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Decentralized risk-sensitive controller design for strict-feedback systems

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Abstract

This paper studies decentralized control of uncertain systems. The class of systems focused on is a set of stochastic strict-feedback systems which interact through their outputs, and performance is measured with respect to a risk sensitive cost criterion. The unknown nonlinear interconnection terms are assumed to be bounded by some known functions of the outputs of the subsystems, multiplied by some unknown parameters. The controllers designed for each subsystem have access only to the information available with regard to the respective subsystem, and they achieve an arbitrarily small value for the risk-sensitive cost for the overall system. Under this completely decentralized control scheme, all closed-loop signals remain bounded in probability.

Keywords: Stochastic strict-feedback systems; Risk sensitive control; Decentralized systems; Backstepping; Uncertain dynamic systems; State feedback

1. Introduction

A recent trend has been the design of feedback control laws to achieve stabilization or tracking for uncertain nonlinear systems where the additive uncertainty is assumed to be random. In the literature, the most common mathematical model used for such systems consists of a set of stochastic differential equations interpreted in the Itô sense; see [14]. Simply, because of the extra quadratic variation terms resulting from the Itô differentiation rule (see [13]), a control law designed for a deterministic system does not lead to a satisfactory solution to the corresponding stochastic control problem. Moreover, different notions of stability and performance indices need to be used to determine the usefulness of the feedback controllers in a stochastic setup. The most natural stochastic counterparts of the "deterministic" concepts such as boundedness and (asymptotic) stability can be found in [11], whereas a more recent concept "noise-to-state stability", where the word "noise" refers to the intensity of the additive random noise, can be found in the recent book [12], which has several chapters on stabilization of stochastic nonlinear systems. Again, in the context of noise-to-state stability, the papers [3,4] have developed adaptive controllers, equipped with state and

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output information respectively, for stochastic strict feedback systems, where the adaptive nature of the controllers is related to the unknown intensity of the additive random disturbances. A more relevant concept to our work, however, is “risk sensitive cost criterion” in which not only the mean value but also the variance of an integral cost is penalized; see [16] in the linear context, and [5–7] in the nonlinear context. The rigorous investigation of the risk-sensitive index presented in [5] revealed that for a nonlinear system the H_2 and H_∞ norms can be recovered as the small risk and the small noise limits (respectively) of the risk sensitive index. Finally, we cite [1,2,15] as the background references that have presented control design schemes for strict feedback systems perturbed by random disturbances, that result in closed-loop signals achieving an arbitrarily small average risk sensitive cost. Nevertheless, to the best of our knowledge, the issue of decentralized adaptive control of a set of nonlinear systems in the face of additive random disturbances has not been studied heretofore.

In this paper, we study the problem of decentralized adaptive control of a set of interacting uncertain strict-feedback systems motivated by applications such as power systems. A pertinent reference for this topic is [9], where the authors solve the problem of decentralized H_∞ almost disturbance decoupling for a class of large-scale nonlinear uncertain systems under the assumption that the uncertain time-varying interconnections are bounded by known nonlinear functions. Our approach, in this paper, is to model the uncertainty in the interactions between the subsystems as linearly parameterized uncertainty, and to construct a set of adaptive controllers, each of which having access to the information available in each subsystem, such that the overall system maintains an arbitrarily small average-risk sensitive cost in the presence of random disturbances. This also constitutes a robust version (with respect to the random external disturbances) of the decentralized design presented in the related work [10].

We should also point out that this paper tackles a stochastic *nonlinear* adaptive control problem under a risk-sensitive cost criterion. Therefore, an optimum risk-sensitive controller where the cost function contains a priori fixed weights on the states and the control action seems to be quite difficult to obtain. The approach taken in this paper leads to suboptimal yet computable finite-dimensional controllers. The design procedure offers substantial flexibility in choosing the weights on the state variables. However, no cost on the control action is imposed, and as a result the proposed controller could be a high-gain controller, particularly when the performance specifications on state variables are tight. Therefore, the design parameters should be picked judiciously to trade off between a good transient response of the states and the control action.

