

ECON607 Fall 2010
University of Hawaii
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Assignment 2

The due date for this assignment is Tuesday, October 12.

1. (*Two-sector growth model*) Consider the following two-sector model of optimal growth. A social planner seeks to maximize the life-time utility of the representative HH given by $\sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$, where c_t is consumption of good 1 at time t , whereas l_t is leisure at time t .

Sector 1 produces consumption good using capital k_{1t} and labor n_{1t} according to the production function $f_1(k_{1t}, n_{1t})$. Sector 2 produces capital good which is going to be used in the production next period according to the production function $f_2(k_{2t}, n_{2t})$. The time endowment for the HH at each period is \bar{l} . The initial capital stock is given by $k_0 > 0$.

- (a) Formulate this problem as a dynamic programming (DP) problem. Display the functional equation (FE) and clearly specify the state and control variables.
- (b) Consider another economy that is similar to the previous one except for the fact that capital is sector specific, i.e., capital stock for sector 1 must be only used for this sector, same for sector 2. The capital-good sector produces capital that is specific to each sector according to the transformation technology

$$g(k_{1,t+1}, k_{2,t+1}) \leq f_2(k_{2t}, n_{2t}).$$

Formulate this problem as a dynamic programming (DP) problem. Display the functional equation (FE) and clearly specify the state and control variables.

2. (*Habit Persistence*) Consider following dynamic problem with habit persistence preference:

$$\begin{aligned} & \max \sum_{t=0}^{\infty} \beta^t (\ln c_t + \gamma \ln c_{t-1}) \\ & \text{s.t.} \\ & c_t + k_{t+1} \leq Ak_t^\alpha \\ & c_t, k_{t+1} \geq 0 \\ & k_0, c_{-1} > 0 \text{ given} \end{aligned}$$

Formulate this problem as a dynamic programming (DP) problem. Display the functional equation (FE) and clearly specify the state and control variables.

3. (*Howard's policy iteration algorithm*) Consider the following optimal growth problem

$$\begin{aligned} \max E_0 \sum_{t=0}^{\infty} \beta^t \ln c_t \\ \text{s.t.} \\ c_t + k_{t+1} \leq Ak_t^\alpha \theta_t \\ k_0 \text{ given, } A > 0, 1 > \alpha > 0 \end{aligned}$$

where the sequence $\{\theta_t\}$ is an i.i.d. shock with $\ln \theta_t$ distributed according to a normal distribution with mean zero and variance σ^2 .

Consider the following algorithm. Guess at a policy of the form

$$k_{t+1} = h_0 Ak_t^\alpha \theta_t$$

for any constant $h_0 \in (0, 1)$. Then form the value function

$$v_0(k_0, \theta_0) = E_0 \sum_{t=0}^{\infty} \beta^t \ln(Ak_t^\alpha \theta_t - h_0 Ak_t^\alpha \theta_t)$$

(Hint: you will see the value function takes the form $v_0(k_0, \theta_0) = H_0 + H_1 \ln \theta_0 + \frac{\alpha}{1-\alpha\beta} \ln k_0$.) Next, we choose a new policy h_1 by maximizing

$$\ln(Ak^\alpha \theta - k') + \beta E v_0(k', \theta')$$

where $k' = h_1 Ak^\alpha \theta$. Then we form the value function again

$$v_1(k_0, \theta_0) = E_0 \sum_{t=0}^{\infty} \beta^t \ln(Ak_t^\alpha \theta_t - h_1 Ak_t^\alpha \theta_t).$$

Continue iterating on this scheme until successive h_j have converged.

Show that, for the present example, this algorithm converges to the optimal policy function in one step.

