

Combining FOCs, we obtain

$$\frac{c_{t+1}}{c_t}=\beta(r_t+1-\delta) \tag{5}$$

$$\frac{w_t}{2c_t}=\alpha\log 2 \tag{6}$$

The firm's problem is

$$\max(k_t^{\theta} n_t^{1-\theta} - r_t k_t - w_t n_t)$$

FOCs are

$$\begin{aligned} r_t &= \theta k_t^{\theta-1} n_t^{1-\theta} = \theta \frac{y_t}{k_t} \\ w_t &= (1-\theta) k_t^{\theta} n_t^{-\theta} = (1-\theta)\frac{y_t}{n_t}. \end{aligned}$$

Now back to the calibration. We have $1-\theta = 0.7$, which implies $\theta = 0.3$. Next, the law of motion for the capital stock implies that at the SS

$$i = \delta k$$

Divide both sides by y, we have

$$\frac{i}{y} = \delta \frac{k}{y}.$$

$c/y = 0.85$ implies

$$\frac{i}{y} = 1 - \frac{c}{y} = 0.15.$$

Thus we have

$$\delta = \frac{i/y}{k/y} = \frac{0.15}{3} = 0.05.$$

Now the EE is

$$\frac{c_{t+1}}{c_t} = \beta(r_t + 1 - \delta) = \beta(\theta \frac{y_t}{k_t} + 1 - \delta)$$

In SS, $c_{t+1} = c_t = c$, $y_t = y$, and $k_t = k$, thus we have

$$\begin{aligned} 1 &= \beta(\theta \frac{y}{k} + 1 - \delta) \\ &= \beta(0.4/3 + 1 - 0.05) \end{aligned}$$

which implies

$$\beta = 0.952.$$

In SS, the FOC w.r.t. π_t is

$$(1-\theta)\frac{y}{n}\frac{1}{2c} = \alpha \log 2$$

Since in SS, $n = \pi \cdot \frac{1}{2} + (1-\pi) \cdot 0$ and $\pi = 0.8$, we obtain $n = 0.4$, we also have $y/c = 1/0.85$, therefore, we know

$$\alpha = 1.485.$$

5. (*Calibrating a RBC model with home production*) Consider an economy with infinitely lived individuals. Individuals maximize

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \{ b \ln c_t + (1-b)\ln(1-l_t) \}$$

where consumption is a aggregator defined by

$$c_t = [\alpha c_{m,t}^e + (1-\alpha)c_{h,t}^e]^{\frac{1}{e}}$$

$c_{m,t}$ is the consumption of market-produced goods, $c_{h,t}$ is consumption of home-produced goods.

The model has two technologies, one for market production and one for home production:

$$\begin{aligned} f(z_{m,t}, k_{m,t}, l_{m,t}) &= e^{z_{m,t}} k_{m,t}^{\theta} l_{m,t}^{1-\theta} \\ g(z_{h,t}, k_{h,t}, l_{h,t}) &= e^{z_{h,t}} k_{h,t}^{\eta} l_{h,t}^{1-\eta}. \end{aligned}$$

where θ and η are the capital share parameters for the two technologies, respectively. The two technology shocks follow the processes:

$$\begin{aligned} z_{m,t} &= \rho z_{m,t-1} + \varepsilon_{m,t} \\ z_{h,t} &= \rho z_{h,t-1} + \varepsilon_{h,t} \end{aligned}$$

where the two innovations $\varepsilon_{m,t}$ and $\varepsilon_{h,t}$ are normally distributed with zero means and standard deviations σ_m and σ_h; have a contemporaneous correlation $\varphi = cov(\sigma_m, \sigma_h)$; and are independent over time. In each period, a capital constraint holds

$$k_t = k_{m,t} + k_{h,t}.$$

The law of motion for the aggregate capital stock is

$$k_{t+1} = (1-\delta_m)k_{m,t} + (1-\delta_h)k_{h,t} + i_t.$$

We also have following time constraint

$$l_{m,t} + l_{h,t} = l_t.$$

And the resource constraints for two sectors

$$\begin{aligned} c_{m,t} + i_t + G_t &= f(z_{m,t}, k_{m,t}, l_{m,t}) \\ c_{h,t} &= g(z_{h,t}, k_{h,t}, l_{h,t}). \end{aligned}$$

