ECON607 Fall 2010
University of Hawaii
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Assignment 1 Suggested Solutions

The due date for this assignment is Thursday, Sep. 23.

1. Consider an stochastic optimal growth model as in the lecture with utility function $u(c) = \ln c$, production function $f(k) = k^\alpha$, and resource constraint $c_t + k_{t+1} \leq e^{z_t} f(k_t)$, where z_t is the i.i.d. productivity shock belongs to a lognormal distribution. (15 points)

   (a) Write the Bellman equation for the social planner's problem.
   Answer:
   $$v(k,z) = \max_{0 \leq k' \leq e^z k^\alpha} \{\ln(e^z k^\alpha - k') + \beta E[v(k', z')]\} \tag{1}$$

   (b) Use "Guess and Verify" method to slove the Bellman equation. (Hint: Guess $v(k,z) = H + F\ln k + Gz$.)
   Answer: Let's guess value function takes form
   $$v(k,z) = H + F\ln k + Gz$$
   What we need to do is to verify this is a right guess and determine the values of constant term H, F and G. Substituting our guess into the RHS of the Bellman equation (BE), we obtain
   $$\begin{aligned} v(k,z) &= \max_{k'}\{\ln(e^z k^\alpha - k') + \beta E[H + F\ln k' + Gz']\} \\ &= \max_{k'}\{\ln(e^z k^\alpha - k') + \beta E[H + F\ln k']\} \quad (\because E(z') = 0) \end{aligned} \tag{2}$$
   FOC w.r.t. k' implies
   $$-\frac{1}{e^z k^\alpha - k'} + \frac{\beta F}{k'} = 0$$
   From it we have the optimal policy function as following
   $$k' = \frac{\beta F}{1+\beta F} e^z k^\alpha \tag{3}$$
   Put this policy function back into equation (2), we have
   $$\begin{aligned} V(k,z) &= H + F\ln k + Gz \\ &= \{\ln(e^z k^\alpha - \frac{\beta F}{1+\beta F} e^z k^\alpha) + \beta H + \beta F \ln \frac{\beta F}{1+\beta F} e^z k^\alpha\} \\ &= \ln(\frac{1}{1+\beta F}) + \beta F \ln(\frac{\beta F}{1+\beta F}) + \beta H + \alpha(1+\beta F)\ln k + (1+\beta F)z \end{aligned}$$

To match coefficients, we need to have

$$\begin{aligned}H &= \ln(\frac{1}{1+\beta F}) + \beta F \ln(\frac{\beta F}{1+\beta F}) + \beta H \\ F &= \alpha(1+\beta F) \\ G &= (1+\beta F)\end{aligned}$$

which implies

$$\begin{aligned}F &= \frac{\alpha}{1-\alpha\beta} \\ H &= \frac{1}{1-\alpha\beta} \\ H &= \frac{1}{1-\beta}[\ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln\alpha\beta].\end{aligned}$$

2. Prove that the budget constraint in sequential market equilibrium (SME) as in the lecture is as same as the one in ADE, i.e., BC in SME $\Rightarrow$ BC in ADE. (15 points)

Answer: BC in SME is

$$c_t + q_t s_{t+1} + i_t = r_t k_t + w_t n_t + s_t + \pi_t, \forall t \tag{4}$$

Let's start from period 0 BC

$$c_0 + q_0 s_1 + i_0 = r_0 k_0 + w_0 n_0 + s_0 + \pi_0$$

Rewrite

$$c_0 + i_0 = r_0 k_0 + w_0 n_0 + \pi_0 + s_0 - q_0 s_1 \tag{5}$$

From the BC above, we know

$$s_1 = c_1 + i_1 + q_1 s_2 - (r_1 k_1 + w_1 n_1 + \pi_1)$$

Substituting s_1 into equation (5), we have

$$(c_0 + i_0) + q_0(c_1 + i_1) + q_0 q_1 s_2 = r_0 k_0 + w_0 n_0 + q_0(r_1 k_1 + w_1 n_1) + \pi_0 + q_0 \pi_1 + s_0$$

Then we can also substitute out s_2. Repeatly doing like this, we end up with

$$\sum_{t=0}^{T} \prod_{s=0}^{t-1} q_s (c_t + i_t) + \prod_{t=0}^{T} q_t s_{T+1} = \sum_{t=0}^{T} \prod_{s=0}^{t-1} q_s (r_t k_t + w_t n_t + \pi_t) + s_0$$

Let $\prod_{s=0}^{t-1} q_s = p_t$ and notice that $s_{T+1} = 0$ and $s_0 = 0$, we have

$$\sum_{t=0}^{T} p_t (c_t + i_t) = \sum_{t=0}^{T} p_t (r_t k_t + w_t n_t + \pi_t)$$

Notice that

$$\sum_{t=0}^{T} p_t \pi_t = \pi$$

Therefore, we have

$$\sum_{t=0}^{T} p_t(c_t + i_t) = \sum_{t=0}^{T} p_t(r_t k_t + w_t n_t) + \pi$$

This is exactly is BC in ADE for finite horizon case. For the infinite horizon case, we need a transversality condition as following

$$\lim_{T\to\infty} \prod_{t=0}^{T} q_t s_{T+1} = 0$$

to guarantee the identical BCs in SME and ADE.

