

**ECON607 Fall 2010**  
**University of Hawaii**  
**Professor Hui He**  
**TA: Xiaodong Sun**  
**Assignment 2**

The due date for this assignment is Tuesday, October 12. ( Total points = 150)

1. (*Two-sector growth model*) Consider the following two-sector model of optimal growth. A social planner seeks to maximize the life-time utility of the representative HH given by  $\sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$ , where  $c_t$  is consumption of good 1 at time  $t$ , whereas  $l_t$  is leisure at time  $t$ .

Sector 1 produces consumption good using capital  $k_{1t}$  and labor  $n_{1t}$  according to the production function  $f_1(k_{1t}, n_{1t})$ . Sector 2 produces capital good which is going to be used in the production next period according to the production function  $f_2(k_{2t}, n_{2t})$ . The time endowment for the HH at each period is  $\bar{l}$ . The initial capital stock is given by  $k_0 > 0$ . (10 points)

- (a) Formulate this problem as a dynamic programming (DP) problem. Display the functional equation (FE) and clearly specify the state and control variables.

**Answer:** DP problem is

$$v(k) = \max_{c, n_1, n_2, l, k_1, k_2, k'} \{u(c, l) + \beta v(k')\}$$

$$s.t.$$

$$c \leq f_1(k_1, n_1), k' \leq f_2(k_2, n_2),$$

$$n_1 + n_2 + l \leq \bar{l}, k_1 + k_2 \leq k.$$

- (b) Consider another economy that is similar to the previous one except for the fact that capital is sector specific, i.e., capital stock for sector 1 must be only used for this sector, same for sector 2. The capital-good sector produces capital that is specific to each sector according to the transformation technology

$$g(k_{1,t+1}, k_{2,t+1}) \leq f_2(k_{2t}, n_{2t}).$$

Formulate this problem as a dynamic programming (DP) problem. Display the functional equation (FE) and clearly specify the state and control variables.

**Answer:** DP problem is

$$v(k_1, k_2) = \max_{c, n_1, n_2, l, k'_1, k'_2} \{u(c, l) + \beta v(k'_1, k'_2)\}$$

$$s.t.$$

$$c \leq f_1(k_1, n_1), g(k'_1, k'_2) \leq f_2(k_2, n_2),$$

$$n_1 + n_2 + l \leq \bar{l}.$$

2. (*Habit Persistence*) Consider following dynamic problem with habit persistence preference:

$$\begin{aligned} & \max \sum_{t=0}^{\infty} \beta^t (\ln c_t + \gamma \ln c_{t-1}) \\ & \text{s.t.} \\ & c_t + k_{t+1} \leq Ak_t^\alpha \\ & c_t, k_{t+1} \geq 0 \\ & k_0, c_{-1} > 0 \text{ given} \end{aligned}$$

Formulate this problem as a dynamic programming (DP) problem. Display the functional equation (FE) and clearly specify the state and control variables. (5 points)

**Answer:** DP problem is

$$\begin{aligned} v(k, c_{-1}) &= \max_{c, k'} \{ \ln c + \gamma \ln c_{-1} + \beta v(k', c) \} \\ & \text{s.t.} \\ & c + k' \leq Ak^\alpha \\ & c \geq 0, k' \geq 0. \end{aligned}$$

3. (*Howard's policy iteration algorithm*) Consider the following optimal growth problem

$$\begin{aligned} & \max E_0 \sum_{t=0}^{\infty} \beta^t \ln c_t \\ & \text{s.t.} \\ & c_t + k_{t+1} \leq Ak_t^\alpha \theta_t \\ & k_0 \text{ given, } A > 0, 1 > \alpha > 0 \end{aligned}$$

where the sequence  $\{\theta_t\}$  is an i.i.d. shock with  $\ln \theta_t$  distributed according to a normal distribution with mean zero and variance  $\sigma^2$ .

Consider the following algorithm. Guess at a policy of the form

$$k_{t+1} = h_0 Ak_t^\alpha \theta_t$$

for any constant  $h_0 \in (0, 1)$ . Then form the value function

$$v_0(k_0, \theta_0) = E_0 \sum_{t=0}^{\infty} \beta^t \ln(Ak_t^\alpha \theta_t - h_0 Ak_t^\alpha \theta_t)$$

(Hint: you will see the value function takes the form  $v_0(k_0, \theta_0) = H_0 + H_1 \ln \theta_0 + \frac{\alpha}{1-\alpha\beta} \ln k_0$ .) Next, we choose a new policy  $h_1$  by maximizing

$$\ln(Ak^\alpha \theta - k') + \beta E v_0(k', \theta')$$

where  $k' = h_1 Ak^\alpha \theta$ . Then we form the value function again

$$v_1(k_0, \theta_0) = E_0 \sum_{t=0}^{\infty} \beta^t \ln(Ak_t^\alpha \theta_t - h_1 Ak_t^\alpha \theta_t).$$

Continue iterating on this scheme until successive  $h_j$  have converged.

Show that, for the present example, this algorithm converges to the optimal policy function in one step. (15 points)

**Answer:** Given this guess  $k_{t+1} = h_0 A k_t^\alpha \theta_t$ , our infinite-horizon value function or life time utility is

$$\begin{aligned}
v_0(k_0, \theta_0) &= E_0 \sum_{t=0}^{\infty} \beta^t \ln c_t \\
&= E_0 \sum_{t=0}^{\infty} \beta^t \ln(Ak_t^\alpha \theta_t - h_0 Ak_t^\alpha \theta_t) \\
&= \sum_{t=0}^{\infty} \beta^t E_0 [\ln(A - h_0) + \alpha \ln k_t + \ln \theta_t] \\
&= \sum_{t=0}^{\infty} \beta^t E_0 [\ln(A - h_0) + \alpha \ln k_t] + \ln \theta_0 \tag{1}
\end{aligned}$$