2. Problem formulation

We consider a set of N nonlinear strict-feedback systems where the behavior of the ith subsystem is governed by the following Itô differential equations:

$$\begin{aligned} \mathrm{d}x_{ij} &= [x_{i,j+1} + f_{ij}(x_{i1},\ldots,x_{ij}) + g_{ij}(y_1,\ldots,y_N)]\,\mathrm{d}t \\ &\quad + h_{ij}^{\mathrm{T}}(y_1,\ldots,y_N)\,\mathrm{d}w, \quad 1 \leqslant j \leqslant n_i - 1, \\ \mathrm{d}x_{in_i} &= [b_i(x_{i1},\ldots,x_{in_i})u_i + f_{in_i}(x_{i1},\ldots,x_{in_i}) \\ &\quad + g_{in_i}(y_1,\ldots,y_N)]\,\mathrm{d}t + h_{in_i}^{\mathrm{T}}(y_1,\ldots,y_N)\,\mathrm{d}w, \\ y_i &= x_{i1}, \end{aligned} \tag{1}$$

where $x_i := [x_{i1},\ldots,x_{in_i}]^{\mathrm{T}}$ is the n_i-dimensional state vector, u_i is the scalar control input, y_i is the scalar output, w is the $\mathscr{R}^q$ valued standard Wiener process, and the initial condition $x_i(0)$ is fixed. The nonlinear functions f_{ij}, h_{ij}, b_i, and $1/b_i$ are known and sufficiently smooth, whereas the nonlinear functions g_{ij} are unknown, but still sufficiently smooth. Furthermore, these nonlinear functions will be assumed to satisfy the following assumptions.

Assumption 1. For each $i \in \{1,\ldots,N\}$, and each $j \in \{1,\ldots,n_i\}$, there exist a scalar unknown parameter $\theta_i \geqslant 0$ which satisfies $\theta_i^2 \leqslant N_i$ for some known constant $N_i > 0$, and known sufficiently smooth functions $G_{ijl}(y_l)$, $l = 1,\ldots,N$, such that

$$|g_{ij}(y_1,\ldots,y_N)| \leqslant \theta_i \left[1 + \sum_{l=1}^{N} |y_l| G_{ijl}(y_l)\right], \quad \forall (y_1,\ldots,y_N) \in \mathscr{R}^N.$$

Assumption 2. For each $i \in \{1,\dots,N\}$, and each $j \in \{1,\dots,n_i\}$, there exist known sufficiently smooth functions $H_{ijl}(y_l)$, $l = 1,\dots,N$, such that

$$|h_{ij}(y_1,\dots,y_N)|^2 \leqslant \sum_{l=1}^{N} y_l^2 H_{ijl}(y_l), \qquad \forall (y_1,\dots,y_N) \in \mathscr{R}^N.$$

In the next section, we will study the problem of picking the control inputs u_i, where u_i has access to the information available in the ith subsystem only, such that the closed-loop signals remain bounded in probability and the following risk-sensitive performance inequality holds:

$$\limsup_{T\to\infty} \frac{2}{\mu T} \ln E \times \exp\left[\frac{\mu}{2}\int_0^T \sum_{i=1}^{N} [y_i^2 + l_i(t,x_i)]\,\mathrm{d}t\right] \leqslant R, \tag{2}$$

where $\mu > 0$ is an arbitrary risk-sensitivity parameter, $R > 0$ is an arbitrary constant representing the average risk-sensitive cost, and $l_i(t,x_i) \geqslant 0$ are some weight functions. The weight functions $l_i(t,x_i)$ cannot be picked in an arbitrary manner; however, the control design procedure presented in the next section still gives us a great degree of freedom in the construction of these weight functions.

3. Controller design

We begin with the following assumption.

