Incomplete Blocks

In an incomplete block design, not all treatments are tested in each block, i.e. there is an incomplete set of treatments in each block.

Designs are complex and are usually looked up in tables.

Advantages
Small blocks are more homogeneous than large blocks, so experimental error is lower.
Use when there is variability within larger blocks.
Use to increase precision.

Disadvantages
Designs can require a fixed number of treatments, a fixed number of reps, or both.
More reps are needed.
The complexity of the analysis is increased.
There is unequal precision for certain comparisons of treatment means.

Uses
Use to reduce block size in single factor experiments when the number of treatments is large.

Balanced Incomplete Block Designs

Every treatment or entry occurs an equal number of times in a block with every other treatment.
All comparisons have equal precision, i.e. the variance is constant.
Treatments and blocks are not orthogonal (independent).
λ = number of times each pair of treatments occurs together

Balanced Incomplete Blocks for 7 Treatments
# of treatments, t = 7   # of blocks, b = 7 $\lambda = 1$ block size, k = 3

Block

| (1) | 1 2 4 | (3) | 3 4 6 | (5) | 1 5 6 | (7) | 1 3 7 |
| (2) | 2 3 5 | (4) | 4 5 7 | (6) | 2 6 7 | | |

Preliminary ANOVA

Source	df	
Total	20	
Treatment	6	Treatment and Block are Not Orthogonal
Block	6	
Error	8	

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Incomplete Latin Squares or Youden Squares

Column balanced relative to treatment
Row orthogonal to treatment

t = 7   # of reps, r = 3   k = 3
First 3 rows of Latin square

Columns (Blocks)

Rows	(1)	(2)	(3)	(4)	(5)	(6)	(7)
(1)	1	2	3	4	5	6	7
(2)	2	3	4	5	6	7	1
(3)	4	5	6	7	1	2	3

Preliminary ANOVA

Source	df
Total	20
Treatment	6
Column	6
Row	2
Error	6

Lattices

Number of treatments, t, is a perfect square, $t = k^2$
The block size, k, is the square root of t, or $k = \sqrt{t}$
Lattice designs can be looked up in tables. After obtaining the design, randomize
   1) reps
   2) block in rep
   3) treatment in block

Balanced Lattice Designs

Each rep has a complete set of treatments
Number of reps, r = k + 1
Each rep has k blocks, each containing k treatments
Number of blocks, b = k*r = k(k + 1)
       t*r = k*b = total number of observations

Every pair of treatments occurs together in the same block exactly once,

$$\lambda = \frac{r(k - 1)}{t - 1} = 1$$

Degree of precision for comparing means is the same for all pairs
Block and treatment are not orthogonal. In each block a treatment occurs either 0 or 1 times, i.e. number of observations is not equal.

Block	Rep I		Rep II		Rep III		Rep IV
(1)	1 2 3	(4)	1 4 7	(7)	1 5 9	(10)	1 8 6
(2)	4 5 6	(5)	2 5 8	(8)	7 2 6	(11)	4 2 9
(3)	7 8 9	(6)	3 6 9	(9)	4 8 3	(12)	7 5 3

Number of treatments, $t = k^2 = 9$
Block size, $k = \sqrt{t} = 3$
Number of reps, $r = k + 1 = 4$
Number of blocks, $b = k*r = k(k + 1) = 12$
Number of observations, $tr = kb = 36$

$$\lambda = \frac{r(k - 1)}{t - 1} = \frac{4(2)}{8} = 1$$

ANOVA for Balanced Lattice - preliminary

Source		df
Total	$k^2(k + 1) - 1$	35
Treatment	$k^2 - 1$	8
Rep	k	3
Block(Rep)	$k^2 - 1$	8
Error	$(k - 1)(k^2 - 1)$	16

Partially Balanced Lattice

Partially balanced lattice has a more flexible number of reps, r
      Simple/double lattice = 2 reps
      Triple lattice = 3 reps
      Quadruple lattice = 4 reps (balanced lattice in the example above)
Lose symmetry where every pair of treatments occurs together once
Comparisons of treatments in the same block have higher precision
Comparisons of treatments not in the same block have lower precision
Data analysis is complicated
To set up partially balance lattice, use first "r" reps of balanced lattice.

Setting up a partially balanced lattice using first r/p reps of a balanced lattice repeated p times is not as good

Lattice Squares

Similar to lattices.
Treatment balanced with respect to both rows and columns
Treatment is not orthogonal to (independent of) Row or Column

$t = 9$ $r = 4$ $k = 3$

	Rep I				Rep II				Rep III				Rep IV		
Cols	(1)	(2)	(3)		(4)	(5)	(6)		(7)	(8)	(9)		(10)	(11)	(12)
Rows															
(1)	1	2	3	(4)	1	4	7	(7)	1	6	8	(10)	1	9	5
(2)	4	5	6	(5)	2	5	8	(8)	9	2	4	(11)	6	2	7
(3)	7	8	9	(6)	3	6	9	(9)	5	7	3	(12)	8	4	3

ANOVA for Lattice Square - preliminary

Source	df	
Total	tr - 1	35
Treatment	$k^2 - 1$	8
Rep	k	3
Row(Rep)	$k^2 - 1$	8
Column(Rep)	$k^2 - 1$	8
Error		8

Intrablock Analysis of Balanced Lattice

Intrablock analysis assumes blocks are fixed effects
Treatments are adjusted to correct for block effects. In comparing 2 treatments, they occur once in the same block, and the rest of the time in different blocks, so treatment differences are affected by block differences. Each treatment total is therefore adjusted for the total of the blocks in which it occurs.

