ECON607 Fall 2010
University of Hawaii
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Assignment 3 Suggested Answer

The due date for this assignment is Tuesday, November 17. (Total 115 points)

1. (*Home production*) Consider an economy with infinitely lived individuals. Individuals maximize

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \{\ln c_t + \gamma \ln(1 - l_t)\}$$

where consumption is a aggregator defined by

$$c_t = [\alpha c_{m,t}^{\eta} + (1-\alpha) c_{h,t}^{\eta}]^{\frac{1}{\eta}}$$

$c_{m,t}$ is the consumption of market-produced goods, $c_{h,t}$ is consumption of home-produced goods.

The model has two technologies, one for market production and one for home production:

$$\begin{aligned} f(z_{m,t}, k_{m,t}, l_{m,t}) &= e^{z_{m,t}} k_{m,t}^{\theta} l_{m,t}^{1-\theta} \\ f(z_{h,t}, k_{h,t}, l_{h,t}) &= e^{z_{h,t}} k_{h,t}^{\nu} l_{h,t}^{1-\nu}. \end{aligned}$$

where θ and ν are the capital share parameters for the two technologies, respectively. The two technology shocks follow the processes:

$$\begin{aligned} z_{m,t} &= \rho z_{m,t-1} + \varepsilon_{m,t} \\ z_{h,t} &= \rho z_{h,t-1} + \varepsilon_{h,t} \end{aligned}$$

where the two innovations $\varepsilon_{m,t}$ and $\varepsilon_{h,t}$ are normally distributed with zero means and standard deviations σ_m and σ_h; have a contemporaneous correlation $\varphi = cov(\sigma_m, \sigma_h)$; and are independent over time. In each period, a capital constraint holds

$$k_t = k_{m,t} + k_{h,t}.$$

The law of motion for the aggregate capital stock is

$$k_{t+1} = (1-\delta) k_t + i_t.$$

We also have following time constraint

$$l_{m,t} + l_{h,t} = l_t.$$

Finally the resource constraints for two sectors

$$\begin{aligned} c_{m,t} + i_t &= f(z_{m,t}, k_{m,t}, l_{m,t}) \\ c_{h,t} &= f(z_{h,t}, k_{h,t}, l_{h,t}). \end{aligned}$$

(20 points)

(a) Write down the Bellman equation for this problem. Be clear what are the state variables, what are control variables.

Answer: The BE for this economy is

$$
\begin{aligned}
v(z_{m,t}, z_{h,t}, k_t) &= \max_{\{c_{m,t}, c_{h,t}, i_t, k_{m,t}, k_{h,t}, l_{m,t}, l_{h,t}\}} \{\ln c_t + \gamma \ln(1-l_t) + \beta E v(z_{m,t+1}, z_{h,t+1}, k_{t+1})\} \\
& \quad s.t. \\
c_t &= [\alpha c_{m,t}^{\eta} + (1-\alpha) c_{h,t}^{\eta}]^{\frac{1}{\eta}} \\
c_{m,t} + i_t &= e^{z_{m,t}} k_{m,t}^{\theta} l_{m,t}^{1-\theta} \\
c_{h,t} &= e^{z_{h,t}} k_{h,t}^{\nu} l_{h,t}^{1-\nu} \\
k_t &= k_{m,t} + k_{h,t} \\
k_{t+1} &= (1-\delta) k_t + i_t \\
l_{m,t} + l_{h,t} &= l_t \\
z_{m,t} &= \rho z_{m,t-1} + \varepsilon_{m,t}, \, z_{h,t} = \rho z_{h,t-1} + \varepsilon_{h,t} \\
& \quad \text{non-neg. constraints, } k_0 \text{ given}
\end{aligned}
$$

(b) Derive the first order conditions of this economy and interpret their economic implications.

Answer: We can use Euler equation approach to solve the equilibrium. Set up the Lagrangian

$$
\begin{aligned}
\mathcal{L} = & E_0 \sum_{t=0}^{\infty} \beta^t \{\ln[\alpha c_{m,t}^{\eta} + (1-\alpha) c_{h,t}^{\eta}]^{\frac{1}{\eta}} + \gamma \ln(1 - l_{m,t} - l_{h,t})\} \\
& + \lambda_t \{e^{z_{m,t}} k_{m,t}^{\theta} l_{m,t}^{1-\theta} - c_{m,t} - (k_{m,t+1} + k_{h,t+1}) + (1-\delta)(k_{m,t} + k_{h,t})\} \\
& + \mu_t \{e^{z_{h,t}} k_{h,t}^{\nu} l_{h,t}^{1-\nu} - c_{h,t}\}.
\end{aligned}
$$

FOCs are

$$
\begin{aligned}
c_{m,t} &: \beta^t \frac{\alpha c_{m,t}^{\eta-1}}{\alpha c_{m,t}^{\eta} + (1-\alpha) c_{h,t}^{\eta}} = \lambda_t & (1) \\
c_{h,t} &: \beta^t \frac{(1-\alpha) c_{h,t}^{\eta-1}}{\alpha c_{m,t}^{\eta} + (1-\alpha) c_{h,t}^{\eta}} = \mu_t & (2) \\
l_{m,t} &: \beta^t \frac{\gamma}{1 - l_{m,t} - l_{h,t}} = \lambda_t (1-\theta) e^{z_{m,t}} k_{m,t}^{\theta} l_{m,t}^{-\theta} & (3) \\
l_{h,t} &: \beta^t \frac{\gamma}{1 - l_{m,t} - l_{h,t}} = \mu_t (1-\nu) e^{z_{h,t}} k_{h,t}^{\nu} l_{h,t}^{-\nu} & (4) \\
k_{m,t} &: \lambda_{t-1} = \beta \lambda_t [\theta e^{z_{m,t}} k_{m,t}^{\theta-1} l_{m,t}^{1-\theta} + 1 - \delta] & (5) \\
k_{h,t} &: \lambda_{t-1} = \beta \{\mu_t (\nu e^{z_{h,t}} k_{h,t}^{\nu-1} l_{h,t}^{1-\nu}) + \lambda_t (1-\delta)\} & (6) \\
k_{m,t+1} &: \lambda_t = \beta E_t \lambda_{t+1} [\theta e^{z_{m,t+1}} k_{m,t+1}^{\theta-1} l_{m,t+1}^{1-\theta} + 1 - \delta] & (7)
\end{aligned}
$$

Economic implications:

(1) says the multiplier associated with resource constraint in market good sector is the discounted marginal utility of market good. $MU_{c_{m,t}} = \lambda_t$. (2) says the

multiplier associated with resource constraint in home-produced good sector is the discounted marginal utility of home-produced good. $MU_{c_{h,t}} = \mu_t$. (3) is the optimal decision on labor supply to market sector $l_{m,t}$. It shows that the marginal cost of extra unit of $l_{m,t}$ is the marginal utility of leisure. The marginal benefit is the increase of marginal product of this extra unit of labor supply translated by the marginal utility of consumption. In other words, this equation says that MRS b/w consumption and leisure in market good sector should be equal to the market wage. (4) is similar for $l_{h,t}$. (5) and (6) implies a non-arbitrage condition (NAC) b/w capital used in market good sector and capital used in home-produced sector. $\mu_t(\nu e^{z_{h,t}} k_{h,t}^{\nu-1} l_{h,t}^{1-\nu}) = \lambda_t \theta e^{z_{m,t}} k_{m,t}^{\theta-1} l_{m,t}^{1-\theta}$, i.e., $MU_{c_{m,t}} MP_{k_{m,t}} = MU_{c_{h,t}} MP_{k_{h,t}}$. (7) is the Euler equation for capital used in market good sector. Notice that due to NAC, the Euler equation for capital used in home-produced sector is redundant.

2. (*Two-sector model of endogenous growth*) Consider the following two-sector growth model:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\begin{aligned} & s.t. \\ c_t &\leq Bk_{1t}^{\alpha} n_t^{1-\alpha} \\ x_t &\leq Ak_{2t} \\ k_{t+1} &= (1-\delta)k_t + x_t \\ k_{1t} + k_{2t} &= k_t \\ n_t &= 1, c_t, x_t \geq 0 \end{aligned}$$

Assume

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \sigma \geq 0$$

(20 points)

(a) Show that this problem is "homogeneous"–that is, doubling k_0 doubles the entire time path of optimal capital stocks. What does this do to the optimal path of k_1's, k_2's, and c's?