Notice that we are using  $E_0 \ln \theta_t = 0, \forall t \geq 1$ . Now let's spend some time on  $\ln k_t$ . We know

$$\begin{aligned}
\ln k_t &= \ln h_0 k_{t-1}^\alpha \theta_{t-1} \\
&= \ln h_0 + \ln \theta_{t-1} + \alpha \ln k_{t-1} \\
&= \ln h_0 + \ln \theta_{t-1} + \alpha \ln h_0 k_{t-2}^\alpha \theta_{t-2} \\
&= (1 + \alpha) \ln h_0 + \ln \theta_{t-1} + \alpha \ln \theta_{t-2} + \alpha^2 \ln k_{t-2} \\
&= (1 + \alpha) \ln h_0 + \ln \theta_{t-1} + \alpha \ln \theta_{t-2} + \alpha^2 \ln h_0 k_{t-3}^\alpha \theta_{t-3} \\
&= (1 + \alpha + \alpha^2) \ln h_0 + \ln \theta_{t-1} + \alpha \ln \theta_{t-2} + \alpha^2 \ln \theta_{t-3} + \alpha^3 \ln k_{t-3} \\
&= \dots \\
&= \frac{1 - \alpha^t}{1 - \alpha} \ln h_0 + \ln \theta_{t-1} + \alpha \ln \theta_{t-2} + \alpha^2 \ln \theta_{t-3} + \dots + \alpha^{t-1} \ln \theta_0 + \alpha^t \ln k_0
\end{aligned}$$

Substituting this expression into equation (1), we get

$$\begin{aligned}
v_0(k_0, \theta_0) &= \sum_{t=0}^{\infty} \beta^t E_0 \left\{ \ln(1 - h_0) + \alpha \left[ \frac{1 - \alpha^t}{1 - \alpha} \ln h_0 + \ln \theta_{t-1} + \alpha \ln \theta_{t-2} + \dots + \alpha^{t-1} \ln \theta_0 + \alpha^t \ln k_0 \right] \right\} + \ln \theta_0 \\
&= \sum_{t=0}^{\infty} \beta^t \left\{ \ln(1 - h_0) + \frac{\alpha}{1 - \alpha} \ln h_0 - \frac{\alpha^{t+1}}{1 - \alpha} \ln h_0 + \alpha^t \ln \theta_0 + \alpha^{t+1} \ln k_0 \right\} + \ln \theta_0 \\
&= \frac{1}{1 - \beta} \ln(1 - h_0) + \frac{\alpha\beta}{(1 - \beta)(1 - \alpha\beta)} \ln h_0 + \frac{2 - \alpha\beta}{1 - \alpha\beta} \ln \theta_0 + \frac{\alpha}{1 - \alpha\beta} \ln k_0 \\
&= H_0 + H_1 \ln \theta_0 + \frac{\alpha}{1 - \alpha\beta} \ln k_0
\end{aligned}$$

where  $H_0, H_1$  are constants. Now we substitute this value function into Bellman equation

$$\begin{aligned}
v_1(k, \theta) &= \max_{k'} \{ \ln(Ak^\alpha \theta - k') + \beta E v_0(k', \theta') \} \\
&= \max_{k'} \left\{ \ln(Ak^\alpha \theta - k') + \beta E \left[ H_0 + H_1 \ln \theta' + \frac{\alpha}{1 - \alpha\beta} \ln k' \right] \right\}
\end{aligned}$$

New policy  $h_1$  is chosen by maximize the equation above. (Note that  $E \ln \theta' = 0$ )

FOC is

$$-\frac{1}{Ak^{\alpha\theta} - k'} + \frac{\alpha\beta}{(1 - \alpha\beta)k'} = 0$$

This implies

$$k' = \alpha\beta k^{\alpha\theta}$$

Therefore, we have  $h_1 = \alpha\beta$ . You can check  $h_2 = h_1$  through doing the procedure again. So in this case, algorithm converges to the optimal policy function just in one step. This is exactly the advantage of policy function iteration method. Its speed of convergence is much faster than that of value function iteration.

4. (*Guess and Verify*) Consider a social planner who faces the following problem:

$$\begin{aligned} & \max_{\{c_t, l_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ & s.t. \\ & c_t + i_t \leq F(k_t, l_t) \\ & k_{t+1} = (1 - \delta)k_t + i_t \\ & c_t \geq 0, 0 < l_t \leq 1 \\ & k_0 \text{ given} \end{aligned}$$

where utility function takes the form

$$u(c_t) = c_t - \theta c_t^2, \theta > 0$$

Assume that  $c$  is always in the range where  $u'(c)$  is positive. Output is linear in capital,  $F(k, l) = Ak$ . Ans we assume  $\delta = 0$ , i.e., no depreciation at all. (10 points)

(a) Write down the Bellman equation for this problem. Be clear what are the state variables, what are control variables.

**Answer:** The Bellman equation is

$$v(k) = \max_{k'} \{u(Ak + (1 - \delta)k - k') + \beta v(k')\}$$

The state variable is  $k$ . the control variables are  $c$  and  $k'$ . Notice that in the equilibrium,  $l_t = 1$ .

(b) Derive the Euler equation relating  $c_t$  and expectations of  $c_{t+1}$ .

**Answer:** The FOC is

$$u'(c) = \beta v'(k')$$

The envelope condition is

$$v'(k) = u'(c)(A + 1 - \delta)$$

Combining together, we obtain EE

$$u'(c) = \beta(A + 1 - \delta)u'(c')$$

$\Rightarrow$

$$1 - 2\theta c = \beta(A + 1)(1 - 2\theta c')$$

- (c) Guess the consumption takes the form  $c_t = H + Fk_t$ . Given this guess, what is the policy function for  $k_{t+1}$ ?

**Answer:** Substituting our guess  $c = H + Fk$  into EE above, we have

$$1 - 2\theta(H + Fk) = \beta(A + 1)(1 - 2\theta(H + Fk'))$$

$\Rightarrow$

$$k' = \frac{(1 - 2\theta H)(\beta(1 + A) - 1)}{2\theta\beta(1 + A)F} + \frac{1}{\beta(1 + A)}k$$

- (d) What is the value for the parameters  $H$  and  $F$ ?