Assumption 3. For each $i \in \{1,\dots,N\}$, the state variables $x_{i1}(t),\dots,x_{in_i}(t)$, which are assumed to exist, are available to the controller u_i.

We then continue with the controller design by applying the backstepping technique on the ith subsystem, where $i \in \{1,\dots,N\}$.

*Step i*1: We introduce the first transformed variable z_{i1} as $z_{i1} := x_{i1}$ whose Itô differential satisfies

$$\mathrm{d}z_{i1} = (x_{i2} + \bar{f}_{i1} + g_{i1})\,\mathrm{d}t + h_{i1}^{\mathrm{T}}\,\mathrm{d}w,$$

where we again use the over-bar symbol to denote the equivalents of the functions in terms of the transformed variables. We then introduce the positive definite function V_{i1} as $V_{i1} := z_{i1}^2 + \tilde{\eta}_{i1}^2$, where $\tilde{\eta}_{i1} := \theta_i^2 - \hat{\eta}_{i1}$ and $\hat{\eta}_{i1}$ is generated by

$$\begin{aligned}\mathrm{d}\hat{\eta}_{i1} &= [(N/2 + n/R)z_{i1}^2 - m_{N_i^2}(\hat{\eta}_{i1}^2)\hat{\eta}_{i1}]\,\mathrm{d}t \\ &=: \delta_{i1}(z_{i1},\hat{\eta}_{i1})\,\mathrm{d}t\end{aligned}$$

with some arbitrary initial condition $\hat{\eta}_{i1}(0)$, where $n = \sum_{i=1}^{N} n_i$. The function $m_{N_i^2}$ is defined as $m_{N_i^2}(\hat{\eta}_{i1}^2) = m(\hat{\eta}_{i1}^2/N_i^2 - 1)$, where the increasing switching function $m(\rho) : \mathscr{R} \mapsto [0,1]$ is sufficiently smooth and satisfies $m(\rho) = 0$ for $\rho \leqslant 0$, and $m(\rho) = 1$ for $\rho \geqslant 1$. The Itô differential of V_{i1} is computed as

$$\begin{aligned}\mathrm{d}V_{i1} &= 2z_{i1}[(x_{i2} + \bar{f}_{i1} + g_{i1})\,\mathrm{d}t + h_{i1}^{\mathrm{T}}\,\mathrm{d}w] \\ &\quad + |h_{i1}|^2\,\mathrm{d}t - 2\tilde{\eta}_{i1}\delta_{i1}\,\mathrm{d}t.\end{aligned} \tag{3}$$

By invoking Assumptions 1 and 2 and using Young's inequality, we derive the following bounds for various terms in (3):

$$2z_{i1}g_{i1} \leqslant z_{i1}^2\theta_i^2(N + 2n/R) + \sum_{l=1}^{N} z_{l1}^2 G_{i1l}^2 + R/(2n),$$

$$\frac{n\mu}{4}|\sigma_{i1}|^2 \leqslant \frac{n^2\mu^2 z_{i1}^4 N}{4} + \sum_{l=1}^{N} z_{l1}^4 H_{i1l}^2,$$

$$-2\tilde{\eta}_{i1}\delta_{i1} \leqslant -\tilde{\eta}_{i1}(N + 2n/R)z_{i1}^2 - m_{4N_i^2}(\tilde{\eta}_i^2)\tilde{\eta}_i^2,$$

where $\sigma_{i1} := 2z_{i1}h_{i1}$. By employing these bounds in (3), we obtain

$$\begin{aligned}\mathrm{d}V_{i1} &\leqslant 2z_{i1}(x_{i2} + \bar{f}_{i1} + (N + 2n/R)z_{i1}\hat{\eta}_{i1}/2 \\ &\quad + n^2\mu^2 z_{i1}^3 N/8)\,\mathrm{d}t - m_{4N_i^2}(\tilde{\eta}_i^2)\tilde{\eta}_i^2\,\mathrm{d}t \\ &\quad + \sum_{l=1}^{N} [z_{l1}^2 G_{i1l}^2 + z_{l1}^4 H_{i1l}^2 + z_{l1}^2 H_{i1l}]\,\mathrm{d}t + \sigma_{i1}^{\mathrm{T}}\,\mathrm{d}w \\ &\quad - \frac{n\mu}{4}|\sigma_{i1}|^2\,\mathrm{d}t + (R/2n)\,\mathrm{d}t.\end{aligned} \tag{4}$$