4. (*Guess and Verify*) Consider a social planner who faces the following problem:

$$\begin{aligned} \max_{\{c_t, l_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t.} \\ c_t + i_t \leq F(k_t, l_t) \\ k_{t+1} = (1 - \delta)k_t + i_t \\ c_t \geq 0, 0 < l_t \leq 1 \\ k_0 \text{ given} \end{aligned}$$

where utility function takes the form

$$u(c_t) = c_t - \theta c_t^2, \theta > 0$$

Assume that c is always in the range where $u'(c)$ is positive. Output is linear in capital, $F(k, l) = Ak$. Ans we assume $\delta = 0$, i.e., no depreciation at all.

- (a) Write down the Bellman equation for this problem. Be clear what are the state variables, what are control variables.
- (b) Derive the Euler equation relating c_t and expectations of c_{t+1} .
- (c) Guess the consumption takes the form $c_t = H + Fk_t$. Given this guess, what is the policy function for k_{t+1} ?
- (d) What is the value for the parameters H and F ?

5. (*Guess and Verify again*) Consider the following problem

$$\begin{aligned} \max E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t.} \\ A_{t+1} &\leq R_t(A_t - c_t) \\ A_0 &> 0 \text{ given.} \end{aligned}$$

where utility function takes the form

$$u(c) = \frac{c^{1-\alpha}}{1-\alpha}, \alpha > 0$$

and R_t is an i.i.d. shock such that $ER_t^{1-\alpha} < \frac{1}{\beta}$, $1 > \beta > 0$. It is assumed that c_t must be chosen before R_t is observed. Show that the optimal policy function takes the form $c_t = \lambda A_t$ and give an explicit formula for λ . (Hint: Guess the value function takes the form $v(A) = BA^{1-\alpha}$ for some constant B .)

6. (*Optimal growth model with two types of agent*) Consider the standard optimal growth model with production function $F(k_t, l_t)$. Suppose now that there are an equal number of two types of HHs with preferences given by $\sum_{t=0}^{\infty} u_1(c_{1t})$ and $\sum_{t=0}^{\infty} u_2(c_{2t})$ respectively. Initial ownership of the capital stock is given by k_{10} and k_{20} .

- (a) Define the social planner's problem for this economy.
- (b) Define an ADE.
- (c) Set up the social planner's problem as a dynamic programming problem.
- (d) Define a SME.

7. Consider a social planner who faces the following problem:

$$\begin{aligned} \max_{\{c_t, l_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t.} \\ c_t + i_t &\leq F(k_t, l_t) \\ k_{t+1} &= (1 - \delta)k_t + i_t \\ c_t &\geq 0, 0 < l_t \leq 1 \\ k_0 &\text{ given} \end{aligned}$$

Now we assume $u(c) = \ln c$, production function $F(k_t, l_t) = k_t^\alpha l_t^{1-\alpha}$ and $\delta = 1$. Establish that you indeed have a solution for this problem. (Hint: I am not asking you to solve the solution, instead I am asking you to prove the existence of a solution to the sequential problem above.)

8. (*Recursive Competitive Equilibrium with labor-leisure choice*) Consider the following optimal growth model with labor-leisure choice

$$\begin{aligned} & \max_{\{c_t, l_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \\ & \text{s.t.} \\ c_t + i_t & \leq F(k_t, n_t) \\ k_{t+1} & = (1 - \delta)k_t + i_t \\ c_t & \geq 0, n_t, l_t > 0, n_t + l_t = 1 \\ & k_0 \text{ given} \end{aligned}$$

where n_t is the labor input and l_t denotes leisure.

- (a) Define a SME for this economy.
 (b) Define a RCE for this economy.

9. (*Value function iteration*) Consider the following optimal growth problem

$$\begin{aligned} & \max \sum_{t=0}^{\infty} \beta^t \ln c_t \\ & \text{s.t.} \\ c_t + k_{t+1} & \leq k_t^\alpha \\ c_t, k_{t+1} & \geq 0, k_0 \text{ given} \end{aligned}$$

We assume $\beta = 0.97$, $\alpha = 0.36$, $\delta = 0.06$. Write a Matlab code that takes $k_0 = 0.75 \times k^*$ and iterates towards k^* , where k^* is the steady state value of capital stock. Using $\varepsilon = 0.0001$ as a convergence criterion. Plot the value function $v(k)$ and the optimal policy function $k' = g(k)$. How many time periods does it take to converge to the steady state? Do the same thing from $k_0 = 0.10 \times k^*$ and compute the number of time periods to converge. Now change α from 0.36 to 0.30, compute the number of time periods to convergence. How does the speed of convergence depend on α ? Why?