**Answer:**

$$\begin{aligned} c_t &= (A + 1)k - k' \\ &= (A + 1)k - \frac{(1 - 2\theta H)(\beta(1 + A) - 1)}{2\theta\beta(1 + A)F} - \frac{1}{\beta(1 + A)}k \\ &= -\frac{(1 - 2\theta H)(\beta(1 + A) - 1)}{2\theta\beta(1 + A)F} + \frac{\beta(1 + A)^2 - 1}{\beta(1 + A)}k \end{aligned}$$

Thus we have

$$\begin{aligned} F &= \frac{\beta(1 + A)^2 - 1}{\beta(1 + A)} \\ H &= -\frac{(1 - 2\theta H)(\beta(1 + A) - 1)}{2\theta\beta(1 + A)F}. \end{aligned}$$

5. (*Guess and Verify again*) Consider the following problem

$$\begin{aligned} &\max E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \\ &s.t. \\ &A_{t+1} \leq R_t(A_t - c_t) \\ &A_0 > 0 \text{ given.} \end{aligned}$$

where utility function takes the form

$$u(c) = \frac{c^{1-\alpha}}{1-\alpha}, \alpha > 0$$

and  $R_t$  is an i.i.d. shock such that  $ER_t^{1-\alpha} < \frac{1}{\beta}$ ,  $1 > \beta > 0$ . It is assumed that  $c_t$  must be chosen before  $R_t$  is observed. Show that the optimal policy function takes the form  $c_t = \lambda A_t$  and give an explicit formula for  $\lambda$ . (Hint: Guess the value function takes the form  $v(A) = BA^{1-\alpha}$  for some constant  $B$ .) (15 points)

**Answer:** The FE for this economy is

$$\begin{aligned} v(A) &= \max_{c, A'} \left\{ \frac{c^{1-\alpha}}{1-\alpha} + \beta E v(A') \right\} \\ &s.t. \\ &A' \leq R(A - c) \end{aligned}$$

Let's guess

$$v(A) = BA^{1-\alpha}$$

Substituting this guess into the FE above, we have

$$\begin{aligned} BA^{1-\alpha} &= \max_c \left\{ \frac{c^{1-\alpha}}{1-\alpha} + \beta EBA^{1-\alpha} \right\} \\ &= \max_c \left\{ \frac{c^{1-\alpha}}{1-\alpha} + \beta EB(R(A-c))^{1-\alpha} \right\} \\ &= \max_c \left\{ \frac{c^{1-\alpha}}{1-\alpha} + \beta B(A-c)^{1-\alpha} E(R^{1-\alpha}) \right\}. \end{aligned} \quad (2)$$

FOC w.r.t.  $c$  is

$$c^{-\alpha} = \beta B(1-\alpha)(A-c)^{-\alpha} E(R^{1-\alpha})$$

which implies

$$c = \frac{H^{-1/\alpha}}{1 + H^{-1/\alpha}} A \quad (3)$$

where  $H \equiv \beta B(1-\alpha)E(R^{1-\alpha})$ .

Substituting the optimal policy function (3) into the RHS of Bellman equation (2) to replace  $c$ , we have

$$BA^{1-\alpha} = \left[ \frac{1}{1-\alpha} \left( \frac{H^{-1/\alpha}}{1 + H^{-1/\alpha}} \right)^{1-\alpha} + \beta BE(R^{1-\alpha}) \left( \frac{1}{1 + H^{-1/\alpha}} \right)^{1-\alpha} \right] A^{1-\alpha}$$

After some manipulations, we can obtain

$$B = \frac{\{1 - \beta^{1/\alpha} [E(R^{1-\alpha})]^{1/\alpha}\}^{-\alpha}}{1 - \alpha}$$

This result verifies our conjecture about the form of the value function. Given the value of  $B$ , we have

$$c = \{1 - \beta^{1/\alpha} [E(R^{1-\alpha})]^{1/\alpha}\} A.$$

6. (*Optimal growth model with two types of agent*) Consider the standard optimal growth model with production function  $F(k_t, l_t)$ . Suppose now that there are an equal number of two types of HHs with preferences given by  $\sum_{t=0}^{\infty} u_1(c_{1t})$  and  $\sum_{t=0}^{\infty} u_2(c_{2t})$  respectively. Initial ownership of the capital stock is given by  $k_{10}$  and  $k_{20}$ . (20 points)

(a) Define the social planner's problem for this economy.

**Answer:** Denote  $\theta_j \in (0, 1), \forall j = 1, 2, \sum_{i=1}^2 \theta_j = 1$  the weight that social planner

assigns to agent  $i$ . The planner's problem is

$$\begin{aligned}
& \max \sum_{t=0}^{\infty} \beta^t [\theta_1 u_1(c_{1t}) + \theta_2 u_2(c_{2t})] \\
& \text{s.t.} \\
c_t + k_{t+1} - (1 - \delta)k_t & \leq F(k_t, l_t) \\
c_{1t} + c_{2t} & = c_t \\
k_{1t} + k_{2t} & = k_t \\
l_{1t} + l_{2t} & = l_t \\
c_{1t}, c_{2t} & \geq 0, 0 \leq l_t \leq 1, \\
& k_{10}, k_{20} \text{ given.}
\end{aligned}$$

(b) Define an ADE.

**Answer:** An ADE for this economy is an allocation  $\{(c_{1t}, c_{2t}, k_{1t+1}, k_{2t+1}, l_{1t}, l_{2t})\}_{t=0}^{\infty}$  for agents, an allocation  $\{k_t, l_t\}_{t=0}^{\infty}$  for the firm, and a price system  $\{(p_t, w_t, r_t)\}_{t=0}^{\infty}$  such that

(i). Given prices,  $\{(c_{it}, k_{it+1}, l_{it})\}_{t=0}^{\infty}, \forall i = 1, 2$  solves Agent  $i$ 's problem

$$\begin{aligned}
& \max \sum_{t=0}^{\infty} \beta^t u_i(c_{it}) \\
& \text{s.t.} \\
\sum_{t=0}^{\infty} p_t(c_{it} + k_{it+1}) & \leq \sum_{t=0}^{\infty} p_t((r_t + 1 - \delta)k_{it} + w_t l_{it}) \\
c_{it}, k_{it+1} & \geq 0, 0 \leq l_{it} \leq \bar{l}_i, k_{i0} \text{ given}
\end{aligned}$$

