ECON607 Fall 2010
University of Hawaii
Professor Hui He
Assignment 3

The due date for this assignment is Thursday, November 11.

1. (*Home production*) Consider an economy with infinitely lived individuals. Individuals maximize

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \{\ln c_t + \gamma \ln(1-l_t)\}$$

where consumption is a aggregator defined by

$$c_t = [\alpha c_{m,t}^{\eta} + (1-\alpha) c_{h,t}^{\eta}]^{\frac{1}{\eta}}$$

$c_{m,t}$ is the consumption of market-produced goods, $c_{h,t}$ is consumption of home-produced goods.

The model has two technologies, one for market production and one for home production:

$$\begin{aligned} f(z_{m,t}, k_{m,t}, l_{m,t}) &= e^{z_{m,t}} k_{m,t}^{\theta} l_{m,t}^{1-\theta} \\ f(z_{h,t}, k_{h,t}, l_{h,t}) &= e^{z_{h,t}} k_{h,t}^{\nu} l_{h,t}^{1-\nu}. \end{aligned}$$

where θ and ν are the capital share parameters for the two technologies, respectively. The two technology shocks follow the processes:

$$\begin{aligned} z_{m,t} &= \rho z_{m,t-1} + \varepsilon_{m,t} \\ z_{h,t} &= \rho z_{h,t-1} + \varepsilon_{h,t} \end{aligned}$$

where the two innovations $\varepsilon_{m,t}$ and $\varepsilon_{h,t}$ are normally distributed with zero means and standard deviations σ_m and σ_h; have a contemporaneous correlation $\varphi = cov(\sigma_m, \sigma_h)$; and are independent over time. In each period, a capital constraint holds

$$k_t = k_{m,t} + k_{h,t}.$$

The law of motion for the aggregate capital stock is

$$k_{t+1} = (1-\delta) k_t + i_t.$$

We also have following time constraint

$$l_{m,t} + l_{h,t} = l_t.$$

Finally the resource constraints for two sectors

$$\begin{aligned} c_{m,t} + i_t &= f(z_{m,t}, k_{m,t}, l_{m,t}) \\ c_{h,t} &= f(z_{h,t}, k_{h,t}, l_{h,t}). \end{aligned}$$

(a) Write down the Bellman equation for this problem. Be clear what are the state variables, what are control variables.

(b) Derive the first order conditions of this economy and interpret their economic implications.

2. (*Two-sector model of endogenous growth*) Consider the following two-sector growth model:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\begin{aligned} & s.t. \\ c_t &\leq Bk_{1t}^{\alpha} n_t^{1-\alpha} \\ x_t &\leq Ak_{2t} \\ k_{t+1} &= (1-\delta)k_t + x_t \\ k_{1t} + k_{2t} &= k_t \\ n_t &= 1, c_t, x_t \geq 0 \end{aligned}$$

Assume

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \sigma \geq 0$$

(a) Show that this problem is "homogeneous"–that is, doubling k_0 doubles the entire time path of optimal capital stocks. What does this do to the optimal path of k_1's, k_2's, and c's?

(b) Show that the value function for this problem is homogeneous of degree $1-\sigma$, i.e., $v(\lambda k) = \lambda^{1-\sigma} v(k)$.

(c) Use your results from part (a) and (b) to derive the time path for the solution (BGP) to this planner's problem.

(d) Define a decentralized Arrow-Debreu equilibrium for this economy.

(e) Solve the ADE. Is your result identical to the social planner's problem?

3. (*Investment-Specific Technological Change, Greenwood, Hercowitz and Krusell (AER, 1997)*) Consider the following infinite-horizon growth model

$$\max E \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

$$\begin{aligned} & s.t. \\ y &= zF(k_e, k_s, l) = zk_e^{\alpha_e} k_s^{\alpha_s} l^{1-\alpha_e-\alpha_s} \\ c_t + i_{e,t} + i_{s,t} &= y_t \\ k_s' &= (1-\delta_s)k_s + i_s \\ k_e' &= (1-\delta_s)k_e + i_e q \\ c_t, k_s', k_e' &\geq 0, 0 \leq l_t \leq 1 \end{aligned}$$

where $0 < \alpha_e, \alpha_s, \alpha_e + \alpha_s < 1, 0 < \delta_e, \delta_s < 1$. And we assume

$$u(c) = \theta \ln c + (1 - \theta) \ln(1 - l), 0 < \theta < 1$$

In this economy, k_e is the capital equipment, k_s is the capital structure. Accordingly, i_e is the investment in equipment, i_s is the investment in structure. Notice that there are two types of technological change in this economy: one is the technological change in total factor productivity (TFP) z, another one is technological change only happens in investment, called "*Investment-Specific Technological Change*" (ISTC), denoted by q. Suppose that z and q grow at the (gross) rates γ_z and γ_q, and let $z_t = \gamma_z^t$ and $q_t = \gamma_q^t$.