3. (*Two-period Emdowment Economy*) Consider following two-period pure exchange economy. There is a single consumption good in each period. There are two HHs who have identical preferences over consumption given by

$$u(c_1, c_2) = \ln c_1 + \beta \ln c_2$$

HH 1 has endowments given by $(w_1, 0)$ and HH 2 has endowments $(0, w_2)$. (20 points)

(a) Define an ADE for this economy.

Answer: An ADE is a set of prices $\{p_1, p_2\}$ and an allocation $\{c_1^1, c_2^1, c_1^2, c_2^2\}$ such that

(i). Given prices, the allocation solves HH's problem:

HH1's problem

$$\begin{aligned} & \max_{c_1^1, c_2^1} \ln c_1^1 + \beta \ln c_2^1 \\ & s.t. \\ p_1 c_1^1 + p_2 c_2^1 &\leq p_1 w_1 \\ c_1^1, c_2^1 &\geq 0 \end{aligned}$$

HH2's problem

$$\begin{aligned} & \max_{c_1^2, c_2^2} \ln c_1^2 + \beta \ln c_2^2 \\ & s.t \\ p_1 c_1^2 + p_2 c_2^2 &\leq p_2 w_2 \\ c_1^2, c_2^2 &\geq 0 \end{aligned}$$

(ii). Markets clear.

$$\begin{aligned} c_1^1 + c_1^2 &= w_1 \\ c_2^1 + c_2^2 &= w_2 \end{aligned}$$

(b) Suppose HHs has incentive to smooth their consumption over time by borrowing and lending. Define a SME for this economy. Show that the ADE and SME give identical allocations.

Answer: A SME is a set of price r, an allocation $\{c_t^i, s_t^i\}_{t=1}^2, i = 1, 2$ such that
(i). Given r, $\{c_t^i, s_t^i\}_{t=1}^2, i = 1, 2$ solves HH's problem.
HH1's problem

$$\begin{aligned} &\max_{c_1^1, c_2^1, s_1^1, s_2^1} \ln c_1^1 + \beta \ln c_2^1 \\ &s.t. \\ c_1^1 + s_1^1 &\leq w_1 \\ c_2^1 + s_2^1 &\leq (1+r)s_1^1 \\ c_1^1, c_2^1 &\geq 0 \end{aligned}$$

HH2's problem

$$\begin{aligned} &\max_{c_1^2, c_2^2, s_1^2, s_2^2} \ln c_1^2 + \beta \ln c_2^2 \\ &s.t. \\ c_1^2 + s_1^2 &\leq 0 \\ c_2^2 + s_2^2 &\leq w_2 + (1+r)s_1^2 \\ c_1^2, c_2^2 &\geq 0 \end{aligned}$$

(ii). Markets clear.

$$\begin{aligned} c_1^1 + c_1^2 &= w_1 \\ c_2^1 + c_2^2 &= w_2 \\ s_1^1 + s_1^2 &= 0 \\ s_2^1 + s_2^2 &= 0 \end{aligned}$$

Now let's solve SME.
First of all, notice that we should have $s_2^1 = s_2^2 = 0$. FOCs for HH1's problem:

$$\begin{aligned} c_1 &: \frac{1}{c_1^1} - \lambda_1 = 0 \\ c_2^1 &: \beta \frac{1}{c_2^1} - \lambda_2 = 0 \\ s_1^1 &: -\lambda_1 + (1+r)\lambda_2 = 0 \end{aligned}$$

Combining FOCs, we have EE

$$\frac{c_2^1}{c_1^1} = \beta(1+r) \tag{6}$$

Similarly, we have EE for HH2

$$\frac{c_2^2}{c_1^2} = \beta(1+r) \tag{7}$$

Combining (6) and (7), we get

$$\frac{c_2^1}{c_1^1} = \frac{c_2^2}{c_1^2}$$

Notice that as in problem 2, we can combine two period BCs into one BC through substituting out s_1^1 in HH1's problem and substituting out s_1^2 in HH2's problem. Hence HH1's BC becomes

$$c_1^1 + \frac{1}{1+r}c_2^1 \leq w_1$$

Similarly HH2's BC is

$$c_1^2 + \frac{1}{1+r}c_2^2 \leq w_1$$

Now let's turn to ADE. FOCs for HH1 in ADE are

$$\begin{aligned} c_1 &: \frac{1}{c_1^1} - \lambda p_1 = 0 \\ c_2^1 &: \beta\frac{1}{c_2^1} - \lambda p_2 = 0 \end{aligned}$$