Answer: Suppose that for a given initial capital stock k_0, denote $\Gamma(k_0)$ be the set of all feasible allocations generated by k_0. That is

$$\Gamma(k_0) = \left\{ \begin{array}{c} (c_t, k_t, k_{1t}, k_{2t}, x_t, n_t)_{t=0}^{\infty} : c_t \leq Bk_{1t}^{\alpha} n_t^{1-\alpha}, x_t \leq Ak_{2t}, \\ k_{t+1} = (1-\delta)k_t + x_t, k_{1t} + k_{2t} = k_t, n_t = 1 \end{array} \right\}$$

Claim 1: If $(c_t^*, k_t^*, k_{1t}^*, k_{2t}^*, x_t^*, n_t^*) \in \Gamma(k_0)$, then $(\lambda c_t^*, \lambda k_t^*, \lambda k_{1t}^*, \lambda k_{2t}^*, \lambda x_t^*, \lambda n_t^*) \in \Gamma(\lambda k_0)$

Proof:

$$\begin{aligned} c_t &\leq Bk_{1t}^{\alpha} n_t^{1-\alpha} \Leftrightarrow \lambda c_t \leq B(\lambda k_{1t})^{\alpha} (\lambda n_t)^{1-\alpha} \\ x_t &\leq Ak_{2t} \Leftrightarrow \lambda x_t \leq A\lambda k_{2t} \\ k_{t+1} &= (1-\delta)k_t + x_t \Longleftrightarrow \lambda k_{t+1} = (1-\delta)\lambda k_t + \lambda x_t \\ k_{1t} + k_{2t} &= k_t \Leftrightarrow \lambda k_{1t} + \lambda k_{2t} = \lambda k_t \\ n_t &= 1 \Leftrightarrow \lambda n_t = \lambda \end{aligned}$$

Claim 2: Suppose $(c_t^*, k_t^*, k_{1t}^*, k_{2t}^*, x_t^*, n_t^*)$ is optimal given k_0, then $(\lambda c_t^*, \lambda k_t^*, \lambda k_{1t}^*, \lambda k_{2t}^*, \lambda x_t^*, \lambda n_t^*)$ is also optimal given λk_0.

Proof: Suppose not, i.e., $\exists$ another feasible allocation $(c_t', k_t', k_{1t}', k_{2t}', x_t', n_t') \in \Gamma(\lambda k_0)$ such that

$$\sum_{t=0}^{\infty} \beta^t \frac{(c_t')^{1-\sigma}}{1-\sigma} > \sum_{t=0}^{\infty} \beta^t \frac{(\lambda c_t^*)^{1-\sigma}}{1-\sigma}$$

This implies

$$(\frac{1}{\lambda})^{1-\sigma} \sum_{t=0}^{\infty} \beta^t \frac{c_t'^{1-\sigma}}{1-\sigma} > \sum_{t=0}^{\infty} \beta^t \frac{(c_t^*)^{1-\sigma}}{1-\sigma}$$

$\Rightarrow$

$$\sum_{t=0}^{\infty} \beta^t \frac{(\frac{c_t'}{\lambda})^{1-\sigma}}{1-\sigma} > \sum_{t=0}^{\infty} \beta^t \frac{(c_t^*)^{1-\sigma}}{1-\sigma}$$

Notice that since $c_t' \in \Gamma(\lambda k_0), \Rightarrow \frac{c_t'}{\lambda} \in \Gamma(k_0)$. We obtain contradiction since c_t^* is the optimal under k_0. Thus when k_0 is double, the optimal pathes of all the variables are also double.

(b) Show that the value function for this problem is homogeneous of degree $1-\sigma$, i.e., $v(\lambda k) = \lambda^{1-\sigma} v(k)$.

Answer: Value function $v(k_0)$ is defined by

$$v(k_0) \equiv \sum_{t=0}^{\infty} \beta^t \frac{(c_t)^{1-\sigma}}{1-\sigma}$$

From part (a), we already know that λc_t is optimal under λk_0. Thus we have

$$v(\lambda k_0) \equiv \sum_{t=0}^{\infty} \beta^t \frac{(\lambda c_t)^{1-\sigma}}{1-\sigma} = \lambda^{1-\sigma} \sum_{t=0}^{\infty} \beta^t \frac{(c_t)^{1-\sigma}}{1-\sigma} = \lambda^{1-\sigma} v(k_0).$$

(c) Use your results from part (a) and (b) to derive the time path for the solution (BGP) to this planner's problem.

Answer: Set up Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \frac{(c_t)^{1-\sigma}}{1-\sigma} + \lambda_t \{B k_{1t}^{\alpha} - c_t\} + \mu_t \{A(k_t - k_{1t}) + (1-\delta) k_t - k_{t+1}\}$$

FOCs are

$$\begin{aligned} c_t &: \beta^t c_t^{-\sigma} = \lambda_t \\ k_{1t} &: \lambda_t B \alpha k_{1t}^{\alpha-1} = A \mu_t \\ k_{t+1} &: \mu_t = \mu_{t+1}(A + 1 - \delta) \end{aligned}$$

Combining these FOCs, we have the following EE

$$\frac{c_t^{-\sigma} k_{1t}^{\alpha-1}}{\beta c_{t+1}^{-\sigma} k_{1t+1}^{\alpha-1}} = A + 1 - \delta$$

Notice that from resource constraint, we have

$$\frac{c_t}{c_{t+1}} = (\frac{k_{1t}}{k_{1t+1}})^\alpha$$

Substituting into the EE above, we have

$$(\frac{k_{1t+1}}{k_{1t}})^{\alpha\sigma-\alpha+1} = \beta(A+1-\delta)$$

$\Rightarrow$

$$g_{k_1} = \frac{k_{1t+1}}{k_{1t}} = [\beta(A+1-\delta)]^{1/(\alpha\sigma-\alpha+1)}$$

Then it is easy to show that along BGP, $g_c = g_{k_1} = g_{k_2} = g_k = g_x = [\beta(A+1-\delta)]^{1/(\alpha\sigma-\alpha+1)}$. Once $\beta(A+1-\delta) > 1$, this economy has a long-run economikc growth.

(d) Define a decentralized Arrow-Debreu equilibrium for this economy.

Answer: The ADE for this economy is a price system $(p_{ct}, p_{kt}, w_t, r_t)$, an allocation (c_t, k_t^s, n_t^s) for HH, an allocation (k_{1t}^d, n_t^d) for consumption good firm, and an allocation (k_{2t}^d) for the investment good firm such that

(1). Given prices, allocation (c_t, k_t^s, n_t^s) solves the HH's problem

$$\begin{aligned} & \sum_{t=0}^{\infty} \beta^t \frac{(c_t)^{1-\sigma}}{1-\sigma} \\ & s.t. \\ \sum_{t=0}^{\infty} \left(p_{ct}c_t + p_{kt}[k_{t+1}^s - (1-\delta)k_t^s]\right) \leq & \sum_{t=0}^{\infty} (r_t k_t^s + w_t n_t^s) \\ c_t, k_{t+1}^s \geq & \; 0, 0 \leq n_t^s \leq 1 k_0 > 0 \text{ given} \end{aligned}$$

(2). Given prices, allocation (k_{1t}^d, n_t^d) solves the consumption good firm's problem

$$\begin{aligned} & \max p_{ct} B (k_{1t}^d)^\alpha (n_t^d)^{1-\alpha} - w_t n_t^d - r_t k_{1t}^d \\ & s.t. \\ k_{1t}^d, n_t^d \geq & \; 0 \end{aligned}$$

(3). Given prices, allocation (k_{2t}^d) solves the invetsment good firm's problem

$$\begin{aligned} & \max p_{kt} A k_{2t}^d - r_t k_{2t}^d \\ & s.t. \\ k_{2t}^d \geq & \; 0 \end{aligned}$$

(4). Markets clear.

$$\begin{aligned} k_{1t}^d + k_{2t}^d &= k_t^s = k_t \\ n_t^d &= n_t^s = n_t = 1 \\ c_t &\leq B k_{1t}^\alpha n_t^{1-\alpha} \\ k_{t+1} &= (1-\delta)k_t + A(k_t - k_{1t}). \end{aligned}$$

(e) Solve the ADE. Is your result identical to the social planner's problem?