(ii). Given prices,  $\{k_t, l_t\}_{t=0}^{\infty}$  solves the firm's problem

$$\begin{aligned}
\max \pi & = \sum_{t=0}^{\infty} p_t [y_t - r_t k_t - w_t l_t] \\
& \text{s.t.} \\
y_t & \leq F(k_t, l_t) \\
k_t & \geq 0, 0 \leq l_t
\end{aligned}$$

(iii). Market clears.

$$\begin{aligned}
\sum_{i=1}^2 c_{it} + \sum_{i=1}^2 k_{it+1} & = F\left(\underbrace{\sum_{i=1}^2 k_{it}, \sum_{i=1}^2 l_{it}}_{\text{normalization}}\right) + (1 - \delta) \sum_{i=1}^2 k_{it} \\
& \underbrace{\bar{l}_1 + \bar{l}_2}_{\text{normalization}} \equiv 1
\end{aligned}$$

(c) Set up the social planner's problem as a dynamic programming problem.

**Answer:** The FE is

$$\begin{aligned}
 v(k_{1t}, k_{2t}) &= \max\{u_1(c_{1t}) + u_2(c_{2t}) + \beta v(k_{1t+1}, k_{2t+1})\} \\
 &\quad s.t. \\
 \sum_{i=1}^2 c_{it} + \sum_{i=1}^2 k_{it+1} &= F\left(\sum_{i=1}^2 k_{it}, \sum_{i=1}^2 l_{it}\right) + (1 - \delta) \sum_{i=1}^2 k_{it} \\
 c_{it}, k_{it+1} &\geq 0, 0 \leq l_{it} \leq \bar{l}_i, k_{i0} \text{ given}, \forall i = 1, 2
 \end{aligned}$$

(d) Define a SME.

**Answer:** A SME for this economy is an allocation  $\{(c_{1t}, c_{2t}, k_{1t+1}, k_{2t+1}, b_{1t+1}, b_{2t+1}, l_{1t}, l_{2t})\}_{t=0}^{\infty}$  for agents, an allocation  $\{k_t, l_t\}_{t=0}^{\infty}$  for the firm, and a price system  $\{(q_t, w_t, r_t)\}_{t=0}^{\infty}$  such that

(i). Given prices,  $\{(c_{it}, k_{it+1}, b_{it+1}, l_{it})\}_{t=0}^{\infty}, \forall i = 1, 2$  solves Agent  $i$ 's problem

$$\begin{aligned}
 &\max \sum_{t=0}^{\infty} \beta^t u_i(c_{it}) \\
 &\quad s.t. \\
 c_{it} + k_{it+1} + q_t b_{it+1} &\leq (r_t + 1 - \delta)k_{it} + w_t l_{it} + b_{it} \\
 c_{it}, k_{it+1} &\geq 0, 0 \leq l_{it} \leq \bar{l}_i, k_{i0}, b_{i0} \text{ given}
 \end{aligned}$$

(ii). Given prices,  $\{k_t, l_t\}_{t=0}^{\infty}$  solves the firm's problem  $\forall t$

$$\begin{aligned}
 &\max F(k_t, l_t) - r_t k_t - w_t l_t \\
 k_t &\geq 0, l_t \geq 0
 \end{aligned}$$

(iii). Markets clear.

$$\begin{aligned}
 \sum_{i=1}^2 c_{it} + \sum_{i=1}^2 k_{it+1} &= F\left(\sum_{i=1}^2 k_{it}, \sum_{i=1}^2 l_{it}\right) + (1 - \delta) \sum_{i=1}^2 k_{it} \\
 \sum_{i=1}^2 b_{it+1} &= 0
 \end{aligned}$$

7. Consider a social planner who faces the following problem:

$$\begin{aligned}
 &\max_{\{c_t, l_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\
 &\quad s.t. \\
 c_t + i_t &\leq F(k_t, l_t) \\
 k_{t+1} &= (1 - \delta)k_t + i_t \\
 c_t &\geq 0, 0 < l_t \leq 1 \\
 &k_0 \text{ given}
 \end{aligned}$$

Now we assume  $u(c) = \ln c$ , production function  $F(k_t, l_t) = k_t^\alpha l_t^{1-\alpha}$  and  $\delta = 1$ . Establish that you indeed have a solution for this problem. (Hint: I am not asking you to solve the

solution, instead I am asking you to prove the existence of a solution to the sequential problem above.) (20 points)

**Answer:** To establish that we indeed have a solution for this sequential problem, we want to first transform the sequential problem into a DP problem. then we want to show that we have a solution for the FE. Next we want to apply Theorem 4.3 in SLP (or Theorem 25 on my lecture notes) to claim that this solution is also a solution for the SP.

The FE for this problem is

$$\begin{aligned} Tv(k) &= \max_{c, k'} \{ \ln(c) + \beta v(k') \} \\ & \text{s.t.} \\ c + k' &\leq F(k, 1) = k^\alpha \\ c, k' &\geq 0 \end{aligned}$$

First, we can use *Blackwell's sufficient conditions for a contraction* (Theorem 3.3 in SLP page 54) to confirm that the operator  $T$  defined above is a contraction (see the lecture notes). We then use *Contraction Mapping Theorem* (CMT, Theorem 3.2 in SLP page 50, or Theorem 15 in my lecture notes) to show that  $T$  has exactly one fixed point  $Tv = v$  which is the solution to FE. In order to apply Theorem 4.3 in SLP, we have to check two assumptions:

A.4.1:  $\Gamma(k) = (0, k^\alpha)$  and  $k_0 > 0 \implies \Gamma(k)$  is non-empty.

A.4.2:  $k_{t+1} \leq k_t^\alpha \implies \ln k_{t+1} \leq \alpha \ln k_t \implies \ln k_t \leq \alpha^t \ln k_0, \forall t$ . This in turn implies

$$F(k_t, k_{t+1}) = \ln(k_t^\alpha - k_{t+1}) < \alpha \ln k_t \leq \alpha^{t+1} \ln k_0, \forall t$$

Since  $0 < \alpha < 1$ , with finite  $k_0$ , it is easy to show  $F(k_t, k_{t+1}) < \ln k_0$ , i.e., is bounded from above. It is also bounded below because the Inada condition on consumption guarantee  $c > 0$  which means  $F(k_t, k_{t+1}) > \ln 0 = -\infty$ . And we know  $0 < \beta < 1$ . So according to SLP page 69, the sufficient condition for A.4.2 is satisfied here.

