1 Some Important Distributions

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1.1 The Gamma Distribution

The gamma function is defined as:

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} \exp(-y) dy.$$

It is shown in advanced Calculus texts that $\Gamma(\alpha) < \infty$ for $\alpha > 0$. In addition, using integration by parts and choosing $v = y^{\alpha-1}$ and $u' = \exp(-y)$ (which implies that $v' = (\alpha-1)y^{\alpha-2}$ and $u = -\exp(-y)$), we obtain

$$\Gamma(\alpha) = (\alpha-1)\int_0^\infty y^{\alpha-2} \exp(-y) dy = (\alpha-1)\Gamma(\alpha-2).$$

Thus, when α is an integer, we will have that

$$\Gamma(\alpha) = (\alpha-1)!.$$

If we choose $y = \frac{x}{\beta}$ for some $\beta > 0$ then we obtain

$$\Gamma(\alpha) = \int_0^\infty \left(\frac{x}{\beta}\right)^{\alpha-1} \exp\left(-\frac{x}{\beta}\right)\left(\frac{1}{\beta}\right) dx \Leftrightarrow 1 = \int_0^\infty \underbrace{\frac{1}{\Gamma(\alpha)}\left(\frac{x}{\beta}\right)^{\alpha-1} \exp\left(-\frac{x}{\beta}\right)\left(\frac{1}{\beta}\right)}_{f(x)} dx.$$

We call $f(x)$ the **gamma distribution.** If we let $\alpha = \frac{r}{2}$ for an integer r and $\beta = 2$ then we get that

$$f(x) = \frac{1}{\Gamma\left(\frac{r}{2}\right) 2^{\frac{r}{2}}} (x)^{\frac{r}{2}-1} \exp\left(-\frac{x}{2}\right) \text{ for } 0 < x < \infty$$

which we call the chi-squared distribution with r degrees of freedom which we denote as χ^2_r.

1.2 Normal Distribution

We now will define the normal distribution. To do this, we start out by defining

$$I = \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2}\right) dy$$

and then calculate

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{y^2 + z^2}{2}\right) dydz.$$

To help us compute this integral, we will use polar coordinates and define the transformation

$$y = r\cos\theta \text{ and } z = r\sin\theta \text{ for } 0 \le r < \infty \text{ and } 0 \le \theta \le 2\pi.$$

The Jacobian of this transformation is

$$J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

which gives us that

$$|J| = r\left(\cos^2\theta + \sin^2\theta\right) = r.$$

Thus, we will have that

$$I^2 = \int_0^{2\pi} \int_0^{\infty} \exp\left(-\frac{r^2}{2}\right) r dr d\theta.$$

The inside integral can be computed by substituting $u = r^2$ (which implies $du = 2rdr$.) This gives us that

$$\int_0^{\infty} \exp\left(-\frac{r^2}{2}\right) rdr = \frac{1}{2}\int_0^{\infty} \exp\left(-\frac{u}{2}\right) du = \frac{1}{2} * -2\exp\left(-\frac{u}{2}\right)|_0^{\infty} = 0 + 1 = 1.$$

Therefore, we can conclude that

$$I^2 = 2\pi$$

which, in turn, gives us that

$$1 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy.$$

Next, we define $y = \frac{x-a}{b}$ for $b > 0$. This change of variables gives us that

$$1 = \int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} \exp\left(-\frac{(x-a)^2}{2b^2}\right) dx.$$

Note that both of the above integrands constitute proper PDF as they are always positive on the real line and integrate to unity. We can now determine a and b by calculating the moment generating function for this random variable:

$$
\begin{aligned}
M(t) &= \int_{-\infty}^{\infty} \exp(tx)\frac{1}{b\sqrt{2\pi}} \exp\left(-\frac{(x-a)^2}{2b^2}\right) dx \\
&= \int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} \exp\left(\frac{2b^2tx - x^2 + 2xa - a^2}{2b^2}\right) dx \\
&= \exp\left(-\frac{a^2 - (a+b^2t)^2}{2b^2}\right) \int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} \exp\left(-\frac{(x-a-b^2t)^2}{2b^2}\right) dx \\
&= \exp\left(at + \frac{b^2t^2}{2}\right).
\end{aligned}
$$

The last equality follows because the integrand in the second to last line will be unity (we just showed this!). So, we will have that

$$M'(t) = \left(a + b^2 t\right) M(t)$$

and

$$M''(t) = \left(a + b^2t\right) M'(t) + b^2 M(t) = M(t) \left[\left(a + b^2t\right)^2 + b^2\right].$$

Noting that $M(0) = 1$, we will have that $M'(0) = a = \mu$ and that $M''(0) = a^2 + b^2$ which gives us that $b^2 = E\left[X^2\right] - \mu^2 = \sigma^2$. Accordingly, we will have that

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

This is the PDF for a **Normal Random Variable** with mean μ and variance σ^2. The notation for such a random variable is $N(\mu, \sigma^2)$. Note that if we substitute $y = \frac{x-\mu}{\sigma}$ in the above distribution, we obtain that $Y \sim N(0,1)$. We call Y a **standard normal random variable**.