10. (*LQ approximation*) Consider a HH who faces the following problem:

$$\begin{aligned} & \max_{\{c_t, i_t\}_{t=0}^{\infty}} - \sum_{t=0}^{\infty} \beta^t \{(c_t - b)^2 + \gamma i_t^2\} \\ & \text{s.t.} \\ c_t + i_t & = ra_t + y_t \\ a_{t+1} & = a_t + i_t \\ y_{t+1} & = \rho_1 y_t + \rho_2 y_{t-1} \\ c_t & \geq 0, y_0, y_{-1} \text{ given} \end{aligned}$$

where c_t, i_t, a_t, y_t are the HH's consumption, investment, asset holdings and exogenous labor income at time t . And we have $b > 0, \gamma > 0, \beta \in (0, 1)$ and ρ_1, ρ_2 are parameters. Assume that ρ_1, ρ_2 are such that $(1 - \rho_1 z - \rho_2 z^2) = 0$ implies $|z| > 1$.

- (a) Write down the Bellman equation for this problem. Be clear what are the state variables, what are control variables.
 - (b) Map this problem into an discounted optimal linear regulator problem.
 - (c) For parameter values $(\beta, (1+r), b, \gamma, \rho_1, \rho_2) = (0.96, 0.96^{-1}, 30, 1, 1.2, -0.3)$, write a Matlab code to compute the value function and the optimal policy function for this DOLRP by using value function iteration method.
 - (d) For the same parameters, write a Matlab code to compute the value function and the optimal policy function for this DOLRP by using policy function iteration method.
11. (*Wealth inequality in a two-period economy*) Suppose that there are two types of consumers distinguished by their initial endowments of capital. In particular, type-1 consumers (who comprise fraction θ of the population) are richer than type-2 consumers (who comprise fraction $1 - \theta$ of the population): type-1 consumers are endowed with k_0^1 units of capital and type-2 consumers are endowed with k_0^2 units of capital, where $k_0^1 > k_0^2$. The two types of consumers are identical in all other respects. Each consumer takes prices as given (in particular, each consumer takes the aggregate, or total, capital stock in period 1 as given) when making savings decisions in period 0. The equilibrium (or consistency) condition is that the total savings of the two types of consumers in period 0 must equal the aggregate capital stock that consumers take as given when deciding how much to save. Assume that each consumer's utility function takes the form $u(c_0) + \beta u(c_1)$ with $u(c) = \log(c)$. The production technology available to firms is: $y = k^\alpha n^{1-\alpha}$, with $1 > \alpha > 0$, where y is the firm's output and k and n are the services of capital and labor, respectively.
- (a) Derive the equilibrium aggregate capital stock in period 1 as a function of primitives (i.e., the parameters α, θ and β , and the initial capital stock k_0^1 and k_0^2).
 - (b) Use your answer to part (a) to show that changes in k_0^1 and k_0^2 that keep aggregate capital in period 0 (i.e., $\theta k_0^1 + (1 - \theta)k_0^2$) constant have no effect either on equilibrium aggregate savings or on equilibrium prices. This is a version of an *aggregation theorem* for this economy: holding the total amount of capital in period 0 constant, the behavior of the aggregates in this economy does not depend on the distribution of capital in period 0.
 - (c) Suppose that the felicity function takes the form: $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$ where $\sigma > 0$. Does an aggregation theorem like the one described in part (b) hold for this economy? Explain why or why not.