We also assume that there is a government imposes proportional taxes each period on labor and capital income (net of depreciation) at the constant rates τ_l and τ_k, transfers the lump-sum T_t back to individuals, and consumes the surplus in terms of government expenditure G_t. Government balances its budget period by period. Calibrate this model economy to match long-run average ratio in the US economy: $k_m/y = 4$, $k_h/y = 5$, $i_m/y = 0.118$, $i_h/y = 0.135$, $l_m = 0.25$, $l_h = 0.33$, $\tau_k = 0.70$, $\tau_l = 0.25$, quarterly growth rate of output is 0.5%, and annual real rate of return on capital is 6%. How many parameter values can be determined by these moments? What are these values? (20 points)

Answer: The HH's problem is

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \{ b \ln[\alpha c_{m,t}^e + (1-\alpha)c_{h,t}^e]^{\frac{1}{e}} + (1-b)\ln(1 - l_{m,t} - l_{h,t}) \}$$

subject to

$$\begin{aligned}
c_{m,t}+i_{m,t}+i_{h,t} &= w_t(1-\tau_h)l_{m,t}+r_t(1-\tau_k)k_{m,t}+\delta_m\tau_k k_{m,t}+T_t \\
c_{h,t} &= e^{z_{h,t}}k_{h,t}^{\eta}l_{h,t}^{1-\eta} \\
k_{m,t+1} &= (1-\delta_m)k_{m,t}+i_{m,t} \\
k_{h,t+1} &= (1-\delta_h)k_{h,t}+i_{h,t}
\end{aligned}$$

FOCs are

$$\begin{aligned}
c_{m,t} &: \beta^t bU_{m,t}=\lambda_t \\
c_{h,t} &: \beta^t bU_{h,t}=\mu_t \\
k_{m,t+1} &: \lambda_t=\lambda_{t+1}[1+(r_{t+1}-\delta_m)(1-\tau_k)] \\
k_{h,t+1} &: \lambda_t=\lambda_{t+1}(1-\delta_h)+\mu_{t+1}g_{k,t+1} \\
l_{m,t} &: \beta^t(1-b)U_{l,t}+\lambda_t w_t(1-\tau_l)=0 \\
l_{h,t} &: \beta^t(1-b)U_{l,t}+\mu_t g_{l,t}=0
\end{aligned}$$

where

$$\begin{aligned}
U_{m,t} &= \frac{\alpha c_{m,t}^{e-1}}{[\alpha c_{m,t}^e+(1-\alpha)c_{h,t}^e]} \\
U_{h,t} &= \frac{(1-\alpha)c_{h,t}^{e-1}}{[\alpha c_{m,t}^e+(1-\alpha)c_{h,t}^e]} \\
U_{l,t} &= -\frac{1}{1-l_{m,t}-l_{h,t}} \\
g_{k,t} &= \eta e^{z_{h,t}}k_{h,t}^{\eta-1}l_{h,t}^{1-\eta} \\
g_{l,t} &= (1-\eta)e^{z_{h,t}}k_{h,t}^{\eta}l_{h,t}^{-\eta}.
\end{aligned}$$

Combining these FOCs, we have

$$\begin{aligned}
\frac{\alpha c_{m,t}^{e-1}}{[\alpha c_{m,t}^e+(1-\alpha)c_{h,t}^e]} &= \beta\frac{\alpha c_{m,t+1}^{e-1}}{[\alpha c_{m,t+1}^e+(1-\alpha)c_{h,t+1}^e]}[1+(r_{t+1}-\delta_m)(1-\tau_k)] \\
\frac{\alpha c_{m,t}^{e-1}}{[\alpha c_{m,t}^e+(1-\alpha)c_{h,t}^e]} &= \beta\frac{\alpha c_{m,t+1}^{e-1}}{[\alpha c_{m,t+1}^e+(1-\alpha)c_{h,t+1}^e]}(1-\delta_h) \\
&\quad +\beta\frac{(1-\alpha)c_{h,t+1}^{e-1}}{[\alpha c_{m,t+1}^e+(1-\alpha)c_{h,t+1}^e]}\eta e^{z_{h,t+1}}k_{h,t+1}^{\eta-1}l_{h,t+1}^{1-\eta} \\
\frac{b}{1-l_{m,t}-l_{h,t}} &= \frac{\alpha c_{m,t}^{e-1}}{[\alpha c_{m,t}^e+(1-\alpha)c_{h,t}^e]}w_t(1-\tau_l) \\
\frac{1-b}{1-l_{m,t}-l_{h,t}} &= \frac{(1-\alpha)c_{h,t}^{e-1}}{[\alpha c_{m,t}^e+(1-\alpha)c_{h,t}^e]}(1-\eta)e^{z_{h,t}}k_{h,t}^{\eta}l_{h,t}^{-\eta}.
\end{aligned}$$