(a) To use Theorem 4.3 we also need to check whether

$$\lim_{n \rightarrow \infty} \beta^n v(k_n, z_n) = 0.$$

But we already know (from the homework 1) that in this case value function takes the form  $v(k) = B + F \ln k$ , where  $B$  and  $F$  are both finite constant numbers and optimal policy function is  $k' = \alpha \beta k^\alpha$ . Thus

$$\begin{aligned} \lim_{n \rightarrow \infty} \beta^n v(k_n, z_n) &= \lim_{n \rightarrow \infty} \beta^n (B + F \ln k_n) \\ &= \lim_{n \rightarrow \infty} \beta^n (B + \alpha F \ln k_{n-1}) \\ &= \lim_{n \rightarrow \infty} \beta^n (B + \alpha^2 F \ln k_{n-2}) \\ &= \dots \\ &= \lim_{n \rightarrow \infty} \beta^n (B + \alpha^n F \ln k_0) \\ &= B \lim_{n \rightarrow \infty} \beta^n + F \ln k_0 \lim_{n \rightarrow \infty} (\alpha \beta)^n \end{aligned}$$

Given finite  $k_0$ , and  $0 < \alpha, \beta < 1$ , it's easy to see  $\lim_{n \rightarrow \infty} \beta^n v(k_n, z_n) = 0$ .

Applying Theorem 4.3 we have  $v = v^*$ , therefore the solution to FE is indeed also a solution to SP. Thus we do have a solution to SP.

8. (*Recursive Competitive Equilibrium with labor-leisure choice*) Consider the following optimal growth model with labor-leisure choice

$$\begin{aligned} & \max_{\{c_t, l_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \\ & \text{s.t.} \\ c_t + i_t & \leq F(k_t, n_t) \\ k_{t+1} & = (1 - \delta)k_t + i_t \\ c_t & \geq 0, n_t, l_t > 0, n_t + l_t = 1 \\ & k_0 \text{ given} \end{aligned}$$

where  $n_t$  is the labor input and  $l_t$  denotes leisure. (15 points)

- (a) Define a SME for this economy.

**Answer:** A SME for this economy is an allocation  $\{c_t, n_t^s, l_t, k_{t+1}^s, s_t\}_{t=0}^{\infty}$  for HHs; an allocation  $\{k_t^d, n_t^d\}_{t=0}^{\infty}$  for the firm; and a price system  $\{q_t, w_t, r_t\}_{t=0}^{\infty}$  such that

- (i). Given prices,  $\{c_t, n_t^s, l_t, k_{t+1}^s, s_t\}_{t=0}^{\infty}, \forall i = 1, 2$  solves HH's problem

$$\begin{aligned} & \max \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \\ & \text{s.t.} \\ c_t + k_{t+1}^s + q_t s_{t+1} & \leq (r_t + 1 - \delta)k_t^s + w_t n_t^s + s_t \\ c_t, k_{t+1} & \geq 0, 0 \leq n_t^s \leq 1, n_t^s + l_t = 1, k_0 > 0 \text{ given} \end{aligned}$$

- (ii). Given prices,  $\{k_t^d, n_t^d\}_{t=0}^{\infty}$  solves the firm's problem  $\forall t$

$$\begin{aligned} & \max F(k_t^d, n_t^d) - r_t k_t^d - w_t n_t^d \\ k_t^d & \geq 0, n_t^d \geq 0 \end{aligned}$$

- (iii). Markets clear.

$$\begin{aligned} c_t + k_{t+1} - (1 - \delta)k_t & = F(k_t, n_t) \\ k_t^d & = k_t^s = k_t \\ n_t^d & = n_t^s = n_t \\ n_t + l_t & = 1. \end{aligned}$$

- (b) Define a RCE for this economy.

**Answer:** A Recursive Competitive Equilibrium (RCE) is a value function  $V : R_+^2 \rightarrow R$ , a policy function  $H : R_+^2 \rightarrow R$  for the representative household, an

economy-wide law of motion  $h : R_+ \rightarrow R_t$  for capital, and factor price functions  $R : R_+ \rightarrow R_+$  and  $\varpi : R_+ \rightarrow R_+$  such that

(i). Given  $h, R, \varpi, V$  solves HH's problem and  $H$  is the optimal policy function of HH's problem

$$\begin{aligned} V(K, k) &= \max_{C, Y, L} \{U(C, 1 - N) + \beta V[Y, h(k)]\} \\ &\quad s.t. \\ C + [Y - (1 - \delta)K] &\leq KR(k) + \varpi(k)N \\ C, Y &\geq 0, 0 \leq N \leq 1 \end{aligned}$$

(ii).  $R$  and  $\varpi$  satisfy the firm's FOCs.

$$\begin{aligned} R(k) &= F_k(k, n) \\ \varpi(k) &= F_n(k, n) \end{aligned}$$

(iii). Individual choice is consistent with the aggregate law of motion (consistency condition) and markets clear.