Mathematical model

$$X_{ijq} = \overline{X}... + r_i + b_{ij} + t_q + e_{ijq}$$

Normal equation for T_a total

$$T_q = (k + 1)\overline{X}... + \sum b_{ij} + (k + 1)t_q$$

Normal equation for B_{ii} total

$$B_{ij} = k\overline{X}... + k r_i + kb_{ij} + \sum t_q$$

Calculations:
Block totals, B
Rep totals, R
Treatment totals, T
Grand total, $G = \Sigma X$

$$cf = \frac{(\Sigma X)^2}{k^2(k + 1)} = \frac{(G)^2}{k^2(k + 1)}$$

ANOVA

Source	df		SS	MS = SS/df
Total	$k^2(k + 1) - 1$	35	$\Sigma X^2 - cf$	
Trt (unadj), T	$k^2 - 1$	8	$\dfrac{\Sigma T^2}{(k + 1)} - cf$	
Rep	k	3	$\dfrac{\Sigma R^2}{k^2} - cf$	
Bl (Rep) (adj)	$k^2 - 1$	8		$E_b = SS/df$
Intrablock Err	$(k-1)(k^2-1)$	16		$E_e = SS/df$
Trt (adj), T'	$k^2 - 1$	8	smaller than for T	
Effective Error	$(k-1)(k^2-1)$	16	larger than for E_e	E_e'

Adjustment calculations
For each treatment, calculate sum of block totals for blocks with treatment t, B_t
      ΣB_t for all treatments = kG
Weights for each treatment (adjustment for block)
      $W = kT - (k + 1)B_t + G$
      ΣW for all treatments = 0
Sums of squares of blocks within reps adjusted for treatment effects

$$SSBlock(adj) = \frac{\Sigma^t_{i=1}W_i^2}{k^3(k + 1)} = \frac{\Sigma(kT - (k + 1)B_i + G)^2}{k^3(k + 1)}$$

    SSIntrablock Error = SSTotal - SSTrt - SSRep - SSBlock(adj)

or   SSIntrablock Error = SSError - SSBlock

Adjustment factor for error and treatment means

$$\mu = \frac{MSBlock(adj) - MSIntrablock\ Error}{k^2 MSBlock(adj)} = \frac{E_b - E_e}{k^2 E_b}$$

If μ is negative, set μ to 0. No further adjustment is needed.

Adjusted totals for each treatment
       $T' = T + \mu W$

Adjusted treatment means

$$M' = \frac{T'}{k+1} = \frac{T'}{r}$$

Adjusted treatment SS

$$SSTrt(adj) = \frac{\sum T'^2}{k+1} - \frac{G^2}{k^2(k+1)}$$

Adjusted treatment MS

$$MSTrt(Adj) = \frac{1}{(k+1)(k^2-1)}\left(\sum T'^2 - \frac{G^2}{k^2}\right)$$

Effective error mean square - to be used for F test

$$E_e' = (MSIntrablockErr)(1 + k\mu) = (E_e)(1 + k\mu)$$

$$CV = \frac{\sqrt{Effective\ Error\ MS}}{\overline{G}} * 100\%$$

$$F = \frac{MSTrt(adj)}{Effective\ Error\ MS}$$

       df numerator = $k^2 - 1$
       df denominator = $(k - 1)(k^2 - 1)$

Relative Efficiency

$$RE = \frac{100(SSBlock(adj) + SSIntrablock\ Error)}{k(k^2 - 1)MSEffective\ Error}$$

Recovery of Inter-block Information

Assumes blocks are random, with mean 0 and variance σ_b^2
Observations in the same block are positively correlated.
Analysis uses maximum likelihood estimation with joint frequency distribution of all observations. This is equivalent to minimization of weighted sums of squares.
For 2 observations in the same block, variance $= 2\sigma_e^2$
For 2 observations in different blocks, variance $= 2(\sigma_e^2 + \sigma_b^2)$
Thus weights are based on

$$\frac{1}{\sigma_e^2} \quad \text{and} \quad \frac{1}{\sigma_e^2 + k\sigma_b^2}$$

Restricted maximum likelihood or REML
Commonly used in animal breeding to compare bull or boar progeny information when progeny are sired by artificial insemination on many different farms. Farms are random incomplete blocks.

$$t_q + m = \frac{T_q}{k+1} + \frac{[kT_q - (k+1)B_t + G]}{k^2(k+1)}$$

where t_q = effect of qth treatment as a deviation form the mean

m = mean

T_q = total for all $(k+1)$ units that receive the qth treatment

k = block size

$G = k^2(k+1)m$ = grand total for the whole experiment

$$\text{MSE} \quad \frac{\sigma_e^2}{\sigma^2} < \frac{k}{k+1}$$

Weights: $\quad w = \dfrac{1}{\sigma_e^2} \qquad w' = \dfrac{1}{\sigma_e^2 + k\sigma_b^2}$

$$t_q + m = \frac{T_q}{k+1} + \frac{(w - w')[kT_q - (k+1)B_t + G]}{k^2(k+1)(kw + w')}$$