Notice that from the firm's problem, we have

$$\begin{aligned}
r_t &= f_{k,t}=\theta e^{z_{m,t}}k_{m,t}^{\theta-1}l_{m,t}^{1-\theta}=g_{k,t} \\
w_t &= f_{l,t}=(1-\theta)e^{z_{m,t}}k_{m,t}^{\theta}l_{m,t}^{-\theta}.
\end{aligned}$$

In order to calibrate the model, we need to derive some properties of the BGP. We assume

$$z_{m,t} = z_{h,t} = \gamma^t.$$

It is easy to show that along the BGP, $l_{m,t} = l_m$, $l_{h,t} = l_h$, and all other endogenous variables grow at rate γ such as

$$\begin{aligned} c_{m,t} &= c_m \gamma^t \\ c_{h,t} &= c_h \gamma^t \\ k_{m,t} &= k_m \gamma^t \\ k_{h,t} &= k_h \gamma^t \\ y_t &= y \gamma^t. \end{aligned}$$

Along the BGP, we have

$$1 = \frac{\beta}{\gamma}[1 + (\theta \frac{y}{k_m} - \delta_m)(1 - \tau_k)] \tag{7}$$

$$1 = \frac{\beta}{\gamma}[(1 - \delta_h) + (\frac{c_h}{c_m})^{e-1} \eta \frac{1-\alpha}{\alpha} \frac{c_h}{k_h}] \tag{8}$$

$$\frac{1-b}{1-l} = \frac{\alpha b c_m^{e-1}}{[\alpha c_m^e + (1-\alpha) c_h^e]} (1-\theta) \frac{y}{l_m} (1 - \tau_l) \tag{9}$$

$$\frac{1-b}{1-l} = \frac{(1-\alpha) b c_h^{e-1}}{[\alpha c_m^e + (1-\alpha) c_h^e]} (1-\eta) \frac{y}{l_h}. \tag{10}$$

Also from the capital accumulation equations, we have

$$\begin{aligned} \frac{i_m}{k_m} &= \gamma - 1 + \delta_m \\ \frac{i_h}{k_h} &= \gamma - 1 + \delta_h. \end{aligned}$$

Now we have $\gamma = 1.005$, therefore, we obtain

$$\begin{aligned} \delta_m &= \frac{i_m}{k_m} - \gamma + 1 = \frac{i_m/y}{k_m/y} - 1.005 + 1 \\ &= \frac{0.118}{4} - 1.005 + 1 = 0.0245. \end{aligned}$$

Similarly we have

$$\begin{aligned} \delta_h &= \frac{i_h}{k_h} - \gamma + 1 = \frac{i_h/y}{k_h/y} - 1.005 + 1 \\ &= \frac{0.135}{5} - 1.005 + 1 = 0.0218. \end{aligned}$$

The annual real rate of returns on assets is 6%, which implies the quarterly real rate is about 1.5%. In the model without any distortion, we should end up with $\beta(1+r) =$

$\beta(1+0.015) = 1$, which implies quarterly discount rate $\beta = 0.985$.[1] Since we now know the values of $\gamma, k_m/y, \delta_m, \beta, \tau_k$, from equation (7), we can obtain $\theta = 0.29$. Capital share is quite close to the one in the benchmark RBC model. Notice that (9)/(10) will yield

$$\frac{\alpha}{1-\alpha}(\frac{c_m}{c_h})^{e-1}\frac{(1-\theta)(1-\tau_l)}{(1-\eta)}\frac{l_h}{l_m} = 1 \tag{11}$$

Combining this equation with (8), and notice $\frac{c_h}{k_h} = \frac{y}{k_m} = 1/4$, and use the fact that $l_m = 0.25$, $l_h = 0.33$, we can obtain $\eta = 0.22$. (Notice from equation 8, we have $(\frac{c_h}{c_m})^{e-1}\eta\frac{1-\alpha}{\alpha} = 0.1659$. Substitute this term into equation 11, rewrite it, we have $\frac{1}{(\frac{c_h}{c_m})^{e-1}\eta\frac{1-\alpha}{\alpha}}\frac{\eta}{1-\eta}(1-\theta)(1-\tau_l)\frac{l_h}{l_m} = 1$ which will pin down η.) If we know the ratio $\frac{c_m}{c_h}$ and parameter value of e, then from equation (11), we can pin down α. Then from equation (10). we will obtain b.