Answer: The HH's FOCs:

$$\begin{aligned} c_t &: \beta^t c_t^{-\sigma} = \lambda p_{ct} && (8) \\ k_{t+1}^s &: p_{kt} = r_{t+1} + (1-\delta) p_{k,t+1}. && (9) \end{aligned}$$

The consumption good firm's FOCs:

$$\begin{aligned} k_{1t}^d &: r_t = p_{ct} \alpha B (k_{1t}^d)^{\alpha-1} (n_t^d)^{1-\alpha} && (10) \\ n_t^d &: w_t = p_{ct}(1-\alpha) B (k_{1t}^d)^{\alpha} (n_t^d)^{-\alpha}. && (11) \end{aligned}$$

The investment good firm's FOC

$$k_{2t}^d : r_t = A p_{kt} \tag{12}$$

Combining (10) and (12), we have

$$p_{ct} \alpha B (k_{1t}^d)^{\alpha-1} (n_t^d)^{1-\alpha} = A p_{kt} \tag{13}$$

Substituting (12) into (9), we obtain

$$p_{kt} = (A + 1 - \delta) p_{k,t+1} \tag{14}$$

Combining (13) and (14), we have

$$\frac{p_{ct}}{p_{ct+1}} \frac{k_{1t}^{\alpha-1}}{k_{1t+1}^{\alpha-1}} = \frac{p_{kt}}{p_{kt+1}} = A + 1 - \delta \tag{15}$$

Notice that by (8), we know

$$\frac{c_{t+1}}{c_t} = \left\{ \beta \left(\frac{p_{ct}}{p_{ct+1}} \right) \right\}^{1/\sigma}$$

Substituting the expression of the ratio $\frac{p_{ct}}{p_{ct+1}}$ into (15), we have

$$\frac{c_{t+1}}{c_t} = \left\{ \beta (A + 1 - \delta) \left(\frac{k_{1t}}{k_{1t+1}} \right)^{\alpha-1} \right\}^{1/\sigma}$$

Also notice that from the resource constraint of consumption good sector, we have

$$\frac{c_t}{c_{t+1}} = (\frac{k_{1t}}{k_{1t+1}})^{\alpha}$$

Substituting into the equation above, we have the gross growth rate of k_{1t}

$$\frac{k_{1t+1}}{k_{1t}} = [\beta(A + 1 - \delta)]^{1/(\alpha\sigma - \alpha + 1)}.$$

It is as same as the one derived from the social planner's problem. But it should not surprise us since we know that the First Welfare and Second Welfare Theorem hold in this case.

3. (*Investment-Specific Technological Change, Greenwood, Hercowitz and Krusell (AER, 1997)*) Consider the following infinite-horizon growth model

$$\max E \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

$$s.t.$$

$$\begin{aligned} y &= zF(k_e, k_s, l) = zk_e^{\alpha_e} k_s^{\alpha_s} l^{1-\alpha_e-\alpha_s} \\ c_t + i_{e,t} + i_{s,t} &= y_t \\ k_s' &= (1-\delta_s)k_s + i_s \\ k_e' &= (1-\delta_e)k_e + i_e q \\ c_t, k_s', k_e' &\geq 0, 0 \leq l_t \leq 1 \end{aligned}$$

where $0 < \alpha_e, \alpha_s, \alpha_e + \alpha_s < 1, 0 < \delta_e, \delta_s < 1$. And we assume

$$u(c) = \theta \ln c + (1-\theta)\ln(1-l), 0 < \theta < 1$$

In this economy, k_e is the capital equipment, k_s is the capital structure. Accordingly, i_e is the investment in equipment, i_s is the investment in structure. Notice that there are two types of technological change in this economy: one is the technological change in total factor productivity (TFP) z, another one is technological change only happens in investment, called "*Investment-Specific Technological Change*" (ISTC), denoted by q. Suppose that z and q grow at the (gross) rates γ_z and γ_q, and let $z_t = \gamma_z^t$ and $q_t = \gamma_q^t$. (20 points)

(a) Define a balanced growth path for this economy. Especially derive the growth rates of output, capital equipment, capital structure, consumption, investment in equipment, and investment in structure along BGP.

Answer: The BGP in this economy is a situation in which all economic variables grow at constant rates (but not necessarily same). Let's define the gross growth rate $g_X = \frac{X_{t+1}}{X_t}$ for variable X.

From the law of motion of structure

$$k_s' = (1-\delta_s)k_s + i_s.$$

Divide both sides by k_s, we have

$$g_{k_s} \equiv \frac{k_s'}{k_s} = 1 - \delta_s + \frac{i_s}{k_s}$$

This implies $\frac{i_s}{k_s}$ =constant, which shows that $g_{i_s} = g_{k_s}$. Now look at the law of motion for the capital equipment

$$k_e' = (1-\delta_e)k_e + i_e q.$$

Divide both sides by k_e

$$g_{k_e} \equiv \frac{k_e'}{k_e} = 1 - \delta_e + \frac{i_e}{k_e}q$$

Let's move one more period ahead

$$g_{k_e} \equiv \frac{k_e''}{k_e'} = 1 - \delta_e + \frac{i_e'}{k_e'} q'$$

which implies

$$\frac{i_e}{k_e} q = \frac{i_e'}{k_e'} q'$$

$\Rightarrow$

$$\gamma_q = \frac{q'}{q} = \frac{k_e'}{k_e} \frac{i_e}{i_e'} = \frac{g_{k_e}}{g_{i_e}}.$$

But from the resource constraint

$$c + i_e + i_s = y$$

we know that output y, consumption c, investment in equipment i_e and in structure i_s all have to grow at the same rate, i.e.

$$g_y = g_c = g_{i_e} = g_{i_s} = g$$

Therefore

$$g_{i_s} = g_{k_s} = g; g_{k_e} = g_{i_e} \gamma_q = g \gamma_q$$

But what is g? From the production $y = z k_e^{\alpha_e} k_s^{\alpha_s} l^{1-\alpha_e-\alpha_s}$, we know (since the labor is constant along BGP)

$$\begin{aligned} g &= \frac{y'}{y} = \frac{z'}{z} (\frac{k_e'}{k_e})^{\alpha_e} (\frac{k_s'}{k_s})^{\alpha_s} \\ &= \gamma_z g_{k_s}^{\alpha_s} g_{k_e}^{\alpha_e} \\ &= \gamma_z g^{\alpha_s} (g \gamma_q)^{\alpha_e} \\ &= \gamma_z g^{\alpha_s + \alpha_e} \gamma_q^{\alpha_e} \end{aligned}$$

$\Rightarrow$

$$g = \gamma_z^{1/(1-\alpha_e-\alpha_s)} \gamma_q^{\alpha_e/(1-\alpha_e-\alpha_s)}.$$

(b) Suppose there is a government that imposes tax on capital income at a tax rate τ_k and labor income at a rate τ, and the tax revenue raised by the government in each period is rebated back to agents in the form of lump-sum transfer payments T, define formally a recursive competitive equilibrium (RCE) for this economy.

Answer: The economy-wide state variables are described by $\lambda = (s, z, q)$ where $s \equiv (k_e, k_s)$. The individual's state variables are s.

The HH faces the following DP problem:

$$\begin{aligned} v(k_e, k_s; s, z, q) &= \max_{c, k_e', k_s', l} \{u(c, l) + \beta E[v(k_e', k_s'; s', z', q') \\ & s.t. \\ & c + \frac{k_e' - (1-\delta_e) k_e}{q} + k_s' - (1-\delta_s) k_s \\ &\leq (1-\tau_k)[R_e(\lambda) k_e + R_s(\lambda) k_s] + (1-\tau_l) W(\lambda) l + T(\lambda) \\ c, k_e', k_s' &\geq 0 \end{aligned} \tag{16}$$

The firm's problem is

$$\max \pi = zk_e^{\alpha_e}k_s^{\alpha_s}l^{1-\alpha_e-\alpha_s} - R_e(\lambda)k_e - R_s(\lambda)k_s - W(\lambda)l \tag{17}$$

Due to the CRS auumption of the production function, the firm makes zero profits in each period $\pi = 0$.

Definition: A RCE in this economy is a set of decision rules $c = C(\lambda), k_e' = K_e(\lambda), k_s' = K_s(\lambda)$, and $l = L(\lambda)$, a set of pricing and transfer functions $w = W(\lambda), r_e = R_e(\lambda), r_s = R_s(\lambda)$, and $T = T(\lambda)$, and an aggregate law of motion for the capital stocks $s' = S(\lambda)$ such that:

(1). Given the the aggregate state of the world $\lambda = (s, z, q)$ and the form of the functions $W(\cdot), R_e(\cdot), R_s(\cdot), T(\cdot)$, and $S(\cdot)$, the decision rules $c = C(\lambda), k_e' = K_e(\lambda), k_s' = K_s(\lambda)$, and $l = L(\lambda)$ solve the HH's DP problem (16).