We now pick the desired value α_{i1} of x_{i2} as

$$\alpha_{i1}(z_{i1},\hat{\eta}_{i1}) := -\bar{f}_{i1} - \frac{(N + 2n/R)z_{i1}\hat{\eta}_{i1}}{2} - \frac{n^2\mu^2 z_{i1}^3 N}{8} - \frac{(\beta_{i1} + \kappa_i + 1)z_{i1}}{2},$$

where $\beta_{i1}(z_{i1},\hat{\eta}_{i1})>0$ is a design function, and $\kappa_i(z_{i1})>0$ is a function that will be specified later to cancel out the effects of the interaction terms g_{ij} and h_{ij}. We then define the second transformed variable z_{i2} as $z_{i2}:=x_{i2}-\alpha_{i1}$. Rewriting (4) in terms of z_{i2} yields

$$\begin{aligned}
\mathrm{d}V_{i1} \leqslant{} & [2z_{i1}z_{i2}-(\beta_{i1}+\kappa_i+1)z_{i1}^2-m_{4N_i^2}(\tilde{\eta}_i^2)\tilde{\eta}_i^2]\,\mathrm{d}t \\
& +\sum_{l=1}^{N}[z_{l1}^2G_{i1l}^2+z_{l1}^4H_{i1l}^2+z_{l1}^2H_{i1l}]\,\mathrm{d}t \\
& +\sigma_{i1}^{\mathrm{T}}\,\mathrm{d}w-\frac{n\mu}{4}|\sigma_{i1}|^2\,\mathrm{d}t+(R/2n)\,\mathrm{d}t, \qquad (5)
\end{aligned}$$

which completes the first step.

Step ik ($k=2,\ldots,n_i-1$): We assume the following structure from the previous step:

$$z_{i1}=x_{i1}, \quad z_{ij}=x_{ij}-\alpha_{i,j-1}(\zeta_{i,j-1}), \quad j=2,\ldots,k,$$

$$\zeta_{ij}=[z_{i1},\ldots,z_{ij},\hat{\eta}_{i1},\ldots,\hat{\eta}_{ij}]^{\mathrm{T}}, \quad j=1,\ldots,k-1,$$

$$\begin{aligned}
\mathrm{d}z_{ij}={} & \Bigg[z_{i,j+1}+\alpha_{ij}(\zeta_{ij})+r_{ij}(\zeta_{i,j-1},z_{ij}) \\
& +\sum_{m=1}^{j}p_{ijm}(\zeta_{i,j-1})g_{im}\Bigg]\,\mathrm{d}t \\
& +\sum_{m=1}^{j}s_{ijm}(\zeta_{i,j-1})h_{im}^{\mathrm{T}}\,\mathrm{d}w \\
& +\sum_{m,q\in\{1,\ldots,j-1\}}t_{ijmq}(\zeta_{i,j-1})h_{im}^{\mathrm{T}}h_{iq}\,\mathrm{d}t, \\
& j=1,\ldots,k-1,
\end{aligned}$$

$$\tilde{\eta}_{ij}=\theta_i^2-\hat{\eta}_{ij}, \quad \mathrm{d}\hat{\eta}_{ij}=\delta_{ij}(\zeta_{i,j})\,\mathrm{d}t,$$