The normal and χ^2 distributions are related in the sense that $V \equiv Y^2 \sim \chi_1^2$. To see this, first, note that

$$G(v) = P\left(Y^2 \leq v\right) = P(-\sqrt{v} \leq Y \leq \sqrt{v}) = 2\int_0^{\sqrt{v}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy.$$

Next, we substitute $y = \sqrt{w}$ which implies that $dy = \frac{1}{2}w^{-\frac{1}{2}}dw = \frac{dw}{2\sqrt{w}}$ which gives us that

$$G(v) = \int_0^v \frac{1}{\sqrt{w}\sqrt{2\pi}} \exp\left(-\frac{w}{2}\right) dw$$

and, thus,

$$G'(v) = g(v) = \frac{1}{\sqrt{2}\sqrt{\pi}} v^{\frac{1}{2}-1} \exp\left(-\frac{v}{2}\right).$$

However, because $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, we will have that $V \sim \chi_1^2$.

1.3 Multivariate Normal Distribution

We now consider the multi-variate distribution

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp(-q/2)$$

where $\sigma_1, \sigma_2 > 0$, $-1 < \rho < 1$ and

$$q = \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right].$$

We will show that

$$f(x,y) \text{ is a PDF} \tag{1}$$

$$X \sim N(\mu_1, \sigma_1^2) \text{ and } Y \sim N(\mu_2, \sigma_2^2) \tag{2}$$

$$Cor(X,Y) = \rho. \tag{3}$$

We start out by writing

$$\left(1-\rho^2\right) q = \left(\frac{y-b}{\sigma_2}\right)^2 + \left(1-\rho^2\right)\left(\frac{x-\mu_1}{\sigma_1}\right)^2$$

where

$$b = \mu_2 + \rho\left(\frac{\sigma_2}{\sigma_1}\right)(x-\mu_1).$$

The calculation is just a lot of tedious algebra. Now, we can write

$$f_1(x) = \frac{1}{\sigma_1\sqrt{2\pi}} \exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right) \int \frac{1}{\sigma_2\sqrt{1-\rho^2}\sqrt{2\pi}} \exp\left(-\frac{(y-b)^2}{2\sigma_2^2(1-\rho^2)}\right) dy.$$

From what we know about the Normal Distribution, we can conclude that the integral is unity and, thus, we will have

$$f_1(x) = \frac{1}{\sigma_1\sqrt{2\pi}} \exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right).$$

This gives us part of (2). Because $f(x, y)$ is symmetric, we can also conclude the second half of (2). Also, because $f_1(x)$ is a Normal PDF, we can conclude that

$$\int\int f(x, y)dxdy = 1$$

and, consequently, we have (1). All that remains is to show that ρ is the correlation between X and Y. Given (1) and (2), we can conclude that

$$g_2(y|x) = \frac{1}{\sigma_2\sqrt{1-\rho^2}\sqrt{2\pi}} \exp\left(-\frac{(y-b)^2}{2\sigma_2^2(1-\rho^2)}\right).$$

Accordingly, we will have that

$$Y|X = x \sim N(b, \sigma_2^2(1-\rho^2))$$

which implies that

$$E[Y|X = x] = b = \mu_2 + \rho\left(\frac{\sigma_2}{\sigma_1}\right)(x - \mu_1)$$

but by Theorem 2.4.1 from HMC, we know that ρ must be the correlation between X and Y.

Finally, referring back to the definition of the bivariate Normal distribution, note that if $\rho = 0$, then the joint distribution can be written as the product of its marginals. This then tells us that, for normal random variables, uncorrelatedness and independence are equivalent. We must emphasize once again that this is not true in general.

1.4 Student's Theorem

We will now state and prove Student's Theorem. While this theorem is interesting on its own, the primary purpose of this exercise will be to further develop the student's faculty with the tools and concepts that we have developed thus far. In the proof, at times, we will appeal to some results which we have not discussed but can be found in the text. We will be careful to point out at which points in the derivation we are appealing to such results. Finally, the theorem will involve a distribution that we have not seen before: the t distribution which is the ratio of a standard normal random variable to the square root of a chi-squared random variable divided by its degrees of freedom.