(2). Given $\lambda = (s, z, q)$ and the form of the functions $W(\cdot), R_e(\cdot), R_s(\cdot)$, k_e, k_s, and l are the solutions to the firm's problem (17).

(3). The government's budget constraint is balanced every period.

$$T = \tau_k(r_e k_e + r_s k_s) + \tau_l w l$$

(4). Resource constraint holds.

$$c + i_e + i_s = zF(k_e, k_s, l).$$

(c) Reinterpret this one sector economy into a two-sector growth model with one sector for consumption good and investment in structure, another one for investment. Feel free to make your assumptions. Under what condition that this two-sector model is equivalent to our original economy?

Answer: Think about the following two-sector model:

$$\max E\sum_{t=0}^{\infty}\beta^t u(c_t, l_t)$$

$$\begin{aligned} & s.t. \\ c + i_s &= zk_{1e}^{\alpha_e}k_{1s}^{\alpha_s}l_1^{1-\alpha_e-\alpha_s} \\ i_e &= zqk_{2e}^{\beta_e}k_{2s}^{\beta_s}l_2^{1-\beta_e-\beta_s} \\ k_{1e} + k_{2e} &= k_e \\ k_{1s} + k_{2s} &= k_s \\ k_s' &= (1-\delta_s)k_s + i_s \\ k_e' &= (1-\delta_e)k_e + i_e q \\ l_1 + l_2 &= l \end{aligned}$$

It is easy to show that when $\alpha_e = \beta_e$ and $\alpha_s = \beta_s$, this economy is essentially equivalent to our original economy since the capital-labor ratios will be equal in the two sectors due to non-arbitrage condition. Thus our two-sector economy will give the same prices (w, r_e, r_s) as in the original economy.

4. Consider the two (human and physical) capital version of the endogenous growth model where the representative consumer sloves

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_t)$$

subject to

$$\begin{aligned}
\sum_t p_t \left[c_t + x_{kt} + x_{ht}\right] &\leq \sum_t p_t \left[(1-\tau_{kt}) r_t k_t + (1-\tau_{nt}) w_t n_t h_t\right] \\
k_{t+1} &\leq (1-\delta_k) k_t + x_{kt} \\
h_{t+1} &\leq (1-\delta_h) k_t + x_{ht} \\
c_t &\geq 0, 0 \leq n_t \leq 1,\ h_0 \text{ and } k_0 \text{ are given}
\end{aligned}$$

Where all variables have their standard interpretations. "effective labor supply" is given by $n_t h_t$. τ_{kt} is the tax rate on capital income and τ_{nt} is the tax rate on labor income. Firms face static problems maximizing profilts under the production function and resource constraint

$$c_t + x_{kt} + x_{ht} + g_t \leq A k_t^{\alpha} (n_t h_t)^{1-\alpha}.$$

Assume that $u(c, l) = c^{1-\sigma}/(1-\sigma)$ so that labor supply is fixed. Assume that $\delta_k = \delta_h = \delta$. Assume that all equilibrium quantities are strictly interior. (15 points)

(a) Define a Arrow-Debreu Competitive Equilibrium.

Answer: To be skipped.

(b) What is the equilibrium value for the ratio of h to k in the absence of taxes and government spending? How is this ratio affected by the presence of capital and labor income taxes?

Answer: When there is no tax,

$$\frac{k}{h} = \frac{\alpha}{1-\alpha}.$$

When there are taxes on both capital and labor income,

$$\frac{k}{h} = \frac{\alpha}{1-\alpha} \frac{1-\tau_{kt}}{1-\tau_{nt}}.$$

5. (*Taxation in AK model*) Consider a standard AK model. But now suppose there is a government that imposes tax every period to finance its expenditure g, which is a constant fraction of η of output every period. We assume throught the tax rate is constant at τ, and that the government maintains a balanced budget each period. We also assmume that after purchasing government expenditure g_t every period using tax revenue, the government throws away the expenditure goods g_t into the sea (forget about environmentalism at this moment:-)). The utility function is standard CRRA. (25 points)

(a) Formulate the social planner's problem.

Answer: The palnner's problem is

$$\max \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

$$s.t.$$

$$\begin{aligned} c_t + i_t + g_t &= y_t = Ak_t \\ k_{t+1} &= (1-\delta)k_t + i_t \\ c_t, k_{t+1} &\geq 0, \ k_0 \text{ given} \end{aligned}$$

(b) Define a decentralized Arrow-Debreu competitive equilibrium for this economy.

Answer: The ADE of this economy is a price system $\{p_t, r_t\}$, an allocation $\{c_t, i_t, k_t^s\}$ for HH, and an allocation $\{y_t, k_t^d\}$ for the firm such that:

(1). Given prices $\{p_t, r_t\}$, allocation $\{c_t, i_t, k_t^s\}$ solves HH's problem

$$\max \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

$$s.t.$$

$$\begin{aligned} \sum_{t=0}^{\infty} p_t(c_t + k_{t+1}^s - (1-\delta)k_t^s) &\leq \sum_{t=0}^{\infty} p_t r_t (1-\tau) k_t^s \\ c_t, k_{t+1}^s &\geq 0 \end{aligned}$$

(2). Given prices $\{p_t, r_t\}$, allocation $\{y_t, k_t^d\}$ solves the firm's problem

$$\max p_t(y_t - r_t k_t^d)$$

$$s.t.$$

$$\begin{aligned} y_t &\leq Ak_t^d \\ k_t^d &\geq 0 \end{aligned}$$

(3). Government balances the budget period by period.

$$g_t = \eta y_t = \tau r_t k_t$$

(3). Markets clear.

$$\begin{aligned} k_t^d &= k_t^s = k_t \\ c_t + i_t + g_t &= Ak_t \end{aligned}$$

(c) What is the realized level of utility if this government spending is financed through taxes on capital income (and all other taxes are zero)?

Answer: From the firm's FOC, we have $r_t = A$. Then the government BC tells us $\tau = \eta$.

Working on HH's problem, we will end up with the EE for capital stock

$$\begin{aligned} \gamma_c \equiv \frac{c_{t+1}}{c_t} &= \{\beta(r_{t+1}(1-\tau) + 1 - \delta)\}^{1/\sigma} \\ &= \{\beta(A(1-\eta) + 1 - \delta)\}^{1/\sigma}. \end{aligned}$$

Given

$$c_0 = y_0 - g_0 - i_0 = (A(1-\eta) + 1 - \delta)k_0 - k_1$$

And notice in AK model $\gamma_k = \gamma_c$, therefore, $k_1 = \gamma_c k_0$,

$$c_0 = \left[(A(1-\eta) + 1 - \delta) - \{\beta(A(1-\eta) + 1 - \delta)\}^{1/\sigma}\right] k_0.$$

The realized level of lifetime utility is

$$\begin{aligned}
\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} &= \sum_{t=0}^{\infty} \beta^t \frac{(\gamma_c^t c_0)^{1-\sigma}}{1-\sigma} = \frac{c_0^{1-\sigma}}{1-\sigma} \sum_{t=0}^{\infty} (\beta \gamma_c^{1-\sigma})^t \\
&= \frac{c_0^{1-\sigma}}{1-\sigma} \frac{1}{1-\beta\gamma_c^{1-\sigma}}.
\end{aligned}$$

(d) What is the realized level of utility if this government spending is financed through taxes on investment (and all other taxes are zero)?