(a) Define a balanced growth path for this economy. Especially derive the growth rates of output, capital equipment, capital structure, consumption, investment in equipment, and investment in structure along BGP.

(b) Suppose there is a government that imposes tax on capital income at a tax rate τ_k and labor income at a rate τ, and the tax revenue raised by the government in each period is rebated back to agents in the form of lump-sum transfer payments T, define formally a recursive competitive equilibrium (RCE) for this economy.

(c) Reinterpret this one sector economy into a two-sector growth model with one sector for consumption good and investment in structure, another one for investment in equipment. Feel free to make your assumptions. Under what condition that this two-sector model is equivalent to our original economy?

4. Consider the two (human and physical) capital version of the endogenous growth model where the representative consumer sloves

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_t)$$

subject to

$$\begin{aligned}
\sum_t p_t \left[c_t + x_{kt} + x_{ht}\right] &\leq \sum_t p_t \left[(1 - \tau_{kt}) r_t k_t + (1 - \tau_{nt}) w_t n_t h_t\right] \\
k_{t+1} &\leq (1 - \delta_k) k_t + x_{kt} \\
h_{t+1} &\leq (1 - \delta_h) k_t + x_{ht} \\
c_t &\geq 0, 0 \leq n_t \leq 1, \ h_0 \text{ and } k_0 \text{ are given}
\end{aligned}$$

Where all variables have their standard interpretations. "effective labor supply" is given by $n_t h_t$. τ_{kt} is the tax rate on capital income and τ_{nt} is the tax rate on labor income. Firms face static problems maximizing profilts under the production function and resource constraint

$$c_t + x_{kt} + x_{ht} + g_t \leq A k_t^{\alpha} (n_t h_t)^{1-\alpha}.$$

Assume that $u(c, l) = c^{1-\sigma}/(1 - \sigma)$ so that labor supply is fixed. Assume that $\delta_k = \delta_h = \delta$. Assume that all equilibrium quantities are strictly interior.

(a) Define a Arrow-Debreu Competitive Equilibrium.

(b) What is the equilibrium value for the ratio of h to k in the absence of taxes and government spending? How is this ratio affected by the presence of capital and labor income taxes?

5. (*Taxation in AK model*) Consider a standard AK model. But now suppose there is a government that imposes tax every period to finance its expenditure g, which is a constant fraction of η of output every period. We assume throught the tax rate is constant at τ, and that the government maintains a balanced budget each period. We also assmume that after purchasing government expenditure g_t every period using tax revenue, the government throws away the expenditure goods g_t into the sea (forget about environmentalism at this moment:-)). The utility function is standard CRRA.

   (a) Formulate the social planner's problem.

   (b) Define a decentralized Arrow-Debreu competitive equilibrium for this economy.

   (c) What is the realized level of utility if this government spending is financed through taxes on capital income (and all other taxes are zero)?

   (d) What is the realized level of utility if this government spending is financed through taxes on investment (and all other taxes are zero)?

   (e) What is the realized level of utility if this government spending is financed through taxes on purchases of consumption goods (and all other taxes are zero)?

   (f) Which taxation scheme is the best for consumers?

6. (*Taxation in Uzawa-Lucas Model*) Consider the following planner's problem:

$$\max \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma}$$

$$\begin{aligned} & && s.t. \\ C_t + I_{K,t} &\leq && AK_t^{\alpha}(u_t H_t)^{1-\alpha} \\ I_{H,t} &\leq && B(1-u_t)H_t \\ K_{t+1} &= && (1-\delta)K_t + I_{K,t} \\ H_{t+1} &= && (1-\delta)H_t + I_{H,t} \\ & && K_0, H_0 \text{ given} \end{aligned}$$

   (a) Rewrite this problem as that of a decentralized economy in which the accumulation of stock of human capital H is non-marketable. Define a ADE.

   (b) Suppose now government imposes a flat-rate tax on capital returns. Will this tax affect the growth rate in the economy?

   (c) How would you answer to part (b) change if the utility function is modified as

$$u(C,l) = \frac{(C^{\gamma} l^{1-\gamma})^{1-\sigma}}{1-\sigma}$$

   where $l \in [0,1]$ is the fraction of time spent on leisure?

7. (*Replicate the Equity Premium*) Read Mehra, R., and E. C. Prescott (1985): "The Equity Premium: A Puzzle," *Journal of Monetary Economics*, vol. 15, 145–162. Following Mehra and Prescott, write a Matlab code to replicate Figure 4 in their paper for CRRA coefficient $10 \geq \sigma \geq 0$ and discount rate $0.999 \geq \beta \geq 0.9$.

