F DISTRIBUTION

The t test is used to compare two means or treatments. Many experiments involve more than two treatments, and for such experiments the F test is used.

The F test is based on the ratio of two variances. It is used to determine whether two independent estimates of variance can be assumed to be estimates of the same variance. If treatments are significantly different, the variation in treatment means will be greater than the variation due to random differences among individuals.

\[
F = \frac{\text{estimate of } \sigma^2 \text{ from means}}{\text{estimate of } \sigma^2 \text{ from individuals}}
\]

\[
F = \frac{\text{Variance between Treatments}}{\text{Variance within Treatments}}
\]

\[
F = \frac{\text{Variance of Treatments}}{\text{Variance of Error}}
\]

\[
df = \frac{\text{df for between Treatments}}{\text{df for within Treatments}}
\]

This ratio was called F by George W. Snedecor in honor of Ronald A. Fisher, a pioneer in the use of mathematical statistics in agriculture. In the analysis of variance, the F test is used to test equality of means; that is, to answer the question, "can it reasonably be assumed that the treatment means resulted from sampling populations with equal means?"

The two variances which are estimates of \( \sigma^2 \) are calculated from sample means and by pooling the variances from the samples. If estimates of F were calculated many times for a series of samples drawn from a population of normally distributed variates (therefore, no difference between the variances would be expected), and the frequency of the F values obtained were plotted, an F distribution would result. Distributions of F can be found on the web at http://members.aol.com/johnp71/pdfs.html

Note that the F distribution is not symmetrical as was the t distribution and that only positive values are considered. If you think about the ratio of the variances, it is apparent that if there is no difference between the variances (Ho is true), the ratio would be 1. If an unusual sample had been drawn so that the sample mean variance was larger than the pooled variance, the ratio would be a number greater than 1.
F Ratio for the Calculating Machines

Step 1. For the numerator, calculate $s$ from the means:

**Calculate $s^2_x$ from the means**

$$SS = \sum (X - \overline{X})^2 = \sum X^2 - \left(\frac{\sum X}{n}\right)^2$$

$$SS = 22^2 + 14^2 - \frac{36^2}{2} = 32$$

$$s^2_x = \frac{SS}{n-1} = \frac{32}{2-1} = 32$$

Calculate $s$ from $s^2_x$

$$s^2_x = \frac{s^2}{r} \text{ where } r \text{ is the number of observations per mean}$$

$$s^2 = rs^2_x = 10(32) = 320$$

Degrees of freedom for the numerator is $n - 1 = 2 - 1 = 1$. $n$ is the number of means.

Step 2. For the denominator, calculate $s$ from the pooled sample variances:

For machine A, $s^2_A = 23.77$

For machine B, $s^2_B = 35.11$

**Pooled variance $s^2 = \frac{s^2_A + s^2_B}{2} = \frac{23.77 + 35.11}{2} = 29.44$$

Degrees of freedom for the denominator $df = df_A + df_B = 9 + 9 = 18$

Step 3. Calculate the F ratio

$$F = \frac{s^2 \text{ from sample means}}{s^2 \text{ by pooling sample variances}}$$

$$F = \frac{320}{29.44} = 10.87$$

For two treatments, $F = t^2$. For the calculators, the observed $t = 3.30 = \sqrt{10.87}$
Standard Analysis of Variance

These ratios are calculated in an analysis of variance or ANOVA. These calculating machines are considered as coming from two populations which we want to compare. We will therefore use what is called the completely randomized design or you may have learned this as the one-way analysis of variance. Therefore, we can set up an analysis of variance table with the source of variation and the df as shown below:

<table>
<thead>
<tr>
<th>Source of Var</th>
<th>df</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
<th>F.05</th>
<th>F.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total</td>
<td>19</td>
<td>850</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Between Mach</td>
<td>1</td>
<td>320</td>
<td>320</td>
<td>10.87**</td>
<td>4.41</td>
<td>8.29</td>
</tr>
<tr>
<td>Within Mach</td>
<td>18</td>
<td>530</td>
<td>29.44</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The formula for calculating the total SS is

Total SS = \sum (X - \bar{X})^2 = \sum X^2 - \frac{(\sum X)^2}{n}

The term \((\sum X)^2 / n\) is called the correction term or cf or cf.

For the example problem:

Total SS = 30^2 + 21^2 + ... + 24^2 - cf = 7330 - 6480 = 850

\(cf = (360)^2 / 20 = 6480\)

Between machines

\[SS_B = \left(\frac{\sum X_A}{r}\right)^2 + \left(\frac{\sum X_B}{r}\right)^2 - cf\]

\[= \frac{220^2 + 140^2}{10} - 6480 = 320\]

Within Mach SS = 214 + 316 = 530 (sum of individual SS)

or within Mach SS = Total SS - Between Mach SS = 850 - 320 = 530

These values are entered into the ANOVA table and the analysis completed by calculating the MS which is SS/df and the F test.

The MS for between Mach and Within Mach are 2 estimates of \(\sigma^2\). Assuming the null hypothesis is true, we would expect the variances to be very similar since both estimate
the same variance. We can determine the probability of obtaining divergent values of 
\( \sigma^2 \) by calculating F and comparing it with the value in the F Table. The variance of the 
Between Means is always in the numerator and the variance of Within Means is always in the 
denominator. If the 2 treatments come from populations having different means, \( s^2_{\text{bet}} \) will 
have a component for this and will be larger than \( s^2_{\text{w/n}} \).

\[
F = \frac{320}{29.44} = 10.87
\]

For df = 1, \( F = t^2 \) or \( \sqrt{F} = t \).
10.87 = 3.30 which is the same as the t value found earlier.

In order to determine whether or not this F value is unusual, we look in the F table 
(A-3) and find the F value associated with the proper degrees of freedom. The MS for 
between means is based on 2 populations so had 1 df. The MS for within means (error) is 
based on the pooled variance for the 2 populations. Each population has 9 df (10-1) so 9 + 
9 =18 df. Look up F value for 1 and 18 df. \( F_{.05} = 4.41, F_{.01} = 8.29 \). In Table A3 note that 
df for denominator (smaller ms) is on the left side of the table while the df for the numerator 
greater MS) is along the top of the table.

Since these data were collected in a series of replicates or pairs, we can arrange the 
analysis of variance to remove the variation due to replications since this is a known source 
of variation. This is the analysis used for the randomized complete block design where 
there is a restriction in the arrangement of the experiment, i.e. reps or blocks. The analysis 
of variance has the form below.

<table>
<thead>
<tr>
<th>Source of Var</th>
<th>df</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
<th>( F_{.05} )</th>
<th>( F_{.01} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reps or Pairs</td>
<td>9</td>
<td>297.0</td>
<td>33.0</td>
<td>1.27</td>
<td>3.18</td>
<td>5.35</td>
</tr>
<tr>
<td>Mach or Trt</td>
<td>1</td>
<td>320.0</td>
<td>320.0</td>
<td>12.36</td>
<td>5.18</td>
<td>10.56</td>
</tr>
<tr>
<td>Error</td>
<td>9</td>
<td>233.0</td>
<td>25.89</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>19</td>
<td>850.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The new SS is the Rep SS which is calculated from the totals for each rep. The 
totals for the reps are in the last column on the right in the table given to you.

\[
\text{Rep SS} = 44^2 + 42^2 + \ldots + 47^2/2 - \text{cf} = 13554/2 - 6480 \\
= 6777 - 6480 = 297
\]

\[
\text{Error SS} = \text{Total SS} - \text{Rep SS} - \text{Mach SS} = 850 - 297 - 320 = 233.0
\]