Answer: Now the HH's problem changes to

$$\begin{aligned}
& \max \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} \\
& s.t. \\
\sum_{t=0}^{\infty} p_t(c_t + (1+\tau)(k_{t+1}^s - (1-\delta)k_t^s)) &\leq \sum_{t=0}^{\infty} p_t r_t k_t^s \\
c_t, k_{t+1}^s &\geq 0
\end{aligned}$$

And the government's BC changes to

$$g_t = \eta y_t = \tau i_t = \tau(k_{t+1} - (1-\delta)k_t)$$

It implies

$$\eta A k_t = \tau(k_{t+1} - (1-\delta)k_t)$$

Divide both sides by k_t, we have

$$\gamma_k \equiv \frac{k_{t+1}}{k_t} = \frac{1}{\tau}(\eta A + 1 - \delta).$$

From the resource constraint (market clearing condition), we have

$$\begin{aligned}
c_t + i_t + g_t &= y_t = A k_t \\
c_t + (1+\tau) i_t &= A k_t \\
c_t + (1+\tau)(k_{t+1} - (1-\delta)k_t) &= A k_t
\end{aligned}$$

Divide both sides by k_t, we obtain

$$\frac{c_t}{k_t} = (A + (1+\tau)(1-\delta)) - (1+\tau)\gamma_k = \text{constant}$$

which implies

$$\gamma_c = \gamma_k.$$

The HH's FOCs lead to the EE

$$\gamma_c \equiv \frac{c_{t+1}}{c_t} = \left\{\beta(\frac{A}{1+\tau} + 1 - \delta)\right\}^{1/\sigma}$$

Along the BGP, we must have

$$\gamma_c = \left\{\beta(\frac{A}{1+\tau} + 1 - \delta)\right\}^{1/\sigma} = \frac{1}{\tau}(\eta A + 1 - \delta) = \gamma_k$$

This determines a relationship between tax rate τ and government spending share η

$$\tau = \tau(\eta) = A(\eta A + 1 - \delta)/(\beta(\frac{A}{1+\tau} + 1 - \delta))^{1/\sigma}.$$

And the growth rate of consumption is

$$\gamma_c = \left\{\beta(\frac{A}{1+\tau(\eta)} + 1 - \delta)\right\}^{1/\sigma}.$$

Given

$$\begin{aligned} c_0 &= y_0 - g_0 - i_0 = Ak_0 - \eta A k_0 - i_0 = A(1-\eta)k_0 - (k_1 - (1-\delta)k_0) \\ &= [A(1-\eta) + 1 - \delta - \gamma_k]k_0 \\ &= [A(1-\eta) + 1 - \delta - \frac{1}{\tau(\eta)}(\eta A + 1 - \delta)]k_0. \end{aligned}$$

The realized level of lifetime utility then is

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} &= \sum_{t=0}^{\infty} \beta^t \frac{(\gamma_c^t c_0)^{1-\sigma}}{1-\sigma} = \frac{c_0^{1-\sigma}}{1-\sigma} \sum_{t=0}^{\infty} (\beta \gamma_c^{1-\sigma})^t \\ &= \frac{c_0^{1-\sigma}}{1-\sigma} \frac{1}{1-\beta\gamma_c^{1-\sigma}}. \end{aligned}$$

Notice that in this case, $\tau \neq \eta$, so the realized level of utility is different from the one in part (c).

(e) What is the realized level of utility if this government spending is financed through taxes on purchases of consumption goods (and all other taxes are zero)?

Answer: Now the HH's problem changes to

$$\max \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

$$s.t.$$

$$\begin{aligned} \sum_{t=0}^{\infty} p_t((1+\tau)c_t + (k_{t+1}^s - (1-\delta)k_t^s)) &\leq \sum_{t=0}^{\infty} p_t r_t k_t^s \\ c_t, k_{t+1}^s &\geq 0 \end{aligned}$$

	tax on capital income	tax on investment	tax on consumption
γ_c	$\{\beta(A(1-\eta)+1-\delta)\}^{1/\sigma}$	$\left\{\beta(\frac{A}{1+\tau(\eta)}+1-\delta)\right\}^{1/\sigma}$	$\{\beta(A+1-\delta)\}^{1/\sigma}$
c_0	$\begin{bmatrix} (A(1-\eta)+1-\delta) \\ -(\beta(A(1-\eta)+1-\delta))^{1/\sigma} \end{bmatrix} k_0$	$\begin{bmatrix} A(1-\eta)+1-\delta \\ -\frac{1}{\tau(\eta)}(\eta A+1-\delta) \end{bmatrix} k_0$	$\begin{bmatrix} A(1-\eta)+1-\delta \\ -(\beta(A+1-\delta))^{1/\sigma} \end{bmatrix} k_0$

And the government's BC changes to

$$g_t = \eta y_t = \eta A k_t = \tau c_t$$

Therefore, we have

$$\frac{k_t}{c_t} = \frac{\tau}{\eta A}$$

This implies

$$\gamma_c = \gamma_k.$$

The EE for the HH is

$$\begin{aligned} \gamma_c &\equiv \frac{c_{t+1}}{c_t} = \{\beta(r_{t+1}+1-\delta)\}^{1/\sigma} \\ &= \{\beta(A+1-\delta)\}^{1/\sigma}. \end{aligned}$$

Given

$$\begin{aligned} c_0 &= y_0 - g_0 - i_0 = Ak_0 - \eta A k_0 - (k_1 - (1-\delta)k_0) \\ &= ((1-\eta)A + 1 - \delta - \gamma_k)k_0 \\ &= ((1-\eta)A + 1 - \delta - (\beta(A+1-\delta))^{1/\sigma})k_0. \end{aligned}$$

The realized level of lifetime utility then is

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} &= \sum_{t=0}^{\infty} \beta^t \frac{(\gamma_c^t c_0)^{1-\sigma}}{1-\sigma} = \frac{c_0^{1-\sigma}}{1-\sigma} \sum_{t=0}^{\infty} (\beta \gamma_c^{1-\sigma})^t \\ &= \frac{c_0^{1-\sigma}}{1-\sigma} \frac{1}{1-\beta\gamma_c^{1-\sigma}}. \end{aligned}$$

(f) Which taxation scheme is the best for consumers?

Answer: The utility level depends on c_0 and γ_c. See Table below for the comparison.

Obviously, tax on consumption has the highest growth rate γ_c. This is because the constant tax rate on consumption across time does not affect consumer's intertemporal marginal rate of substitution, thus giving least distortion.

6. (*Taxation in Uzawa-Lucas Model*) Consider the following planner's problem. (15 points)

$$
\begin{aligned}
& \max \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma} \\
& s.t. \\
C_t + I_{K,t} & \leq AK_t^{\alpha}(u_t H_t)^{1-\alpha} \\
I_{H,t} & \leq B(1-u_t)H_t \\
K_{t+1} & = (1-\delta)K_t + I_{K,t} \\
H_{t+1} & = (1-\delta)H_t + I_{H,t} \\
& K_0, H_0 \text{ given}
\end{aligned}
$$

(a) Rewrite this problem as that of a decentralized economy in which the accumulation of stock of human capital H is non-marketable. Define a ADE.

Answer: The ADE of this economy is a price system $\{p_t, r_t, w_t\}$, an allocation $\{C_t, I_{K,t}, u_t, K_t^s, H_t^s\}$ for HH, and an allocation $\{Y_t, K_t^d, H_t^d\}$ for the firm such that:
(1). Given prices $\{p_t, r_t, w_t\}$, allocation $\{C_t, I_{K,t}, u_t, K_t^s, H_t^s\}$ solves HH's problem

$$
\begin{aligned}
& \max \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma} \\
& s.t. \\
\sum_{t=0}^{\infty} p_t(C_t + I_{K,t}) & \leq \sum_{t=0}^{\infty} p_t(r_t K_t^s + w_t u_t H_t^s) \\
K_{t+1}^s & = (1-\delta)K_t^s + I_{K,t} \\
H_{t+1}^s & = (1-\delta)H_t^s + B(1-u_t)H_t^s
\end{aligned}
$$

(2). Given prices $\{p_t, r_t\}$, allocation $\{Y_t, K_t^d, H_t^d\}$ solves the firm's problem:

$$
\begin{aligned}
& \max p_t(Y_t - r_t K_t^d - w_t u_t H_t^d) \\
& s.t. \\
Y_t & \leq A(K_t^d)^{\alpha}(u_t H_t^d)^{1-\alpha}
\end{aligned}
$$

(3). Markets clear.

$$
\begin{aligned}
K_t^s & = K_t^d = K_t \\
H_t^s & = H_t^d = H_t \\
C_t + I_{K,t} & = AK_t^{\alpha}(u_t H_t)^{1-\alpha}.
\end{aligned}
$$

After tedious algebra, we end up with along BGP

$$\gamma_C = \gamma_K = \gamma_Y = \gamma_H = \gamma$$

And the growth rate γ is

$$\gamma = \{\beta(B + 1 - \delta)\}^{1/\sigma}.$$

(b) Suppose now government imposes a flat-rate tax on capital returns. Will this tax affect the growth rate in the economy?