Combining two FOCs, we obtain EE

$$\frac{c_2^1}{\beta c_1^1} = \frac{p_1}{p_2}$$

Similarly EE for HH2 is

$$\frac{c_2^2}{\beta c_1^2} = \frac{p_1}{p_2}$$

Hence we also have

$$\frac{c_2^1}{c_1^1} = \frac{c_2^2}{c_1^2}$$

Notice that it is easy to show

$$\frac{c_2^1}{c_1^1} = \frac{c_2^2}{c_1^2} = \frac{c_2^1 + c_2^2}{c_1^1 + c_1^2} = \frac{w_2}{w_1}$$

Comparing EEs in ADE and SME, we conclude

$$\frac{p_1}{p_2} = 1 + r$$

Normailze $p_1 = 1$, then $p_2 = \frac{1}{1+r}$, substituting these expression into BCs in ADE, we end up the same BCs as in SME. Thus, SME and ADE share the identical objective function, identical BCs and identical EEs, it implies the solotions to ADE and SME must be same.

$$\begin{aligned} c_1^1 &= \frac{\omega_1}{1+\beta}, c_1^2 = \frac{\beta\omega_1}{1+\beta} \\ c_2^1 &= \frac{\omega_2}{1+\beta}, c_1^2 = \frac{\beta\omega_2}{1+\beta}. \end{aligned}$$

(c) Suppose $w_1 = w_2 = 1$ and $\beta = 1$, calculate the competitive equilibrium.

Answer:

$$\begin{aligned} c_1^1 &= c_2^1 = c_1^2 = c_2^2 = \frac{1}{2} \\ s_1^1 &= \frac{1}{2}, s_1^2 = -\frac{1}{2} \\ p_1 &= p_2 = 1 \\ r &= 0 \end{aligned}$$

(d) Suppose HHs cannot borrow or lend from each other. Try to define a competitive equilibrium in this case. Show that a policy which transfers goods from the rich to the poor in each period in Pareto-improving.

Answer: A CE is a price system $\{p_1, p_2\}$, an allocation $\{c_t^i\}_{t=1}^2, i = 1, 2$ such that

(i). Given prices, the allocation solves HH's problem:

HH1's problem

$$\begin{aligned} &\max_{c_1^1, c_2^1} \ln c_1^1 + \beta \ln c_2^1 \\ &s.t. \\ c_1^1 &\le w_1 \\ c_2^1 &\le 0 \\ c_1^1, c_2^1 &\ge 0 \end{aligned}$$

HH2's problem

$$\begin{aligned} &\max_{c_1^2, c_2^2} \ln c_1^2 + \beta \ln c_2^2 \\ &s.t. \\ c_1^2 &\le 0 \\ c_2^2 &\le w_2 \\ c_1^2, c_2^2 &\ge 0 \end{aligned}$$

(ii). Markets clear.

$$\begin{aligned} c_1^1 + c_1^2 &= w_1 \\ c_2^1 + c_2^2 &= w_2 \end{aligned}$$

Obviously the solution is an autarky equilibrium

$$c_1^1 = w_1, c_2^1 = 0, c_1^2 = 0, c_2^2 = w_2$$

Now we show that any transfer mechanism will Pareto dominates the autarky equilibrium. Suppose in each period we trnasfer $0 < T < w_1$ from rich to poor. Now for HH1, life time utility is $\ln(w_1 - T) + \beta \ln T$. Obviously we have

$$\ln(w_1 - T) + \beta \ln T > \ln w_1 + \ln 0 = -\infty$$

Simiarly, we have for HH2

$$\ln T + \beta \ln(w_1 - T) > \ln 0 + \beta \ln w_1 = -\infty.$$

4. (*When Robinson Crusoe meets Arrow-Debreu*) Consider a Robinson-Crusoe economy. Crusoe as the single HH on the island has a utility function over consumption of fish c_t and leisure $(1 - l_t)$ as following

$$\sum_{t=0}^{T} \beta^t [\ln c_t + \gamma \ln(1 - l_t)]$$

where $\beta \in (0,1)$ and $\gamma > 0$. Crusoe has a technology transforming labor into output of fish by

$$y_t = A_t l_t^{\alpha}$$

where $\alpha \in (0,1)$, $\{A_t\}_{t=0}^{T}$ is a sequence of numbers which measures Crusoe's state of productivity. The fish can only be used for consumption (Robinson does not have a refrigerator). (25 points)

(a) Set up the social planner's problem for this economy.
Answer: Social planner's problem is

$$\begin{aligned} & \max_{\{c_t, l_t\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t [\ln c_t + \gamma \ln(1 - l_t)] \\ & s.t. \\ c_t &\leq A_t l_t^{\alpha}, t = 0, 1, 2, .., T \\ c_t &\geq 0, 0 \leq l_t \leq 1 \end{aligned}$$