See Greenwood, Rogerson and Wright (1995): "*Household Production in Real Business Cycle Theory,*" Chapter 6 in Cooley and Prescott (1995) for further details.

6. Consider the planning problem for a neoclassical growth model with logarithmic utility, full depreciation of the capital stock in one period, and a production function of the form $y = zk^\alpha$, where z is a random shock to productivity. The shock z is observed before making the current-period savings decision. Assume that the capital stock can take on only two values: i.e., k is restricted to the set $\{\bar{k}_1, \bar{k}_2\}$. In addition, assume that z takes on values in the set $\{\bar{z}_1, \bar{z}_2\}$ and that z follows a Markov chain with transition probabilities $p_{ij} = \Pr(z' = \bar{z}_j \mid z = \bar{z}_i)$. (20 points)

   (a) Let $\bar{z}_1 = 0.9$, $\bar{z}_2 = 1.1$, $p_{11} = 0.95$, $p_{22} = 0.9$. Find the invariant distribution associated with the Markov chain for z. Use the invariant distribution to compute the long-run (or unconditional) expected value of z; that is, compute $E(z) = \pi_1\bar{z}_1 + \pi_2\bar{z}_2$, where π_1 and π_2 determines the invariant distribution.
   Answer: Given the transition matrix

$$P = \begin{bmatrix} 0.95 & 0.05 \\ 0.1 & 0.9 \end{bmatrix}$$

   we can calculate the stationary distribution πacording to the formula

$$\begin{aligned} \pi' &= \pi' P \\ &\Rightarrow \left\{\begin{smallmatrix} \pi_1 = 0.95\pi_1 + 0.1\pi_2 \\ \pi_2 = 0.05\pi_1 + 0.9\pi_2 \end{smallmatrix}\right. \end{aligned}$$

   the solution to this equation is

$$\pi_1 = 2\pi_2$$

   Imposin gthe conditin that $\pi_1 + \pi_2 = 1$, the solution is

$$\begin{aligned} \pi_1 &= 2/3 \\ \pi_2 &= 1/3 \end{aligned}$$

[1] If we stick to the equation (7), we should end up with $\beta = 1.0255$. One might want to think the real interest rate in the data is actually the interest rate already take into account the tax rate and depreciation rate. In other words, 1.5% is the after-tax after-depreciation effective interest rate.

Correspondingly, the long run expected value is

$$E(z) = \pi' z = \frac{2}{3} \times 0.9 + \frac{1}{3} \times 1.1 = \frac{29}{30}$$

(b) Let $\beta = 0.9$, $\alpha = 0.36$, $\bar{k}_1 = 0.95k_{ss}$, $\bar{k}_2 = 1.05k_{ss}$, where k_{ss} is the steady-state capital stock in a version of this model without shocks and with no restrictions on capital. Show that $k_{ss} = (\alpha\beta)^{\frac{1}{1-\alpha}}$. Let $g(k,z)$ denote the planner's optimal decision rule. Prove that $g(k_i, z_j) = k_j$ for all i and j.

Answer: For the dynamic programming problem as

$$v\left(\overline{k}_i, z_i\right) = \max_{\overline{k}' \in \{k_1, k_2\}} \ln\left(z_i \overline{k}_i^{\alpha} - \overline{k}'\right) + \beta\left(p_{i1} v\left(\overline{k}', z_1\right) + p_{i2} v\left(\overline{k}', z_2\right)\right)$$

Since (k_i, z_i) takes on only 4 values, we assume that the policy function takes the following form.

$$\begin{aligned} g(k_1, z_1) &= k_1 \\ g(k_2, z_1) &= k_1 \\ g(k_1, z_2) &= k_2 \\ g(k_2, z_2) &= k_2 \end{aligned}$$

We need to prove taht this is true. The way we are going to proceed is that we are going to calculate the value fucntion values associated with this policy function and then verify that they are indeed maximum.