Answer: Now the HH's problem changes to

$$\max \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma}$$

$$s.t.$$

$$\begin{aligned}
\sum_{t=0}^{\infty} p_t(C_t + K_{t+1}^s - (1-\delta)K_t^s) &\leq \sum_{t=0}^{\infty} p_t(r_t(1-\tau)K_t^s + w_t u_t H_t^s) \\
H_{t+1}^s &= (1-\delta)H_t^s + B(1-u_t)H_t^s
\end{aligned}$$

FOCs are

$$\begin{aligned}
C_t &: \beta^t C_t^{-\sigma} = \lambda p_t \\
K_{t+1} &: p_t = p_{t+1}[(1-\tau)r_{t+1} + 1 - \delta)] \\
H_{t+1} &: \mu_t = \lambda p_{t+1} w_{t+1} u_{t+1} + \mu_{t+1}[B(1-u_{t+1}) + 1 - \delta] \\
u_t &: \lambda p_t w_t H_t = \mu_t B H_t.
\end{aligned}$$

We end up with two EEs. EE for K is

$$\left(\frac{C_{t+1}}{C_t}\right)^{\sigma} = \beta[(1-\tau)r_{t+1} + 1 - \delta)]$$

And EE for H is

$$\frac{p_t}{p_{t+1}} = \frac{w_{t+1}}{w_t}(B + 1 - \delta).$$

From the firm's FOC, we have

$$\frac{w_{t+1}}{w_t} = \gamma_K^{\alpha} \gamma_H^{-\alpha}.$$

The two EEs imply that

$$(1-\tau)r_{t+1} + 1 - \delta = \gamma_K^{\alpha} \gamma_H^{-\alpha}(B + 1 - \delta)$$

Notice that from the firm's problem, we obtain

$$r_{t+1} = \alpha A(u^*)^{1-\alpha} \left(\frac{K_{t+1}}{H_{t+1}}\right)^{\alpha-1}.$$

This implies that $\frac{K_{t+1}}{H_{t+1}}$ = constant. Therefore, $\gamma_K = \gamma_H$. Now it is easy to show that along BGP

$$\gamma_C = \gamma_K = \gamma_Y = \gamma_H = \gamma$$

and

$$\gamma = \{\beta(B + 1 - \delta)\}^{1/\sigma}.$$

Therefore, the flat-rate tax on capital income does not affect the long-run growth rate in this economy. The reason is in this economy, the source of economic growth comes from the AH technology in the human capital sector. The tax on capital does not affect the accumulation of the human capital. Hence the growth rate is as same as before.

(c) How would you answer to part (b) change if the utility function is modified as

$$u(C,l) = \frac{(C^{\gamma} l^{1-\gamma})^{1-\sigma}}{1-\sigma}$$

where $l \in [0,1]$ is the fraction of time spent on leisure?

Answer: Now the HH's problem changes to

$$\begin{aligned} & \max \sum_{t=0}^{\infty} \beta^t \frac{(C_t^{\gamma} l_t^{1-\gamma})^{1-\sigma}}{1-\sigma} \\ & s.t. \\ \sum_{t=0}^{\infty} p_t(C_t + K_{t+1}^s - (1-\delta)K_t^s) \quad \leq & \sum_{t=0}^{\infty} p_t(r_t(1-\tau)K_t^s + w_t u_t H_t^s) \\ H_{t+1}^s \quad = & (1-\delta)H_t^s + B(1-u_t-l_t)H_t^s \end{aligned}$$

After tedious algebra, we can show that along the BGP, we still have

$$\begin{aligned} \gamma_C &= \gamma_K = \gamma_Y = \gamma_H = \gamma \\ u_t &= u^* \\ l_t &= l^* \end{aligned}$$

The growth rate for the economy without tax and with tax are same.

$$\gamma = \{\beta(B(1-l^*) + 1 - \delta)\}^{1/\sigma}.$$

7. (*Replicate the Equity Premium*) Read Mehra, R., and E. C. Prescott (1985): "The Equity Premium: A Puzzle," *Journal of Monetary Economics*, vol. 15, 145–162. Following Mehra and Prescott, write a Matlab code to replicate Figure 4 in their paper for CRRA coefficient $10 \geq \sigma \geq 0$ and discount rate $0.99 \geq \beta \geq 0.9$. (10 points)

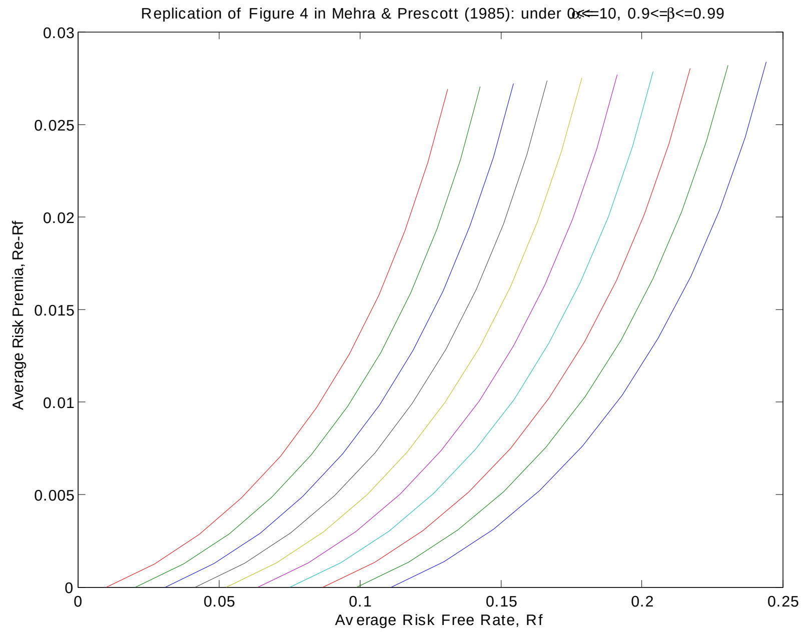
   Answer: We should end up with a graph looks like the following one.

8. (*Growth slowdowns and stock market crashes*) Consider a simple one-tree pure exchange economy. The only source of consumption is the fruit that grows on the tree. This fruit is called dividends by the tribe inhabiting this island. The stochastic process for dividend d_t is described as follows: If d_t is not equal to d_{t-1}, then $d_{t+1} = \gamma d_t$ with probability π, and $d_{t+1} = d_t$ with probability $1-\pi$. If in any pair of periods j and $j+1$, $d_j = d_{j+1}$, then for all $t > j$, $d_t = d_j$. In words, if not stopped, the process grows at a rate γ in every period. However, once it stops growing for one period, it remains constant forever after. Let $d_0 = 1$. We assume only the share purchased at time t can claim the dividend at time t. And we cannot purchase and sell the shares within the same period.

   Preferences over stochastic processes for consumption are given by

   $$E_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}.$$

   Assume that $1 > \sigma > 0, 0 < \beta < 1, \gamma > 1$ and $\beta\gamma^{1-\sigma} < 1$. (20 points)



(a) Define a SME in which shares to this tree are traded.

Answer: First let's define the HH's problem

$$\begin{aligned} & \max E_0 \sum_{t=0}^{\infty} \sum_{d^t} \beta^t u(c_t(d^t)) \\ & s.t. \\ c_t(d^t) + p_t(d^t)s_t(d^t) \leq\ & d_t s_t(d^t) + p_t(d^t)s_{t-1}(d^{t-1}) \\ c_t(d^t), s_t(d^t) \geq\ & 0, s_{-1} = 1 \end{aligned}$$

A SME in this economy is a price process $\{p_t(d^t)\}_{t=0}^{\infty}$, an allocation $\{c_t(d^t), s_t(d^t)\}_{t=0}^{\infty}$ such that:

(1). Given the price, $\{c_t(d^t), s_t(d^t)\}_{t=0}^{\infty}$ solves the HH's problem as above.

(2). Markets clear.

$$c_t(d^t) = d_t.$$

The FOCs lead to the following EE (asset pricing formula):

$$p_t = d_t + \beta E_t \left[\frac{u'(c_{t+1})}{u'(c_t)} p_{t+1} \right]$$

Imposing the equilibrium condition $c_t = d_t$, we have

$$p_t = d_t + \beta E_t \left[\frac{u'(d_{t+1})}{u'(d_t)} p_{t+1} \right].$$

(b) Display the equilibrium process for the price of shares in this tree p_t as a function of the history of dividends. Is the price process a Markov process in the sense that it depends just on the last period's dividends?