Solving this problem using FOC ends up with

$$\frac{\alpha}{l_t} = \frac{\gamma}{1 - l_t}$$

$\Rightarrow$

$$\begin{aligned} l_t^* &= \frac{\alpha}{\alpha + \gamma} \\ c_t^* &= A_t (l_t^*)^{\alpha} = A_t \left(\frac{\alpha}{\alpha + \gamma}\right)^{\alpha}. \end{aligned}$$

(b) One interpretation of the production technology is there is a fixed supply of one boat (or catamaran, denoted by k) so that the technology is

$$y_t = A_t k_t^{1-\alpha} l_t^{\alpha}.$$

Assume that the boat is owned by Crusoe and the technology is operated by a firm (Dole Fishing Co.) which rents capital k_t and labor l_t in a competitive market at factor prices r_t and w_t. Let p_t denote the Arrow-Debreu price of one unit of consumption at time t.

1. Set up Crusoe's decison problem as a HH problem.
   Answer: Crusoe's problem is

$$\begin{aligned} & \max_{\{c_t, l_t\}_{t=0}^T} \sum_{t=0}^{T} \beta^t [\ln c_t + \gamma \ln(1 - l_t)] \\ & s.t. \\ \sum_{t=0}^{T} p_t c_t & \leq \sum_{t=0}^{T} p_t (r_t k_t^s + w_t l_t^s) \\ c_t & \geq 0, 0 \leq l_t^s \leq 1, 0 \leq k_t^s \leq k \end{aligned}$$

2. Set up the firm's problem.
   Answer: The firm's problem is

$$\begin{aligned} \max_{\{k_t^d, l_t^d\}_{t=0}^T} \pi & = \sum_{t=0}^{T} p_t [y_t - r_t k_t^d - w_t l_t^d] \\ & s.t. \\ y_t & \leq A_t (k_t^d)^{1-\alpha} (l_t^d)^\alpha \\ l_t^d & \geq 0, k_t^d \geq 0 \end{aligned}$$

   This is equivalent to a one period problem

$$\begin{aligned} \max_{k_t^d, l_t^d} \pi_t & = p_t [y_t - r_t k_t^d - w_t l_t^d] \\ & s.t. \\ y_t & \leq A_t (k_t^d)^{1-\alpha} (l_t^d)^\alpha \\ l_t^d & \geq 0, k_t^d \geq 0. \end{aligned}$$

3. What are the resource constraint?
   Answer:

$$c_t \leq A_t k_t^{1-\alpha} l_t^\alpha.$$

4. Define a competitive equilibrium.
   Answer: A CE is a set of prices $\{p_t, r_t, w_t\}_{t=0}^T$, an allocation $\{k_t^d, l_t^d, y_t\}_{t=0}^T$ for the firm, and an allocation $\{c_t, k_t^s, l_t^s\}_{t=0}^T$ for the HH, such that
   (i). Given prices, $\{c_t, k_t^s, l_t^s\}_{t=0}^T$ solves HH's problem.
   (ii). Given prices, $\{k_t^d, l_t^d, y_t\}_{t=0}^T$ solves the firm's problem.
   (iii). All markets clear.

$$\begin{aligned} k_t^d & = k_t^s = k \\ l_t^d & = l_t^s = l_t \\ c_t & = A_t k^{1-\alpha} l_t^\alpha. \end{aligned}$$

5. Solve for the CE allocations.
Answer: Firm's FOCs

$$r_t = A_t(1-\alpha)k^{-\alpha}l_t^{\alpha} \quad (8)$$
$$w_t = A_t\alpha k^{1-\alpha}l_t^{\alpha-1} \quad (9)$$

HH's EE

$$\frac{w_t}{c_t} = \frac{\gamma}{1-l_t} \quad (10)$$

Substituting (9) into equation (10) to replace w_t, and using resource constraint $c_t = A_t k^{1-\alpha} l_t^{\alpha}$, we have

$$l_t^e = \frac{\alpha}{\alpha+\gamma}$$
$$c_t^* = A_t(l_t^*)^{\alpha} = A_t(\frac{\alpha}{\alpha+\gamma})^{\alpha}.$$

6. How does the CE change when A_0 rises? When A_1 rises?
Answer: When $A_0 \uparrow$, obviously $c_0 \uparrow$, but $(c_1, c_2, ..., c_T)$ are unchanged. (Because there is no storage technology to transform higher productivity today to smooth consumption.) r_0 and w_0 will also increase at the same rate as c_0. $\{l_t\}_{t=0}^{T}$ remains unchanged. Same things happen to the rise in A_1.