with some arbitrary $\hat{\eta}(0)$,

$j=1,\ldots,k-1$

$$V_{ij}=\Xi_{ij}(\zeta_{i,j-1})z_{ij}^2+\tilde{\eta}_{ij}^2, \quad j=1,\ldots,k-1,$$

where $\Xi_{i1}:=1$, and

$\Xi_{ij}(\zeta_{i,j-1}), \quad j=2,\ldots,k-1,$

are some smooth functions,

$$\begin{aligned}
\mathrm{d}V_{i1} \leqslant{} & [2z_{i1}z_{i2}-(\beta_{i1}+\kappa_i+1)z_{i1}^2-m_{4N_i^2}(\tilde{\eta}_i^2)\tilde{\eta}_i^2]\,\mathrm{d}t \\
& +\sum_{l=1}^{N}[z_{l1}^2G_{i1l}^2+z_{l1}^4H_{i1l}^2+z_{l1}^2H_{i1l}]\,\mathrm{d}t \\
& +\sigma_{i1}^{\mathrm{T}}\,\mathrm{d}w-\frac{n\mu}{4}|\sigma_{i1}|^2\,\mathrm{d}t+(R/2n)\,\mathrm{d}t,
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}V_{ij} \leqslant{} & [2\Xi_{ij}z_{ij}z_{i,j+1}-2\Xi_{i,j-1}z_{i,j-1}z_{ij}-\beta_{ij}z_{ij}^2 \\
& -m_{4N_i^2}(\tilde{\eta}_{ij}^2)\tilde{\eta}_{ij}^2]\,\mathrm{d}t \\
& +\sigma_{ij}^{\mathrm{T}}\,\mathrm{d}w-\frac{n\mu}{4}|\sigma_{ij}|^2\,\mathrm{d}t \\
& +\frac{R}{n}\,\mathrm{d}t+\sum_{m=1}^{j}\sum_{l=1}^{N}[z_{l1}^2G_{iml}^2+3jNz_{l1}^4H_{iml}^2/2]\,\mathrm{d}t, \\
& j=1,\ldots,k-1.
\end{aligned}$$

From this, we can compute the Itô differential of z_{ik} as

$$\begin{aligned}
\mathrm{d}z_{ik}={} & \left[x_{i,k+1}+r_{ik}+\sum_{m=1}^{k}p_{ikm}g_{im}\right]\mathrm{d}t \\
& +\sum_{m=1}^{k}s_{ikm}h_{im}^{\mathrm{T}}\,\mathrm{d}w \\
& +\sum_{m,q\in\{1,\ldots,k-1\}}t_{ikmq}h_{im}^{\mathrm{T}}h_{iq}\,\mathrm{d}t,
\end{aligned}$$

where

$$\begin{aligned}
r_{ik}(\zeta_{i,k-1},z_{ik}):={} & \bar{f}_{ik}-\sum_{j=1}^{k-1}\frac{\partial\alpha_{i,k-1}}{\partial z_{ij}}(z_{i,j+1}+\alpha_{ij}+r_{ij}) \\
& -\sum_{j=1}^{k-1}\frac{\partial\alpha_{i,k-1}}{\partial\hat{\eta}_{ij}}\delta_{ij},
\end{aligned}$$

$$p_{ikm}(\zeta_{i,k-1}):=-\sum_{j=m}^{k-1}\frac{\partial\alpha_{i,k-1}}{\partial z_{ij}}p_{ijm}, \quad m=1,\ldots,k-1,$$

$$p_{ikk}=1,$$

$$s_{ikm}(\zeta_{i,k-1}):=-\sum_{j=m}^{k-1}\frac{\partial\alpha_{i,k-1}}{\partial z_{ij}}s_{ijm}, \quad m=1,\ldots,k-1,$$

$$s_{ikk}=1,$$

$$t_{ikmq}(\zeta_{i,k-1}):=-\sum_{j=\max\{m+1,q+1\}}^{k-1}\frac{\partial\alpha_{i,k-1}}{\partial z_{ij}}t_{ijmq}$$