Answer: Following the lecture notes, it is easy to show that the share price p_t is an expected discounted stream of future dividends.

$$p_t = E_t \sum_{j=0}^{\infty} \beta^j \frac{u'(d_{t+j})}{u'(d_t)} d_{t+j}.$$

If the growth process is stopped since time t, i.e., $d_{t+j} = d_t, \forall j \geq 0$, we will have

$$p_t^s = \sum_{j=0}^{\infty} \beta^j d_t = d_t \sum_{j=0}^{\infty} \beta^j = \frac{d_t}{1-\beta}.$$

If the growth process is not stopped then two things can happen tomorrow. First, with probability π the economy grows. In this event $\frac{c_{t+1}}{c_t} = \gamma$ and the dividend of the tree is $d_{t+1} = \gamma d_t$. Second, with probability $1-\pi$ the economy stops growing. In this event $\frac{c_{t+1}}{c_t} = 1$ and the dividend of the tree is $d_{t+1} = d_t$. Let's guess in this case, the share price is a linear function of d_t, i.e., $p_t^g = p^g d_t$ where p^g is a constant. Thus, the (cum dividend) price of the tree at time t is

$$\begin{aligned} p^g d_t &= p_t^g = d_t + \beta \left\{ \pi (\frac{c_{t+1}}{c_t})^{-\sigma} p^g d_{t+1} + (1-\pi) p_{t+1}^s \right\} \\ &= d_t + \beta \left\{ \pi \gamma^{-\sigma} p^g \gamma d_t + (1-\pi) \frac{d_t}{1-\beta} \right\} \end{aligned}$$

We end up with

$$p^g = (1 - \pi\beta\gamma^{1-\sigma})^{-1} \left[1 + \frac{\beta(1-\pi)}{1-\beta} \right].$$

Thus our guess is right. We have

$$p_t^g = (1 - \pi\beta\gamma^{1-\sigma})^{-1} \left[1 + \frac{\beta(1-\pi)}{1-\beta} \right] d_t.$$

The price of the tree is Markov provided we expand the state to (d_{t-1}, d_t). Specifically, if $d_{t-1} \neq d_t$, we have $p_t = p_t^g$. Otherwise, we have $p_t = p_t^s$. In this sense, the price process is a Markov process because it depends just on the last period's dividends.

(c) Let T be the first period in which $d_{T-1} = d_T = \gamma^{(T-1)}$. Is $P_{T-1} > P_T$? Show conditions under which this is true. What is the economic intuition for this result? What does it say about stock market declines or crashes?

Answer: The question asks under what condition we will have $p_t^g > p_t^s$. It is equivalent to

$$p^g = (1 - \pi\beta\gamma^{1-\sigma})^{-1} \left[1 + \frac{\beta(1-\pi)}{1-\beta} \right] > \frac{1}{1-\beta} = p^s.$$

Again, this is equivalent to

$$\frac{1-\beta\pi}{1-\pi\beta\gamma^{1-\sigma}} > 1$$

in turn is equivalent to

$$\gamma > 1.$$

As long as there is economic growth, the stock price is also growing. But when the growth stops, the stock price drops from p_t^g to p_t^s. We can interpret this as stock market crashes.

(d) If this model is correct, what does it say about the behavior of the aggregate value of the stock market in economies that switched from high to low growth (e.g., Japan)?

Answer: See answer to part (c).

9. (*Pricing bonds with different maturities*) Consider a Lucas' tree economy in which the representative consumer maximizes

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

subject to

$$c_t + \sum_{i=1}^{n} L_{i,t} + p_t s_{t+1} \leq (p_t + d_t)s_t + \sum_{j=1}^{n} R_{j,t-j} L_{j,t-j}.$$

and given $d_0, s_0, \{L_{1,-1}R_{1,-1}, L_{2,-2}R_{2,-2}, ..., L_{n,-n}R_{n,-n}\}$. L_{it} is the quantity of i-period pure discount bonds purchased at time t, with a certain payoff of $R_{i,t}$ per period in i periods. s_t is the number of shares in tree purchased at time $t-1$, and p_t is the tree price at time t. There is exactly one tree per capita in the economy.

Dividends d_t grow stochastically, where $d_{t+1} = \gamma_t d_t$, and the evolution of γ follows a stationary Markov chain. Agents learn the realization of γ_t at beginning of time $t+1$. The dividend is perishable, and can be used only for consumption. (20 points)

(a) Write down the Bellman equation for this economy. What are the state variables? What are the choice variables?

Answer: The Bellman equation is

$$\begin{aligned}
v(d_t, s_t, \gamma_{t-1}, \{L_{j,t-j}\}_{j=1}^n) &= \max_{c_t, s_{t+1}, \sum_{i=1}^n L_{i,t}} \{u(c_t) + \beta E_t v(d_{t+1}, s_{t+1}, \gamma_t, \{L_{j,t+1-j}\}_{j=1}^n)\} \\
&\quad s.t. \\
c_t + \sum_{i=1}^{n} L_{i,t} + p_t s_{t+1} &\leq (p_t + d_t)s_t + \sum_{j=1}^{n} R_{j,t-j} L_{j,t-j} \\
c_t, s_{t+1} &\geq 0 \\
&\quad d_0, s_0, \gamma_{-1}, \{L_{1,-1}R_{1,-1}, L_{2,-2}R_{2,-2}, ..., L_{n,-n}R_{n,-n}\} \text{ given}
\end{aligned}$$

(b) Write down the Lagrangian for this problem. Derive the first order necessary conditions and market claering conditions.

Answer: The Lagrangian is

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t) + \lambda_t \left[(p_t + d_t)s_t + \sum_{j=1}^{n} R_{j,t-j} L_{j,t-j} - c_t - \sum_{i=1}^{n} L_{i,t} - p_t s_{t+1} \right] \right\}$$

FOCs are

$$\begin{aligned} c_t &: c_t^{-\sigma} = \lambda_t \\ s_{t+1} &: \lambda_t p_t = \beta E_t \lambda_{t+1}(p_{t+1} + d_{t+1}) \\ L_{i,t} &: \lambda_t = \beta^i E_t \lambda_{t+i} R_{i,t} \end{aligned}$$

Combining FOCs, we have the usual asset pricing formula for the stock

$$p_t = \beta E_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} (p_{t+1} + d_{t+1}) \right] \tag{18}$$

and the asset pricing formula for the i-period bond

$$R_{i,t}^{-1} = \beta^i E_t \left[\left(\frac{c_{t+i}}{c_t} \right)^{-\sigma} \right]. \tag{19}$$

The market clearing conditions are

$$\begin{aligned} c_t &= d_t \\ s_t &= 1 \\ L_{i,t} &= 0, \forall i, \forall t \end{aligned}$$

(c) Derive an expression for the equilibrium returns R_{3t}. Show that this expression is independent of d_t.

Answer: From the FOCs, we have EEs for 1-, 2-, and 3-period bonds:

$$\begin{aligned} E_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} R_{1,t} \right] &= 1 \\ E_t \left[\beta^2 \left(\frac{c_{t+2}}{c_t} \right)^{-\sigma} R_{2,t} \right] &= 1 \\ E_t \left[\beta^3 \left(\frac{c_{t+3}}{c_t} \right)^{-\sigma} R_{3,t} \right] &= 1 \end{aligned}$$

Notice that $\left(\frac{c_{t+3}}{c_t} \right)^{-\sigma} = \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} \cdot \left(\frac{c_{t+2}}{c_{t+1}} \right)^{-\sigma} \cdot \left(\frac{c_{t+3}}{c_{t+2}} \right)^{-\sigma}$. Hence we have

$$\begin{aligned} E_t \left[\beta^3 \left(\frac{c_{t+3}}{c_t} \right)^{-\sigma} R_{3,t} \right] &= E_t \left[\beta^3 \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} \cdot \left(\frac{c_{t+2}}{c_{t+1}} \right)^{-\sigma} \cdot \left(\frac{c_{t+3}}{c_{t+2}} \right)^{-\sigma} R_{3,t} \right] \\ &= E_t \left[\beta^3 \left(\frac{d_{t+1}}{d_t} \right)^{-\sigma} \cdot \left(\frac{d_{t+2}}{d_{t+1}} \right)^{-\sigma} \cdot \left(\frac{d_{t+3}}{d_{t+2}} \right)^{-\sigma} R_{3,t} \right] \\ &= E_t \left[\beta^3 \left(\gamma_t \gamma_{t+1} \gamma_{t+2} \right)^{-\sigma} R_{3,t} \right] = 1 \end{aligned}$$

Thus it is easy to see that $R_{3,t}$ is independent of d_t.