5. (*Irrelevance of capital ownership*) Exercise 2.9 in SLP. (10 points)

Answer: Under this new setup. the HH's problem is

$$\max_{\{c_t, n_t^s\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$
$$s.t.$$
$$\sum_{t=0}^{\infty} p_t c_t \leq \sum_{t=0}^{\infty} p_t w_t n_t^s + \pi$$
$$c_t \geq 0, 0 \leq n_t^s \leq 1$$

The firm's problem changes to

$$\max_{\{y_t, i_t, n_t^d\}_{t=0}^{\infty}} \pi = \sum_{t=0}^{\infty} p_t[y_t - w_t n_t^d - i_t]$$
$$s.t.$$
$$i_t = k_{t+1} - (1-\delta)k_t$$
$$y_t \leq F(k_t, n_t^d)$$
$$k_t \geq 0, n_t^d \geq 0$$

FOC for the HH is

$$\beta_t U'(c_t) = \mu p_t \quad (11)$$

FOCs for the firm are

$$\begin{aligned} n_t &: \quad w_t = F_n(k_t, n_t) && (12) \\ k_{t+1} &: \quad p_t = p_{t+1}[F_k(k_{t+1}, n_{t+1}) + 1 - \delta] && (13) \end{aligned}$$

Substituting (11) into (13), we obtain EE

$$U'(c_t) = \beta U'(c_{t+1})[F_k(k_{t+1}, n_{t+1}) + 1 - \delta]$$

This is the same EE as in the CE which HHs have the capital ownership. Besides, we can substitute the expression of firm's profits π into HH's BC

$$\begin{aligned} \sum_{t=0}^{\infty} p_t c_t &= \sum_{t=0}^{\infty} p_t w_t n_t + \pi \\ &= \sum_{t=0}^{\infty} p_t w_t n_t^s + \sum_{t=0}^{\infty} p_t [y_t - w_t n_t^d - i_t] \\ &= \sum_{t=0}^{\infty} p_t [y_t - i_t] \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} \sum_{t=0}^{\infty} p_t (c_t + i_t) &= \sum_{t=0}^{\infty} p_t y_t \\ &= \sum_{t=0}^{\infty} p_t [r_t k_t + w_t n_t] \end{aligned}$$

Therefore, these two problems are identical.

6. *(Continuity of contraction mapping)* Exercise 3.8 in SLP. (5 points)

   Answer: The function $T : S \to S$ is uniformly continuous if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that for all $x, y \in S$ with $|x - y| < \delta$ we have that $|Tx - Ty| < \varepsilon$.

   We already know that if T is a contraction, then for some $\beta \in (0, 1)$, we have

   $$\frac{|Tx - Ty|}{|x - y|} \leq \beta < 1, \; forall x, y \in S \text{ with } x \neq y$$

   Now let $\delta \equiv \frac{\varepsilon}{\beta} > 0$, then for any arbitray $\varepsilon > 0$, if $|x - y| < \delta$, then

   $$|Tx - Ty| \leq \beta |x - y| < \beta\varepsilon = \varepsilon.$$

7. *(An example of contraction mapping)* Exercise 3.10 in SLP. (10 points)

   Answer: a). Since v is bounded, so the continuous function f is bounded, say by M, on $[-\|v\|, +\|v\|]$. Then it is easy to show that $0 < |Tv(s)| \leq |c| + s\|v\|$, which the right hand side is clearly positive and bounded. Therefore, $\forall t > 0$, Tv is bounded on $[0, t]$. The continuity of TV is straightforward.

b). Note that

$$\begin{aligned}
|Tv(s) - Tw(s)| &\leq \int_0^s |f(v(z)) - f(w(z))| \, dz \\
&\leq \int_0^s B \, |v(z) - w(z)| \, dz \\
&\leq Bs \, \|v - w\| \, .
\end{aligned}$$

Choose $\tau = \beta/B$, where $\beta \in (0,1)$, then $0 \leq s \leq \tau$ implies that $Bs \|v - w\| \leq \beta \|v - w\|$. We then have

$$|Tv(s) - Tw(s)| \leq \beta \|v - w\| \, , \forall v, w \in C[0, t]$$

Put the sup on both side, we have

$$\|Tv(s) - Tw(s)\| \leq \beta \|v - w\| \, , \forall v, w \in C[0, t].$$

c). Suppose the fixed point is $x \in C[0, \tau]$, with

$$x(s) = c + \int_0^s f(x(z)) dz,$$

hence for $0 \leq s, s' \leq \tau$,

$$\begin{aligned}
x(s) - x(s') &= \int_{s'}^s f(x(z)) dz \\
&= f(x(\hat{z}))(s - s'), \text{ for some } \hat{z} \in [s', s].
\end{aligned}$$

Therefore, we have

$$\frac{x(s) - x(s')}{s - s'} = f(x(\hat{z}))$$

Let $s' \to s$, then $\hat{z} \to s$, so we have $x'(s) = f(x(s))$. The uniqueness comes directly from the Contraction Mapping Theorem.