$$-\frac{1}{2}\sum_{j=m}^{k-1}\sum_{l=q}^{k-1}\frac{\partial^2\alpha_{i,k-1}}{\partial z_{ij}\partial z_{il}}s_{ijm}s_{ilq},$$

$$m,q\in\{1,\dots,k\}.$$

We introduce the function $V_{ik} := \Xi_{ik}z_{ik}^2+\tilde\eta_{ik}^2$, where $\tilde\eta_{ik} := \theta_i^2 - \hat\eta_{ik}$ and

$$\Xi_{ik}(\zeta_{i,k-1}) := \frac{R}{R+n\sum_{m,q\in\{1,\dots,k\}}s_{ikm}^2 s_{ikq}^2}. \tag{6}$$

The signal $\hat\eta_{ik}$ is generated by $\mathrm{d}\hat\eta_{ik}=\delta_{ik}(\zeta_{ik})\,\mathrm{d}t$, with some arbitrary initial condition $\hat\eta_{ik}(0)$, and the function $\delta_{ik}(\zeta_{ik})$ will be specified shortly. We calculate the Itô differential of V_{ik} as

$$\begin{aligned}\mathrm{d}V_{ik} &= 2\Xi_{ik}z_{ik}(x_{i,k+1}+m_{ik})\,\mathrm{d}t + z_{ik}\sum_{m=1}^{k}\check p_{ikm}g_{im}\,\mathrm{d}t\\ &\quad + z_{ik}\sum_{m=1}^{k}\check s_{ikm}h_{im}^{\mathrm T}\,\mathrm{d}w\\ &\quad + z_{ik}\sum_{m,q\in\{1,\dots,k\}}\check t_{ikmq}h_{im}^{\mathrm T}h_{iq}\,\mathrm{d}t\\ &\quad + \Xi_{ik}\sum_{m,q\in\{1,\dots,k\}}s_{ikm}s_{ikq}h_{im}^{\mathrm T}h_{iq}\,\mathrm{d}t\\ &\quad -2\tilde\eta_{ik}\delta_{ik}\,\mathrm{d}t,\end{aligned} \tag{7}$$

where

$$\begin{aligned}&m_{ik}(\zeta_{i,k-1},z_{ik})\\ &\quad := r_{ik} + \frac{z_{ik}}{2\Xi_{ik}}\sum_{j=1}^{k-1}\frac{\partial\Xi_{ik}}{\partial z_{ij}}(z_{i,j+1}+\alpha_{ij}+r_{ij})\\ &\qquad + \frac{z_{ik}}{2\Xi_{ik}}\sum_{j=1}^{k-1}\frac{\partial\Xi_{ik}}{\partial\hat\eta_{ij}}\delta_{ij},\end{aligned}$$

$$\check p_{ikm}(\zeta_{i,k-1},z_{ik}) := 2\Xi_{ik}p_{ikm} + z_{ik}\sum_{j=m}^{k-1}\frac{\partial\Xi_{ik}}{\partial z_{ij}}p_{ijm},$$

$$m=1,\dots,k,$$

$$\check s_{ikm}(\zeta_{i,k-1},z_{ik}) := 2\Xi_{ik}s_{ikm} + z_{ik}\sum_{j=m}^{k-1}\frac{\partial\Xi_{ik}}{\partial z_{ij}}s_{ijm},$$

$$m=1,\dots,k,$$

$$\begin{aligned}\check t_{ikm}(\zeta_{i,k-1},z_{ik}) &:= 2\Xi_{ik}t_{ikmq}\\ &\quad + z_{ik}\sum_{j=\max\{m+1,q+1\}}^{k-1}\frac{\partial\Xi_{ik}}{\partial z_{ij}}t_{ijmq}\\ &\quad + \frac{z_{ik}}{2}\sum_{j=m}^{k-1}\sum_{l=q}^{k-1}\frac{\partial^2\Xi_{ik}}{\partial z_{ij}\partial z_{il}}s_{ijm}s_{ilq}\\ &\quad + 2s_{ikm}\sum_{j=q}^{k-1}\frac{\partial\Xi_{ik}}{\partial z_{ij}}s_{ijq},\\ &\quad m,q\in\{1,\dots,k\}.\end{aligned}$$