(d) Suppose the government announces that at time 0 beginning in three years, it will introduce a permanent tax on dividends at a constant proportional rate τ (i.e., the stockholder receives $(1-\tau)d_t$ each period thereafter). The proceeds of the tax will be distributed in a lump-sum payment back to the agent, whether or not he owns the stock. What is the effect on the stock price at time 0 (write down an expression for the price)? What happens to the short-term interest rates? What happens to long-term (greater than three years) interest rates?

Answer: The consumer's BC after $t \geq 3$ changes to

$$c_t + \sum_{i=1}^{n} L_{i,t} + p_t s_{t+1} \leq (p_t + (1-\tau)d_t)s_t + \sum_{j=1}^{n} R_{j,t-j}L_{j,t-j} + \tau d_t, \forall t \geq 3.$$

Following the lecture notes, it is easy to show that if there is no tax

$$\begin{aligned}
p_0 &= E_0 \sum_{j=1}^{\infty} \beta^j (\gamma_{j-1}\gamma_{j-2}...\gamma_0)^{-\sigma} d_j \\
&= E_0 \sum_{j=1}^{\infty} \beta^j (\gamma_{j-1}\gamma_{j-2}...\gamma_0)^{-\sigma} (\gamma_{j-1}\gamma_{j-2}...\gamma_0) d_0 \\
&= d_0 E_0 \sum_{j=1}^{\infty} \beta^j (\gamma_{j-1}\gamma_{j-2}...\gamma_0)^{1-\sigma}
\end{aligned}$$

If there is tax, the price of shares at time 0 changes to

$$\begin{aligned}
p_0 &= E_0 \left[\beta\gamma_0^{1-\sigma} d_0 + \beta^2 (\gamma_1\gamma_0)^{1-\sigma} d_0 + \sum_{t=1}^{\infty} \beta^t (\prod_{s=0}^{t-1} \gamma_s)^{1-\sigma} (1-\tau) d_0 \right] \\
&= d_0 E_0 \left[\beta\gamma_0^{1-\sigma} + \beta^2 (\gamma_1\gamma_0)^{1-\sigma} + \sum_{t=1}^{\infty} \beta^t (\prod_{s=0}^{t-1} \gamma_s)^{1-\sigma} (1-\tau) \right].
\end{aligned}$$

Thus with tax, the stock price will decline. It is not surprising because the lecture notes showed that the share price is an expected discounted stream of future dividends. Now with tax, the stream of future dividends goes down, so does the stock price.

But for interest rates, it is a different story since in the equilibrium, the representative consumer still consumes the same amount $c_t = (1-\tau)d_t + \tau d_t = d_t$. EE for bonds (19) does not change. Thus the interest rates (short-term and long-term) will not change at all.

10. (*Two-tree Model) Consider a version of "Lucas tree" economy in which there are two types of trees. Both types of trees are perfectly durable; a type-i* tree($i = 1, 2$) yields a random amount of dividends equal to d_{it} in period t. Assume that $\{d_{1t}\}_{t=0}^{\infty}$ and $\{d_{2t}\}_{t=0}^{\infty}$ are i.i.d. sequences of random variables and that d_{1t} and d_{2s} are statistically independent for all t and s. In addition, for $i = 1, 2$, assume that $d_{it} = d_L$ with probability π_i and equals $d_H > d_L$ with probability $1 - \pi_i$. the economy is populated by a continuum

(of measure one) of identical consumers with preferences over consumption streams given by:

$$E_0 \sum_{t=0}^{\infty} \beta^t \log(c_t)$$

where c_t is consumption in period t. In period 0, each consumer wons one tree of each type. Dividends are non-storable and are the only source of consumption goods. There are competitive markets in which consumers can buy and sell both types of trees.

(a) Define a sequential competitive equilibrium in which the only assets that consumers trade are the two (types of) trees.

Answer: Skipped

(b) Find an algebraic expression for the equilibrium price of a type-1 tree (measured in terms of today's consumption goods), assuming that the dividends of both types of trees are equal to d_l today. Your expression should depend only on primitives (i.e., on the parameters describing preferences and technology).

Answer: Follow the model or lecture notes:

$$\begin{aligned} p_{1t} &= E_t \sum_{s=1}^{\infty} \beta^s \frac{u'(c_{t+s})}{u'(c_t)} d_{1t+s} \\ &= E_t \sum_{s=1}^{\infty} \beta^s \frac{d_{1t}+d_{2t}}{d_{1t+s}+d_{2t+s}} d_{1t+s} \\ &= (d_{1t}+d_{2t}) \sum_{s=1}^{\infty} \beta^s E(\frac{d_{1t+s}}{d_{1t+s}+d_{2t+s}}) \end{aligned}$$

So there are 4 states of the world:

State 1: $(d_{l,}d_l)$ with probability of $\pi_1\pi_2$, the outcome of $\frac{d_{1t+s}}{d_{1t+s}+d_{2t+s}}$ is $\frac{d_l}{d_l+d_l} = 1/2$

State 2: $(d_{l,}d_h)$ with probability of $\pi_1(1-\pi_2)$, the outcome of $\frac{d_{1t+s}}{d_{1t+s}+d_{2t+s}}$ is $\frac{d_l}{d_l+d_h}$

State 3: $(d_{h,}d_l)$ with probability of $(1-\pi_1)\pi_2$, the outcome of $\frac{d_{1t+s}}{d_{1t+s}+d_{2t+s}}$ is $\frac{d_h}{d_h+d_l}$

State 4: $(d_{h,}d_h)$ with probability of $(1-\pi_1)(1-\pi_2)$, the outcome of $\frac{d_{1t+s}}{d_{1t+s}+d_{2t+s}}$ is $\frac{d_h}{d_h+d_h} = 1/2$

So

$$E(\frac{d_{1t+s}}{d_{1t+s}+d_{2t+s}}) = \frac{1}{2}(\pi_1\pi_2 + (1-\pi_1)(1-\pi_2)) + \pi_1(1-\pi_2)\frac{d_l}{d_l+d_h} + (1-\pi_1)\pi_2\frac{d_h}{d_h+d_l}$$

then we have

$$p_1 = 2d_l\frac{\beta}{1-\beta}\left[\frac{1}{2}(\pi_1\pi_2 + (1-\pi_1)(1-\pi_2)) + \pi_1(1-\pi_2)\frac{d_l}{d_l+d_h} + (1-\pi_1)\pi_2\frac{d_h}{d_h+d_l}\right]$$

for the case of both dividends equal to d_l today.

(c) How many Arrow securities are there in this economy? Express the prices of these securities in terms of primitives.

Answer:

$$\begin{aligned} & \beta \frac{u'(c_{t+1})}{u'(c_t)} p(d_{1t+1}, d_{2t+1}) \\ = \quad & \beta \frac{d_{1t} + d_{2t}}{d_{1t+1} + d_{2t+1}} p(d_{1t+1}) p(d_{2t+1}) \end{aligned}$$

$p(d_{1t+1}) p(d_{2t+1})$ here do not depend on d_{1t} or d_{2t}
then the arrow securities' prices are:

$$\begin{aligned} & \beta \pi_1 \pi_2 \frac{d_{1t} + d_{2t}}{2d_l} \\ & \beta \pi_1 (1 - \pi_2) \frac{d_{1t} + d_{2t}}{d_l + d_h} \\ & \beta (1 - \pi_1) \pi_2 \frac{d_{1t} + d_{2t}}{d_l + d_h} \\ & \beta (1 - \pi_1)(1 - \pi_2) \frac{d_{1t} + d_{2t}}{2d_h} \end{aligned}$$

corresponding to each 4 states.

(d) Use your answer from part (c) to find the price (expressed in terms of today's consumption goods) of an asset that pays one unit of the consumption good in the next period if the dividends of the two trees (in the next period) are not equal to each other and pays zero otherwise.

Answer: the price here will be

$$\beta[\pi_1 (1 - \pi_2) + (1 - \pi_1) \pi_2] \frac{d_{1t} + d_{2t}}{d_l + d_h}$$