8. (a) We start with the first period's budget constraint:

$$\begin{aligned}
c_0 + qb_2 &= b_0 + w \\
b_1 &= q^{-1}(b_0 + w - c_0)
\end{aligned}$$

For $t = 1$ we have:

$$\begin{aligned}
c_1 + qb_2 &= b_1 + w \\
b_2 &= q^{-1}(b_1 + w - c_1) \\
&= q^{-1}(q^{-1}(b_0 + w - c_0) + w - c_1) \\
&= q^{-2}b_0 + q^{-2}w + q^{-1}w - q^{-2}c_0 - q^{-1}c_1
\end{aligned}$$

For $t = 2$ we have:

$$\begin{aligned} c_2 + qb_3 &= b_2 + w \\ b_3 &= q^{-1}(b_2 + w - c_2) \\ &= q^{-1}(q^{-2}b_0 + q^{-2}w + q^{-1}w - q^{-2}c_0 - q^{-1}c_1 + w - c_2) \\ &= q^{-3}b_0 + q^{-3}w + q^{-2}w + q^{-1}w - q^{-3}c_0 - q^{-2}c_1 - q^{-1}c_2 \end{aligned}$$

and so on, until we have:

$$\begin{aligned} b_{(t+1)} &= q^{-(t+1)}b_0 + w(q^{-(t+1)} + ... + q^{-1}) - (q^{-(t+1)}c_0 + q^{-t}c_1 + ... + q^{-1}c_t) \\ q^{(t+1)}b_{(t+1)} &= b_0 + w(1 + ... + q^t) - (c_0 + qc_1 + ... + q^t c_t) \\ &= b_0 + w\sum_{k=0}^{t} q^k - \sum_{k=0}^{t} q^k c_k \end{aligned}$$

taking the limit gives us:

$$\lim_{t\to\infty} q^{(t+1)}b_{(t+1)} = b_0 + w \lim_{t\to\infty} \sum_{k=0}^{t} q^k - \lim_{t\to\infty} \sum_{k=0}^{t} q^k c_k$$

$$0 = b_0 + \frac{w}{1-q} - \sum_{t=0}^{\infty} q^t c_t$$

where we used the no-Ponzi game condition on the LHS. So the consumer's consolidated (or lifetime) budget constraint is

$$\sum_{t=0}^{\infty} q^t c_t = b_0 + \frac{w}{1-q}$$

(b) The transversality condition is:

$$\lim_{t\to\infty} \beta^t u'(c_t^*)b_t^* = 0$$

or

$$\lim_{t\to\infty} \beta^t u'(b_t - qb_{(t+1)} + w)b_t = 0$$

the Euler equation is

$$qu'(b_t - qb_{t+1} + w) = \beta u'(b_{t+1} - qb_{t+2} + w)$$

We want to show

$$\sum_{t=0}^{\infty} q^t b_t = 0.$$

Note that the Euler equation holds $\forall t$

$$\begin{aligned} qu'(b_0 - qb_1 + w) &= \beta u'(b_1 - qb_2 + w) \\ qu'(b_1 - qb_2 + w) &= \beta u'(b_2 - qb_3 + w) \\ q^2 u'(b_0 - qb_1 + w) &= \beta^2 u'(b_2 - qb_3 + w) \end{aligned}$$

So we have

$$q^t u'(b_0 - qb_1 + w) = \beta^t u'(b_t - qb_{t+1} + w)$$

Multiplying both sides by b_t and taking limits:

$$\begin{aligned}
\lim_{t\to\infty} q^t u'(b_0 - qb_1 + w)b_t &= \lim_{t\to\infty} \beta^t u'(b_t - qb_{t+1} + w)b_t \\
u'(b_0 - qb_1 + w) \lim_{t\to\infty} q^t b_t &= 0 \text{ (using the transversality condition)}
\end{aligned}$$

and so

$$\lim_{t\to\infty} q^t b_t = 0$$

(c) We want to prove that a sequence $\{b_t^*\}_{t=0}^{\infty} = 0$ that satisfies the transversality condition and the Euler equation maximizes the consumer's objective function, subject to the sequence of budget constraints and the nPg condition.

Modified proof:

Consider any alternative feasible sequence $b = \{b_t^*\}_{t=0}^{\infty}$.We want to show that for any such sequence,

$$\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t [u(b_t^* - qb_{t+1}^* + w) - u(b_t - qb_{t+1} + w)] \geq 0$$

Define

$$A_T(b) = \sum_{t=0}^{T} \beta^t [u(b_t^* - qb_{t+1}^* + w) - u(b_t - qb_{t+1} + w)] \geq 0$$

We will show that, as T goes to infinity $A_T(b)$ is bounded below by zero.