Theorem 1 *(Student's Theorem) Let $X_1, ..., X_n$ be an i.i.d. sample where $X_i \sim N(\mu, \sigma^2)$. Define $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n-1}\sum (X - \overline{X})^2$. Then we will have that (i) $\overline{X}$ is independent of S^2 (ii) $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$ (iii) $T = \frac{\overline{X} - \mu}{s/\sqrt{n}} \sim t_{n-1}$*

Proof. *Define $X = (X_1, ..., X_n)'$. This gives us that*

$$X \sim N(\mu\mathbf{1}, \sigma^2\mathbf{I})$$

where $\mathbf{1}$ is an $nX1$ vector of ones and $\mathbf{I}$ is the n-dimensional identity matrix. Next, we define $\mathbf{v}' = \left(\frac{1}{n}, ..., \frac{1}{n}\right) = \frac{1}{n}\mathbf{1}'$. Thus, we will have that

$$\overline{X} = \mathbf{v}'\mathbf{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Next, we define $Y = \left(X_1 - \overline{X}, ..., X_n - \overline{X}\right)'$ *and consider*

$$W = \begin{bmatrix} \overline{X} \\ \\ Y \end{bmatrix} = \begin{bmatrix} \mathbf{v}' \\ \\ \mathbf{I} - \mathbf{1}\mathbf{v}' \end{bmatrix} X.$$

Note that

$$\mathbf{1}\mathbf{v}' = \begin{bmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \\ \\ \vdots & \ddots & \vdots \\ \\ \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix}.$$

Now, appealing to Theorem 3.5.1 from the HMC, we will have that

$$W \sim N\left(\mu_W, \mathbf{V}_W\right)$$

where

$$\mu_W = \left[\mu, \underbrace{\mathbf{O}_n}_{1Xn}\right]'$$

and

$$\begin{aligned} \mathbf{V}_W &= \begin{bmatrix} \mathbf{v}' \\ \\ \mathbf{I} - \mathbf{1}\mathbf{v}' \end{bmatrix} \sigma^2 \mathbf{I} \begin{bmatrix} \mathbf{v}' \\ \\ \mathbf{I} - \mathbf{1}\mathbf{v}' \end{bmatrix}' \\ &= \sigma^2 \begin{bmatrix} \frac{1}{n} & \mathbf{O}_n \\ \\ \mathbf{O}_n' & \mathbf{I} - \mathbf{1}\mathbf{v}' \end{bmatrix}. \end{aligned}$$

To better understand this calculation, note the following:

$$\mathbf{v}'\mathbf{v} = n * \frac{1}{n^2} = \frac{1}{n}$$

and

$$(\mathbf{I} - \mathbf{1v}')(\mathbf{I} - \mathbf{v1}') = \mathbf{I} - 2\mathbf{1v}' + \mathbf{1v}'\mathbf{v1}' = \mathbf{I} - 2\mathbf{1v}' + \frac{1}{n} * \mathbf{11}' = \mathbf{I} - \mathbf{1v}'.$$

We leave it as an exercise to show that the off-diagonal elements are zeros. Because the covariance of $\overline{X}$ *and* Y *is zero and because they are jointly normal, we can also conclude that they are independent. Next note that because* $S^2 = \frac{1}{n-1}Y'Y$*, we obtain that* $\overline{X}$ *is independent of* S^2 *which is part (i). Next, we consider*

$$V = \sum_{i=1}^{n} \left(\frac{x_i - \mu}{\sigma}\right)^2.$$

Note that the summands are all χ_1^2 *and, thus,* $V \sim \chi_n^2$ *since the sum of* n *independent* χ_1^2 *random variables is* χ_n^2*. Now, we note that*

$$\begin{aligned} V &= \sum_{i=1}^{n} \left(\frac{x_i - \overline{x} + \overline{x} - \mu}{\sigma}\right)^2 \\ &= \sum_{i=1}^{n} \left(\frac{x_i - \overline{x}}{\sigma}\right)^2 + n\left(\frac{\overline{x} - \mu}{\sigma}\right)^2 \\ &= \underbrace{\frac{(n-1)S^2}{\sigma^2}}_{A} + \underbrace{\left(\frac{\overline{x} - \mu}{\sigma / \sqrt{n}}\right)^2}_{B}. \end{aligned}$$

But,

$$\frac{\overline{x} - \mu}{\sigma / \sqrt{n}} \sim N(0, 1)$$

and, thus, $B \sim \chi_1^2$. *Next, we take the MGF of* V *and obtain*

$$\begin{aligned} (1-2t)^{-n/2} &= E\left[\exp\left(\frac{t\,(n-1)\,S^2}{\sigma^2}\right)\right](1-2t)^{-1/2} \Rightarrow \\ E\left[\exp\left(\frac{t\,(n-1)\,S^2}{\sigma^2}\right)\right] &= (1-2t)^{-(n-1)/2}. \end{aligned}$$

Note that we used the independence of A *and* B *in the above calculation. This gives us that* $A \sim \chi_{n-1}^2$ *which is part (ii) of the theorem. Finally, we define*

$$\begin{aligned} T &= \frac{\overline{x}-\mu}{s/\sqrt{n}} \\ &= \frac{\frac{\overline{x}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{(n-1)S^2}{\sigma^2(n-1)}}}, \end{aligned}$$

but the numerator is simply is a standard normal random variable and the denominator is the square root of a chi-squared random variable divided by its degrees of freedom. Thus, we have that $T \sim t_{n-1}$. *This completes the proof.* ■