$$\begin{aligned} H(k, k) &= h(k) \\ N &= n \\ K &= k. \end{aligned}$$

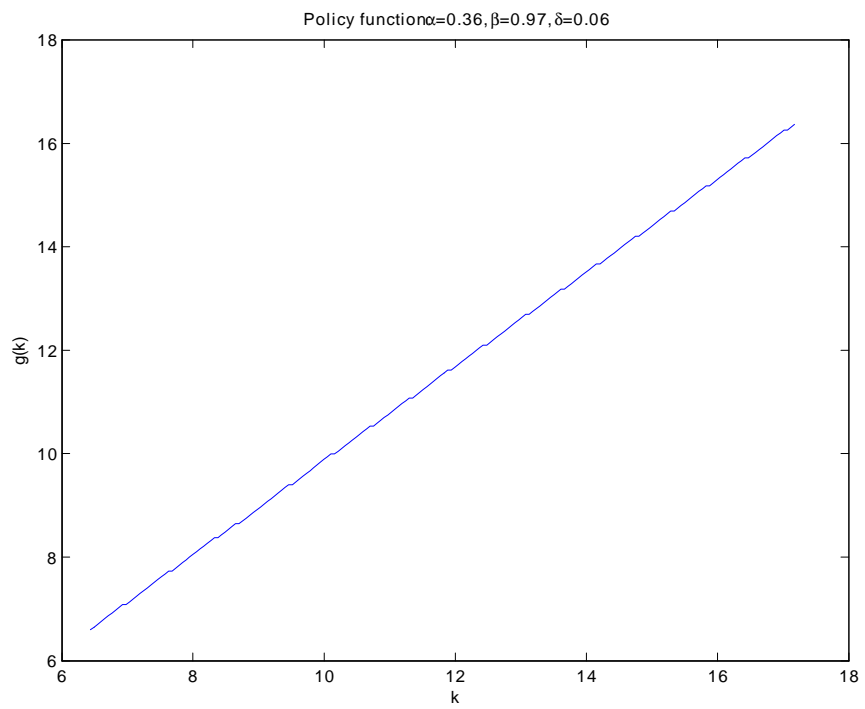
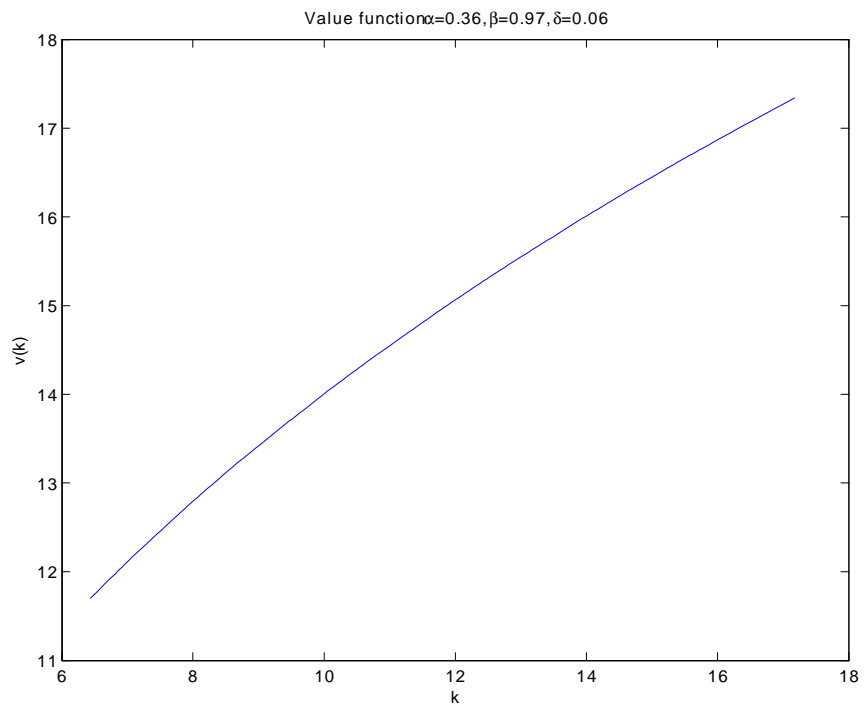
9. (*Value function iteration*) Consider the following optimal growth problem

$$\begin{aligned} \max \sum_{t=0}^{\infty} \beta^t \ln c_t \\ s.t. \\ c_t + k_{t+1} &\leq k_t^\alpha \\ c_t, k_{t+1} &\geq 0, k_0 \text{ given} \end{aligned}$$

We assume  $\beta = 0.97, \alpha = 0.36, \delta = 0.06$ . Write a Matlab code that takes  $k_0 = 0.75 \times k^*$  and iterates towards  $k^*$ , where  $k^*$  is the steady state value of capital stock. Using  $\varepsilon = 0.0001$  as a convergence criterion. Plot the value function  $v(k)$  and the optimal policy function  $k' = g(k)$ . How many time periods does it take to converge to the steady state? Do the same thing from  $k_0 = 0.10 \times k^*$  and compute the number of time periods to converge. Now change  $\alpha$  from 0.36 to 0.30, compute the number of time periods to converge. How does the speed of convergence depend on  $\alpha$ ? Why? (15 points)

**Answer:** Under the parameter values defined above, when  $k_0 = 0.75k^*$ , using value function iteration, we achieve convergence by 12 iterations. We plot the value function and optimal policy function in the figures below.

When  $k_0 = 0.10 \times k^*$ , we achieve the convergence by 12 iterations. When  $\alpha$  is changed from 0.36 to 0.30, no matter we start from  $k_0 = 0.75 \times k^*$  or  $0.10 \times k^*$ , we achieve the convergence by 10 iterations, less than  $\alpha = 0.36$  case. This seems to suggest that the



convergence speed is negatively related to  $\alpha$ . What's the economic intuition behind this phenomenon? Let's take a deeper look at the behavior of capital stock over time:

$$\dot{k} = k' - k = \alpha\beta k^\alpha - k$$

Here we use the result of optimal policy function  $k' = \alpha\beta k^\alpha$ .

Doing the first-order Taylor Expansion around the steady state value  $k^*$ :

$$\dot{k} \simeq (\alpha\beta k^{*\alpha} - k^*) + (\alpha^2\beta k^{*\alpha-1} - 1)(k_t - k^*) = (\alpha^2\beta k^{*\alpha-1} - 1)(k_t - k^*) \quad (4)$$

Substituting  $k^* = (\alpha\beta)^{\frac{1}{1-\alpha}}$  into (4), we get

$$\dot{k} \simeq (\alpha - 1)(k_t - k^*)$$

Define  $x(t) = k_t - k^*$  as the distance between current capital stock and the steady state capital, so  $\dot{x}(t) = \dot{k}$

$\Rightarrow$

$$\dot{x}(t) \simeq (\alpha - 1)x(t) = -(1 - \alpha)x(t)$$

$\Rightarrow$

$$\frac{\dot{x}(t)}{x(t)} \simeq -(1 - \alpha)$$

From this, it's easy to see

$$x(t) \simeq x(0)e^{-(1-\alpha)t}$$

Transforming back to  $k$

$$k_t - k^* \simeq e^{-(1-\alpha)t}(k_0 - k^*)$$

That means the speed of convergence is nearly  $1 - \alpha$ . Each iteration capital stock removes  $0 < 1 - \alpha < 1$  proportion of the reminded distance toward  $k^*$ . Now we understand why when  $\alpha \uparrow$  say from 0.30 to 0.36, the speed of convergence decreases.