By invoking Assumptions 1 and 2 and using Young's inequality, we obtain the following bounds for various terms in (7):

$$\begin{aligned}z_{ik}\sum_{m=1}^{k}\check p_{ikm}g_{im} &\leqslant \frac{z_{ik}^2\theta_i^2(N+2n/R)}{4}\sum_{m=1}^{k}\check p_{ikm}^2\\ &\quad + \sum_{m=1}^{k}\sum_{l=1}^{N}z_{l1}^2G_{iml}^2 + R/(2n),\end{aligned}$$

$$\begin{aligned}\frac{n\mu}{4}|\sigma_{ik}|^2 &\leqslant \frac{n^2\mu^2z_{ik}^4}{32}\sum_{m,q\in\{1,\dots,k\}}\check s_{ikm}^2\check s_{ikq}^2\\ &\quad + \frac{kN}{2}\sum_{m=1}^{k}\sum_{l=1}^{N}z_{l1}^4H_{iml}^2,\end{aligned}$$

$$\begin{aligned}z_{ik}\sum_{m,q\in\{1,\dots,k\}}\check t_{ikmq}h_{im}^{\mathrm T}h_{iq} &\leqslant \frac{z_{ik}^2}{2}\sum_{m,q\in\{1,\dots,k\}}\check t_{ikmq}^{\,2}\\ &\quad + \frac{kN}{2}\sum_{m\in\{1,\dots,k\}}\sum_{l\in\{1,\dots,N\}}z_{l1}^4H_{iml}^2,\end{aligned}$$

$$\begin{aligned}&\Xi_{ik}\sum_{m,q\in\{1,\dots,k\}}s_{ikm}s_{ikq}h_{im}^{\mathrm T}h_{iq}\\ &\quad\leqslant \frac{R}{2n} + \frac{kN}{2}\sum_{m\in\{1,\dots,k\}}\sum_{l\in\{1,\dots,N\}}z_{l1}^4H_{iml}^2,\end{aligned}$$

where $\sigma_{ik} := z_{ik}\sum_{m=1}^{k}\check s_{ikm}h_{im}$. From these bounds and (7), we can write

$$\mathrm{d}V_{ik} \leqslant 2\Xi_{ik}z_{ik}(x_{i,k+1}+\check m_{ik})\,\mathrm{d}t$$

$$+2\tilde{\eta}_{ik}\left[\frac{z_{ik}^2(N+2n/R)}{8}\sum_{m=1}^{k}\check{p}_{ikm}^2-\delta_{ik}\right]\mathrm{d}t$$

$$+\sigma_{ik}^{\mathrm{T}}\,\mathrm{d}w-\frac{n\mu}{4}\,|\sigma_{ik}|^2\,\mathrm{d}t+\frac{R}{n}\,\mathrm{d}t$$

$$+\sum_{m=1}^{k}\sum_{l=1}^{N}[z_{l1}^2G_{iml}^2+3kNz_{l1}^4H_{iml}^2/2]\,\mathrm{d}t, \tag{8}$$

where

$$\check{m}_{ik}(\zeta_{ik}):=m_{ik}+\frac{z_{ik}\hat{\eta}_{ik}(N+2n/R)}{8\Xi_{ik}}\sum_{m=1}^{k}\check{p}_{ikm}^2$$

$$+\frac{n^2\mu^2z_{ik}^3}{64\Xi_{ik}}\sum_{m,q\in\{1,\dots,k\}}\check{s}_{ikm}^2\check{s}_{ikq}^2$$