8. (*Growth slowdowns and stock market crashes*) Consider a simple one-tree pure exchange economy. The only source of consumption is the fruit that grows on the tree. This fruit is called dividends by the tribe inhabiting this island. The stochastic process for dividend d_t is described as follows: If d_t is not equal to d_{t-1}, then $d_{t+1} = \gamma d_t$ with probability π, and $d_{t+1} = d_t$ with probability $1 - \pi$. If in any pair of periods j and $j+1$, $d_j = d_{j+1}$, then for all $t > j$, $d_t = d_j$. In words, if not stopped, the process grows at a rate γ in every period. However, once it stops growing for one period, it remains constant forever after. Let $d_0 = 1$. We assume only the share purchased at time t can claim the dividend at time t. And we cannot purchase and sell the shares within the same period.

   Preferences over stochastic processes for consumption are given by

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}.$$

   Assume that $1 > \sigma > 0, 0 < \beta < 1, \gamma > 1$ and $\beta\gamma^{1-\sigma} < 1$.

   (a) Define a SME in which shares to this tree are traded.

   (b) Display the equilibrium process for the price of shares in this tree p_t as a function of the history of dividends. Is the price process a Markov process in the sense that it depends just on the last period's dividends?

   (c) Let T be the first period in which $d_{T-1} = d_T = \gamma^{(T-1)}$. Is $P_{T-1} > P_T$? Show conditions under which this is true. What is the economic intuition for this result? What does it say about stock market declines or crashes?

   (d) If this model is correct, what does it say about the behavior of the aggregate value of the stock market in economies that switched from high to low growth (e.g., Japan)?

9. (*Pricing bonds with different maturities*) Consider a Lucas' tree economy in which the representative consumer maximizes

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

   subject to

$$c_t + \sum_{i=1}^{n} L_{i,t} + p_t s_{t+1} \leq (p_t + d_t) s_t + \sum_{j=1}^{n} R_{j,t-j} L_{j,t-j}.$$

   and given $d_0, s_0, \{L_{1,-1}R_{1,-1}, L_{2,-2}R_{2,-2}, ..., L_{n,-n}R_{n,-n}\}$. L_{it} is the quantity of i-period pure discount bonds purchased at time t, with a certain payoff of $R_{i,t}$ per period in i

periods. s_t is the number of shares in tree purchased at time $t-1$, and p_t is the tree price at time t. There is exactly one tree per capita in the economy.

Dividends d_t grow stochastically, where $d_{t+1} = \gamma_t d_t$, and the evolution of γ follows a stationary Markov chain. Agents learn the realization of γ_t at beginning of time $t+1$. The dividend is perishable, and can be used only for consumption.

(a) Write down the Bellman equation for this economy. What are the state variables? What are the choice variables?

(b) Write down the Lagrangian for this problem? Derive the first order necessary conditions.

(c) Derive an expression for the equilibrium returns R_{3t}. Show that this expression is independent of d_t.

(d) Suppose the government announces at time 0 that beginning in three years, it will introduce a permanent tax on dividends at a constant proportional rate τ (i.e., the stockholder receives $(1-\tau)d_t$ each period thereafter). The proceeds of the tax will be distributed in a lump-sum payment back to the agent, whether or not he owns the stock. What is the effect on the stock price at time 0 (write down an expression for the price)? What happens to the short-term interest rates? What happens to long-term (greater than three years) interest rates?

10. (*Two-tree Model*) Consider a version of a "Lucas tree" economy in which there are two types of trees. Both types of trees are perfectly durable; a type-i tree ($i = 1, 2$) yields a random amount of dividends equal to d_{it} in period t. Assume that $\{d_{1t}\}_{t=0}^{\infty}$ and $\{d_{2t}\}_{t=0}^{\infty}$ are i.i.d. sequences of random variables and that d_{1t} and d_{2s} are statistically independent for all t and s. In addition, for $i = 1, 2$, assume that $d_{it} = d_L$ with probability π_i and equals $d_H > d_L$ with probability $1 - \pi_i$.

The economy is populated by a continuum (of measure one) of identical consumers with preferences over consumption streams given by:

$$E_0 \sum_{t=0}^{\infty} \beta^t \log(c_t),$$

where c_t is consumption in period t. In period 0, each consumer owns one tree of each type. Dividends are non-storable and are the only source of consumption goods. There are competitive markets in which consumers can buy and sell both types of trees.

(a) Define a sequential competitive equilibrium in which the only assets that consumers trade are the two (types of) trees.

(b) Find an algebraic expression for the equilibrium price of a type-1 tree (measured in terms of today's consumption goods), assuming that the dividends of both types of trees are equal to d_L today. Your expression should depend only on primitives (i.e., on the parameters describing preferences and technology).

(c) How many Arrow securities are there in this economy? Express the prices of these securities in terms of primitives.

(d) Use your answer from part (c) to find the price (expressed in terms of today's consumption goods) of an asset that pays one unit of the consumption good in the next period if the dividends of the two trees (in the next period) are not equal to each other and pays zero otherwise.