10. (*LQ approximation*) Consider a HH who faces the following problem:

$$\begin{aligned} & \max_{\{c_t, i_t\}_{t=0}^{\infty}} - \sum_{t=0}^{\infty} \beta^t \{(c_t - b)^2 + \gamma i_t^2\} \\ & s.t. \\ c_t + i_t &= ra_t + y_t \\ a_{t+1} &= a_t + i_t \\ y_{t+1} &= \rho_1 y_t + \rho_2 y_{t-1} \\ c_t &\geq 0, y_0, y_{-1} \text{ given} \end{aligned}$$

where  $c_t, i_t, a_t, y_t$  are the HH's consumption, investment, asset holdings and exogenous labor income at time  $t$ . And we have  $b > 0, \gamma > 0, \beta \in (0, 1)$  and  $\rho_1, \rho_2$  are parameters. Assume that  $\rho_1, \rho_2$  are such that  $(1 - \rho_1 z - \rho_2 z^2) = 0$  implies  $|z| > 1$ . (15 points)

- (a) Write down the Bellman equation for this problem. Be clear what are the state variables, what are control variables.

**Answer:** The Bellman equation is

$$\begin{aligned} v(a_t, y_t, y_{t-1}) &= \max_{c_t, i_t} \{ -(c_t - b)^2 - \gamma i_t^2 + \beta v(a_{t+1}, y_{t+1}, y_t) \} \\ &\quad s.t. \\ c_t + i_t &= r a_t + y_t \\ a_{t+1} &= a_t + i_t \\ y_{t+1} &= \rho_1 y_t + \rho_2 y_{t-1} \\ c_t &\geq 0, y_0, y_{-1} \text{ given} \end{aligned}$$

- (b) Map this problem into an discounted optimal linear regulator problem.

**Answer:** Using the BC  $c_t = r a_t + y_t - i_t$ , the maximization problem is

$$\max - \sum_{t=1}^{\infty} \beta^t \{ (r a_t + y_t - i_t - b)^2 + \gamma i_t^2 \}. \quad (5)$$

Extending it we can observe the existence of the cross terms of the control variable ( $i_t$ ) and state variables ( $a_t, y_t$ ), therefore, we wish to map it into the form

$$\begin{aligned} \max \sum_{t=1}^{\infty} \beta^t (x_t' R x_t + u_t' Q u_t + 2u_t' W x_t) \\ s.t. \\ x_{t+1} &= A x_t + B u_t \end{aligned}$$

where we have

$$\begin{aligned} x_{t+1} &= (1, a_{t+1}, y_{t+1}, y_t)' \\ A &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \rho_1 & \rho_2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ B &= [0 \ 1 \ 0 \ 0]' \\ u_t &= i_t \end{aligned}$$

Extending (5) and after some algebra, we have

$$\begin{aligned} R &= \begin{bmatrix} -b^2 & rb & b & 0 \\ rb & -r^2 & -r & 0 \\ b & -r & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ Q &= -(1 + \gamma) \\ W &= [-b \ r \ 1 \ 0], \end{aligned}$$

therefore completes the transformation.

- (c) For parameter values  $(\beta, (1+r), b, \gamma, \rho_1, \rho_2) = (0.96, 0.96^{-1}, 30, 1, 1.2, -0.3)$ , write a Matlab code to compute the value function and the optimal policy function for this DOLRP by using value function iteration method.

**Answer:** To compute the optimal policy function we can first transform the above problem into the standard form, using

$$\begin{aligned}\bar{u}_t &= u_t + Q^{-1}Wx_t \\ \bar{R} &= R - W'Q^{-1}W \\ \bar{A} &= A - BQ^{-1}W\end{aligned}$$

and we will have

$$\begin{aligned}P &= \bar{R} + \beta\bar{A}'P\bar{A} - \beta\bar{A}PB(Q + \beta B'PB)^{-1}\beta B'P\bar{A} \\ \bar{F} &= (Q + \beta B'PB)^{-1}\beta B'P\bar{A} \\ \bar{u}_t &= -\bar{F}x_t \\ \bar{F} &= F - Q^{-1}W,\end{aligned}$$

and the problem is changed to

$$\begin{aligned}\max \sum_{t=0}^{\infty} \beta^t \{x_t' \bar{R} x_t + \bar{u}_t' Q \bar{u}_t\} \\ s.t. x_{t+1} &= \bar{A}x_t + Bu_t.\end{aligned}$$

Given the tolerance degree of 0.0001, we can get the value function as

$$P = 1.0e + 004 * \begin{bmatrix} -1.0737 & 0.0001 & 0.0130 & -0.0038 \\ 0.0001 & -0.0000 & -0.0000 & 0.0000 \\ 0.0130 & -0.0000 & -0.0004 & 0.0001 \\ -0.0038 & 0.0000 & 0.0001 & -0.0000 \end{bmatrix}$$

and optimal policy function

$$F = [14.6876 \quad -0.0208 \quad -0.4871 \quad -0.0031].$$

That is

$$i_t = -14.6876 + 0.0208a_t + 0.4871y_t + 0.0031y_{t-1}.$$

- (d) For the same parameters, write a Matlab code to compute the value function and the optimal policy function for this DOLRP by using policy function iteration method.