Using the parameter values given in the problem we get:

$$\begin{aligned} k_{ss} &= 0.1719 \\ k_1 &= 0.1633 \\ k_2 &= 0.18047 \end{aligned}$$

Let's try out all 4 cases and substitute in for the assumed policy function. For $(k_i, z_i) = (k_1, z_1)$ we have:

$$\begin{aligned} v\left(\overline{k}_1, \overline{z_1}\right) &= \ln\left(z_1 \overline{k}_1^{\alpha} - \overline{k}_1^{\alpha}\right) + \beta\left(p_{11} v\left(\overline{k}_1, z_1\right) + p_{12} v\left(\overline{k}_1, z_2\right)\right) \Leftrightarrow \\ v\left(\overline{k}_1, \overline{z_1}\right) &= \ln\left(0.9 * 0.1633^{0.36} - 0.1633\right) + 0.9(0.95 v\left(\overline{k}_1, \overline{z_1}\right) + 0.05 v\left(\overline{k}_1, \overline{z_2}\right) \Leftrightarrow \\ v\left(\overline{k}_1, \overline{z_1}\right) &= -1.1861 + 0.855 v\left(\overline{k}_1, \overline{z_1}\right) + 0.045 v\left(\overline{k}_1, \overline{z_2}\right) \Leftrightarrow \\ v\left(\overline{k}_1, \overline{z_1}\right) &= -8.18 + 0.3103 v\left(\overline{k}_1, \overline{z}_2\right) \end{aligned}$$

Similarly we get

$$\begin{aligned} v\left(\overline{k}_1, \overline{z}_2\right) &= -0.93543 + 0.855 v\left(\overline{k}_2, \overline{z}_1\right) + 0.045 v\left(\overline{k}_2, \overline{z}_2\right) \\ v\left(\overline{k}_2, \overline{z}_1\right) &= -0.13134 + 0.855 v\left(\overline{k}_1, \overline{z}_1\right) + 0.045 v\left(\overline{k}_1, \overline{z}_2\right) \\ v\left(\overline{k}_2, \overline{z}_2\right) &= -0.8833 + 0.855 v\left(\overline{k}_2, \overline{z}_1\right) + 0.045 v\left(\overline{k}_2, \overline{z}_2\right) \\ v\left(\overline{k}_2, \overline{z}_2\right) &= -0.9249 + 0.8953 v\left(\overline{k}_2, \overline{z}_1\right) \end{aligned}$$

Essentially we have a system of 4 equations with 4 unknowns. Plugging in the first equation into the third we get:

$$v\left(\overline{k}_2, \overline{z}_1\right) = -8.12524 + 0.3103 v\left(\overline{k}_1, \overline{z}_2\right)$$

Similarly, plugging in the last equation into the second one we get:

$$v\left(\overline{k}_1, \overline{z}_2\right) = 0.89381 + 0.89529 v\left(\overline{k}_2, \overline{z}_1\right)$$

Solving out the above system of 2 equations and 2 unknowns, we get:

$$\begin{aligned} v\left(\overline{k}_1, \overline{z}_2\right) &= -8.8351 \\ v\left(\overline{k}_2, \overline{z}_1\right) &= -10.8668 \end{aligned}$$

Substituting into equations 1 and 4 from above we get:

$$\begin{aligned} v\left(\overline{k}_1, \overline{z}_1\right) &= -10.9215 \\ v\left(\overline{k}_2, \overline{z}_2\right) &= -10.6539 \end{aligned}$$

We now need to check that this decision rule is optimal. We will go about checking this changing the decision rule in each of the 4 cases and then calculating the resulting value function. For example, when we have $\left(\overline{k}_1, \overline{z}_1\right)$ we will assume that the planner chooses $\overline{k}_2$ instead of $\overline{k}_1$. In that case we would get:

$$\begin{aligned} v^{alt}\left(\overline{k}_1, \overline{z}_1\right) &= \ln\left(z_1 \overline{k}^{\alpha} - \overline{k}_2\right) + \beta(p_{11} v\left(\overline{k}_2, z_1\right) + p_{12} v\left(\overline{k}_2, z_2\right)) \Rightarrow \\ v^{alt}\left(\overline{k}_1, \overline{z}_1\right) &= \ln\left(0.9 * 0.1633^{0.36} - 0.18047\right) + 0.9\left(0.95 * (-10.8668) + 0.05 * (-10.6539)\right) \Rightarrow \\ v^{alt}\left(\overline{k}_1, \overline{z}_1\right) &= -11.0145 < -10.9215 = v\left(\overline{k}_1, \overline{z}_1\right) \end{aligned}$$