By convavity of u,

$$\begin{aligned}
A_T(b) &\geq \sum_{t=0}^{T} \beta^t [u'(b_t^* - qb_{t+1}^* + w)(b_t^* - b_t) - qu'(b_t^* - qb_{t+1}^* + w)(b_{t+1}^* - b_{t+1})] \\
&= [\sum_{t=0}^{T} \beta^t u'(b_t^* - qb_{t+1}^* + w)(b_t^* - b_t)] - [\sum_{t=0}^{T} \beta^t qu'(b_t^* - qb_{t+1}^* + w)(b_{t+1}^* - b_{t+1})] \\
&= u'(b_0^* - qb_1^* + w)(b_0^* - b_0)] + \sum_{t=1}^{T} \beta^t u'(b_t^* - qb_{t+1}^* + w)(b_t^* - b_t) - \\
&\quad [\sum_{t=0}^{T-1} \beta^t qu'(b_t^* - qb_{t+1}^* + w)(b_{t+1}^* - b_{t+1}) - \beta^T qu'(b_T^* - qb_{T+1}^* + w)(b_{T+1}^* - b_{T+1})] \\
&= [u'(b_0^* - qb_1^* + w)(b_0^* - b_0)] + \sum_{t=0}^{T-1} \beta^{t+1} u'(b_{t+1}^* - qb_{t+2}^* + w)(b_{t+1}^* - b_{t+1}) - \\
&\quad [\sum_{t=0}^{T-1} \beta^t qu'(b_t^* - qb_{t+1}^* + w)(b_{t+1}^* - b_{t+1}) - \beta^T qu'(b_T^* - qb_{T+1}^* + w)(b_{T+1}^* - b_{T+1})] \\
&= u'(b_0^* - qb_1^* + w)(b_0^* - b_0) - \beta^T qu'(b_T^* - qb_{T+1}^* + w)(b_{T+1}^* - b_{T+1}) + \\
&\quad \sum_{t=0}^{T-1} \beta^t (b_{t+1}^* - b_{t+1})[\beta u'(b_{t+1}^* - qb_{t+2}^* + w) - qu'(b_t^* - qb_{t+1}^* + w)]
\end{aligned}$$

Note that:

$$
\begin{aligned}
(b_0^* - b_0) &= 0 \Longrightarrow u^{'}(b_0^* - qb_1^* + w)(b_0^* - b_0) = 0 \\
\beta u'(b_{t+1}^* - qb_{(t+2)}^* + w) - qu'(b_t^* - qb_{t+1}^* + w) &= 0 \quad \text{(Euler Equation).}
\end{aligned}
$$

And so we have

$$
\begin{aligned}
A_T(b) &\geq -\beta^T qu^{'}(b_T^* - qb_{T+1}^* + w)(b_{T+1}^* - b_{T+1}) \\
&= \beta^T qu^{'}(b_T^* - qb_{T+1}^* + w)(-b_{T+1}^* + b_{T+1}) \\
&= \beta^T \beta u^{'}(b_{T+1} - qb_{T+2} + w)(-b_{T+1}^* + b_{T+1})
\end{aligned}
$$

Taking limits we have

$$
\lim_{T\to\infty} A_T(b) \geq \lim_{T\to\infty} \beta^{T+1} u'(b_{T+1}^* - qb_{T+2}^* + w)(-b_{T+1}^* + b_{T+1}^*)
$$

Recalling that

$$
\begin{aligned}
q^t u'(b_0^* - qb_1^* + w) &= \beta^t u'(b_t^* - qb_{t+1}^* + w) \\
u'(b_0^* - qb_1^* + w) &= \frac{\beta^t}{q^t} u'(b_t^* - qb_{t+1}^* + w) \\
\lim_{t\to\infty} q^t b_t &= 0
\end{aligned}
$$

and

$$
\lim_{t\to\infty} \beta^t u^{'}(c_t^*) b_t^* = 0
$$

We have

$$
\begin{aligned}
\lim_{T\to\infty} A_T(b) &\geq \lim_{T\to\infty} \beta^{T+1} u^{'}(b_{T+1}^* - qb_{T+2}^* + w)(-b_{T+1} + b_{T+1}^*) \\
&= \lim_{T\to\infty} \frac{\beta^{T+1}}{q^{T+1}} q^{T+1} u^{'}(b_{T+1}^* - qb_{T+2}^* + w) b_{T+1} - \lim_{T\to\infty} \beta^{T+1} u^{'}(b_{T+1}^* - qb_{T+2}^* + w) b_{T+1}^* \\
&= \lim_{T\to\infty} u^{'}(b_0^* - qb_1^* + w) q^{T+1} b_{T+1} = 0
\end{aligned}
$$

9. (a) Carefully define a sequential competitive equilibrium for this economy.