**Answer:** We should obtain the identical results as in part (c).

11. (*Wealth inequality in a two-period economy*) Suppose that there are two types of consumers distinguished by their initial endowments of capital. In particular, type-1 consumers (who comprise fraction  $\theta$  of the population) are richer than type-2 consumers (who comprise fraction  $1 - \theta$  of the population): type-1 consumers are endowed with  $k_0^1$  units of capital and type-2 consumers are endowed with  $k_0^2$  units of capital, where

$k_0^1 > k_0^2$ . The two types of consumers are identical in all other respects. Each consumer takes prices as given (in particular, each consumer takes the aggregate, or total, capital stock in period 1 as given) when making savings decisions in period 0. The equilibrium (or consistency) condition is that the total savings of the two types of consumers in period 0 must equal the aggregate capital stock that consumers take as given when deciding how much to save. Assume that each consumer's utility function takes the form  $u(c_0) + \beta u(c_1)$  with  $u(c) = \log(c)$ . The production technology available to firms is:  $y = k^\alpha n^{1-\alpha}$ , with  $1 > \alpha > 0$ , where  $y$  is the firm's output and  $k$  and  $n$  are the services of capital and labor, respectively. (10 points)

- (a) Derive the equilibrium aggregate capital stock in period 1 as a function of primitives (i.e., the parameters  $\alpha$ ,  $\theta$  and  $\beta$ , and the initial capital stock  $k_0^1$  and  $k_0^2$ ).
- (b) Use your answer to part (a) to show that changes in  $k_0^1$  and  $k_0^2$  that keep aggregate capital in period 0 (i.e.,  $\theta k_0^1 + (1 - \theta)k_0^2$ ) constant have no effect either on equilibrium aggregate savings or on equilibrium prices. This is a version of an *aggregation theorem* for this economy: holding the total amount of capital in period 0 constant, the behavior of the aggregates in this economy does not depend on the distribution of capital in period 0.

**Answer:** In this question we have two types of consumers: there are a fraction  $\theta$  of type 1 consumers, who are endowed with  $k_0^1$  unit of capital in period 0. The rest of the consumers, a fraction of  $1 - \theta$ , are endowed with  $k_0^2$ , where  $k_0^1 > k_0^2$ . Then, given the log utility function, the problem that a consumer of type  $i = 1, 2$  must solve is:

$$\begin{aligned} & \max_{c_0^i, c_1^i} \log(c_0^i) + \beta \log(c_1^i) \\ & s.t. \\ c_0^i &= r_0 k_0^i + w_0 - k_1^i \\ c_1^i &= r_1 k_1^i + w_1 \end{aligned}$$

which implies the FOC:

$$r_1 k_1^i + w_1 = \beta r_1 (r_0 k_0^i + w_0 - k_1^i) \quad i = 1, 2$$

On the other hand, each period  $i = 0, 1$ , the firm must solve:

$$\max_{\bar{K}_i, \bar{N}_i} \bar{K}_i^\alpha \bar{N}_i^{1-\alpha} - \bar{K}_i r_i - \bar{N}_i w_i$$

which implies the competitive prices:

$$w_1 = (1 - \alpha) \bar{K}_1^\alpha \quad \text{and} \quad r_1 = \alpha \bar{K}_1^{\alpha-1}$$

Also, the CRS production technology implies that :

$$r_0 \bar{K}_0 + w_0 \bar{N}_0 = y_0 = \bar{K}_0^\alpha \bar{N}_0^{1-\alpha}$$

(1)

Finally, the equilibrium conditions are:  $\bar{K} = \theta k_i^1 + (1 - \theta) k_i^2$  for  $i = 1, 2$  and as labor is inelastically supplied,  $\bar{N}_i = 1$  for  $i = 0, 1$ .

Now we are ready to solve for the equilibrium. First, multiplying each type's FOC equation by its fraction in the population and summing up both equations, we get:

$$r_1 (\theta k_1^1 + (1 - \theta) k_1^2) + w_1 = \beta r_1 [r_0 (\theta k_0^1 + (1 - \theta) k_0^2) + w_0 - (\theta k_1^1 + (1 - \theta) k_1^2)]$$

Using the equilibrium conditions and eq.(1) :

$$r_1 \bar{K}_1 + w_1 = \beta r_1 [\bar{K}_0^\alpha - \bar{K}_1]$$

then, replacing the competitive prices,

$$\alpha \bar{K}_1^{\alpha-1} \bar{K}_1 + (1 - \alpha) \bar{K}_1^\alpha = \beta \alpha \bar{K}_1^{\alpha-1} [\bar{K}_0^\alpha - \bar{K}_1]$$

which implies

$$\bar{K}_1 = \frac{\alpha \beta \bar{K}_0^\alpha}{1 + \alpha \beta}$$

Thus, as period 1 aggregate savings only depends on aggregate period 0 savings, and not on how much has each consumer, if this aggregate is not changed,  $\bar{K}_1$  won't change either. And as prices depend also only on aggregates, changes in the distribution of capital endowments, that leave the aggregate equal, won't affect them.

- (c) Suppose that the felicity function takes the form:  $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$  where  $\sigma > 0$ . Does an aggregation theorem like the one described in part (b) hold for this economy? Explain why or why not.

**Answer:** So now we just have to change the consumers utility function. Each type's problem is:

$$\begin{aligned} \max_{c_0^i, c_1^i} & \frac{(c_0^i)^{1-\sigma} - 1}{1-\sigma} + \beta \frac{(c_1^i)^{1-\sigma} - 1}{1-\sigma} \\ \text{s.t.} & \\ c_0^i &= r_0 k_0^i + w_0 - k_1^i \\ c_1^i &= r_1 k_1^i + w_1 \end{aligned}$$

which implies the FOC:

$$r_0 k_0^i + w_0 - k_1^i = (\beta r_1)^{-1/\sigma} (r_1 k_1^i + w_1)$$

The firm's problem has not changed, so we have the same equations for prices. Following the same steps of part (a), we get that:

$$r_0 \bar{K}_0 + w_0 - \bar{K}_1 = (\beta r_1)^{-1/\sigma} (r_1 \bar{K}_1 + w_1)$$

then, remembering that wages and interest rates only depend on aggregates, from eq. above, which implicitly defines  $\bar{K}_1$ , we can conclude that if  $\bar{K}_0$  does not change then  $\bar{K}_1$  won't change also, and thus, the aggregation results holds here too.