Similarly we get:

$$\begin{aligned} v^{alt}\left(\overline{k}_1, \overline{z}_2\right) &= -10.7126 < -8.8351 = v\left(\overline{k}_1, \overline{z}_2\right) \\ v^{alt}\left(\overline{k}_2, \overline{z}_1\right) &= -11.4859 < -10.8668 = v\left(\overline{k}_2, \overline{z}_1\right) \\ v^{alt}\left(\overline{k}_2, \overline{z}_2\right) &= -10.7581 < -10.6539 = v\left(\overline{k}_2, \overline{z}_2\right) \end{aligned}$$

and we see that in each case the original decision rule performs better. The rational we use to conclude that our decision rule is indeed optimal is the following: our guess about the decision rule implies a certain value for our value function. On the other hand changing the decision rule with theonly other possible alternative for each case would imply a different value fucntion. If this alternative value function was actually closer to the true value function (which is the optimal) then it would give a higher value than our guess. But that is not the case. Therefore our guess is closer to the optimal. You can think of this procedure as a value function iteration. Given however that for each decision rule there is only one other alternative, this implies that our value function is indeed optimal and that our guess is indeed the correct decision rule.

(c) The decision rule from part (b) and the law of motion for z jointly determine an invariant distribution over (k, z)-pairs. Find this distribution. (That is, find probabilities $\pi_{ij} = \Pr(k = k_i, z = z_j)$ that "reproduce" themselves: if π_{ij} is the unconditional probability that the economy is in state (k_i, z_j) today, then it is also the unconditional probability that the economy is in this state tomorrow.) Use your answer to compute the long-run (or unconditional) expected values of the capital stock and of output.

Answer: Based on policy function $g(k_i, z_i)$ and transition matrix of z, the pair (k, z) follows a Markov process with the transition matrix

	(z_1, k_1)	(z_1, k_2)	(z_2, k_1)	(z_2, k_2)
(z_1, k_1)	0.95	0	0.05	0
(z_1, k_2)	0.95	0	0.05	0
(z_2, k_1)	0	0.1	0	0.9
(z_2, k_2)	0	0.1	0	0.9

Now we can calculate the stationary distribution either on the computer or by hand. The result is

$$\begin{aligned} p(k_1, z_1) &= \frac{19}{30} = 0.63333 \\ p(k_2, z_1) &= \frac{1}{30} = 0.03333 \\ p(k_1, z_2) &= \frac{1}{30} = 0.03333 \\ p(k_2, z_2) &= \frac{3}{10} = 0.3 \end{aligned}$$

Using the stationary distribution, the long-run (or unconditional) expected values of the capital stock and of output are

$$\begin{aligned} Ek &= p(k_1, z_1)k_1 + p(k_1, z_2)k_1 + p(k_2, z_1)k_2 + p(k_2, z_2)k_2 = \frac{2}{3}k_1 + \frac{1}{3}k_2 = 0.1747 \\ Ey &= p(k_1, z_1) * z_1 k_1^{\alpha} + p(k_1, z_2) * z_2 k_1^{\alpha} + p(k_2, z_1) * z_1 k_2^{\alpha} + p(k_2, z_2) * z_2 k_2^{\alpha} = 0.51031 \end{aligned}$$

(d) In Matlab, use the optimal decision rule, the law of motion for z, and a random number generator to create a simulated time series $\{k_t, y_t\}_{t=0}^{T}$ given an initial condition (k_0, z_0). Compute $\frac{\sum_{t=0}^{T} k_t}{T}$ and $\frac{\sum_{t=0}^{T} y_t}{T}$ for a suitably large value of T and confirm that these sample means are close to the corresponding population means that you computed in part (c).

Answer: Depending on the realization of each simulation, the result will differ a little bit. For example, one possible result based on T=10000 is

$$\begin{aligned} \frac{1}{T}\sum_{t=1}^{T} k_t &= 0.1747 \\ \frac{1}{T}\sum_{t=1}^{T} y_t &= 0.4687 \end{aligned}$$