A sequential competitive equilibrium for the economy $\{u_A, u_B, w\}$, is a sequence $\{c_{it}^*\}_{t=0}^{\infty}, \{a_{i,t+1}^*\}_{t=0}^{\infty}, \{q_t^*\}_{t=0}^{\infty}$ (where q_t^*means price of Arrow security) for $i = A, B$ such that

1. For $i = A, B$,

$$
\begin{aligned}
\left\{c_{it}^*, a_{i,t+1}^*\right\}_{t=0}^{\infty} &= \arg\max \sum_{t=0}^{\infty} \beta_i^t \log(c_{it}) \\
& \quad s.t. \\
c_{it} + q_t^* a_{i,t+1} &= a_{i,t} + \omega \\
\lim_{t\to\infty} a_{i,t+1} \left(\prod_{t=0}^{\infty} q_t \right) &\geq 0 \\
a_{i,0} &= 0, \ c_{it} \geq 0
\end{aligned}
$$

2. $\lambda c^*_{At} + (1-\lambda)c^*_{Bt} = w$ for $t = 0, 1, 2...$
3. $\lambda a^*_{A,t+1} + (1-\lambda)a^*_{B,t+1} = 0$ for $t = 0, 1, 2...$

(b) Carefully define a recursive competitive equlibrium for this economy.

A Recursive Competitive Equilibrium (RCE) for the economy $\{u_A, u_{B,} w\}$ is a set of functions: price funtion: $q(K)$; policy function: $a'_i = g_i(a_i, K)$; value function: $v_i(a_{i,} K)$; transition function: $K' = G(K)$ ($K_t = \sum a_{A,t}$ is the aggregate asset holdings at time t by type-A agents; a_i is the individual asset holdings for the representative agent in each type i) such that:

1. For $i = A, B$, $a'_i = g_i(a_{i,} K)$ and $v_i(a_i, K)$ solve

$$\begin{aligned} v_i(a_{i,} K) &= \max_{\{c_i, a'_i\}} \log(c_i) + \beta_i v_i(a'_i, K') \\ & \quad s.t. \\ c_i + q(K)a'_i &= a_i + w \\ K' &= G(K) \end{aligned}$$

2. Consistency

$$\begin{aligned} G(K) &= g_A(K, K) \\ -\frac{\lambda}{1-\lambda} G(K) &= g_B(-\frac{\lambda}{1-\lambda} K, K) \end{aligned}$$

(c) Show that this economy has steady state: in particular, show that the type-B agents become poorer and poorer over time and consume zero in the limit.

To solve this problem, we first get Euler equation. In recursive formulation, consumers solve

$$\begin{aligned} v_i(a_i, K) &= \max_{\{c_i, a'_i)} \log(a_i + w - q(K)a'_i) + \beta_i v_i(a'_i, K') \\ & \quad s.t. \\ K' &= G(K) \end{aligned}$$

Solve for F.O.C. and use envelope condition, we get the euler equation

$$\beta_i \frac{u'(c_{i,t+1})}{u'(c_{i,t})} = q(K)$$

or equivalently,

$$\beta_A \frac{u'(c_{A,t+1})}{u'(c_{A,t})} = \beta_B \frac{u'(c_{B,t+1})}{u'(c_{B,t})}$$

Now we can see that $c_{i,t+1} \neq c_{i,t} (\forall i, \forall t)$. Suppose not, without loss of generality let $c_{A,t+1} = c_{A,t}$. By feasibility condition, we know that $c_{B,t+1} = c_{B,t}$. Plug into the equation we get $\beta_A = \beta_B$, a contradiction. As a result, there cannot be any steady state in this economy.

We start to prove the convergence property of consumption path. First, we want to show that $[C_{At}]_{t=0}^{\infty}$ ($\{c_{Bt}\}_{t=0}^{\infty}$) is an increasing (decreasing) sequence. We already know that $c_{A,t+1} \neq c_{A,t}(\forall t)$. Now suppose that $c_{A,t+1} < c_{A,t}$ for some t. By the feasibility condition, we know that $c_{B,t+1} > c_{B,t}$. From the strict concavity of felicity function, we have

$$\frac{u'(c_{A,t+1})}{u'(c_{A,t})} > 1 > \frac{u'(c_{B,t+1})}{u'(c_{B,t})}$$

$\Rightarrow$

$$\beta_A \frac{u'(c_{A,t+1})}{u'(c_{A,t})} > \beta_B \frac{u'(c_{B,t+1})}{u'(c_{B,t})}$$

which contradicts Euler equation.

Since bounded montone sequence has a limit, we have $c_{At} \longrightarrow \bar{c}$ for $t \longrightarrow \infty$. But we have shown that the economy has no steady state, so c_{At} can converge to nowhere but the boundary, i.e. $c_{At} \longrightarrow w$ and $c_{Bt} \rightarrow 0$.