$$+\frac{z_{ik}}{4\Xi_{ik}}\sum_{m,q\in\{1,\dots,k\}}\check{t}_{ikmq}^2.$$

We now set

$$\delta_{ik}(\zeta_{ik}):=\frac{z_{ik}^2(N+2n/R)}{8}\sum_{m=1}^{k}\check{p}_{ikm}^2-m_{N_i^2}(\hat{\eta}_{ik}^2)\hat{\eta}_{ik},$$

$$\alpha_{ik}(\zeta_{ik}):=-\check{m}_{ik}-\beta_{ik}z_{ik}/(2\Xi_{ik})-\Xi_{i,k-1}z_{i,k-1}/\Xi_{ik},$$

$$z_{i,k+1}:=x_{i,k+1}-\alpha_{ik}, \tag{9}$$

where $\beta_{ik}(\zeta_{ik})>0$ is a design function. From (8) and (9), we can finally write

$$\mathrm{d}V_{ik}\leqslant[2\Xi_{ik}z_{ik}z_{i,k+1}-2\Xi_{i,k-1}z_{i,k-1}z_{ik}$$

$$-\beta_{ik}z_{ik}^2-m_{4N_i^2}(\tilde{\eta}_{ik}^2)\tilde{\eta}_{ik}^2]\,\mathrm{d}t+\sigma_{ik}^{\mathrm{T}}\,\mathrm{d}w$$

$$-\frac{n\mu}{4}|\sigma_{ik}|^2\,\mathrm{d}t+\frac{R}{n}\,\mathrm{d}t$$

$$+\sum_{m=1}^{k}\sum_{l=1}^{N}[z_{l1}^2G_{iml}^2+3kNz_{l1}^4H_{iml}^2/2]\,\mathrm{d}t.$$

This completes the kth step. Since all of the relevant definitions and results of Step *ik* are consistent with the induction hypothesis, we conclude that the induction hypothesis holds true $\forall k\in\{1,\dots,n_i-1\}$.

Step in_i: The results of Step k hold true also for $k=n_i$ if x_{i,n_i+1} is defined as $x_{i,n_i+1}:=\bar{b}_iu_i$, where u_i is the actual control input for the ith subsystem. We pick the actual control input u_i as

$$u_i=\alpha_{in_i}(\zeta_{in_i})/\bar{b}(\zeta_{in_i}), \tag{10}$$

where the desired value α_{in_i} of x_{i,n_i+1} is obtained by setting $k=n_i$ in (9). The control input (10) renders the Itô differential of $V_{in_i}=\Xi_{in_i}z_{in_i}^2+\tilde{\eta}_{in_i}^2$, where Ξ_{in_i} is obtained by setting $k=n_i$ in (6), as

$$\mathrm{d}V_{in_i}\leqslant[-2\Xi_{i,n_i-1}z_{i,n_i-1}z_{in_i}$$

$$-\beta_{in_i}z_{in_i}^2-m_{4N_i^2}(\tilde{\eta}_{in_i}^2)\tilde{\eta}_{in_i}^2]\,\mathrm{d}t+\sigma_{in_i}^{\mathrm{T}}\,\mathrm{d}w$$

$$-\frac{n\mu}{4}|\sigma_{in_i}|^2\,\mathrm{d}t+\frac{R}{n}\,\mathrm{d}t$$

$$+\sum_{m=1}^{n_i}\sum_{l=1}^{N}[z_{l1}^2G_{iml}^2+3n_iNz_{l1}^4H_{iml}^2/2]\,\mathrm{d}t.$$

To obtain a risk-sensitive performance inequality for the overall system, we define a positive definite function V as

$$V:=\sum_{i=1}^{N}\sum_{j=1}^{n_i}V_{ij}.$$

From the results of Steps *ik* and in_i, we conclude that the Itô differential of V satisfies

$$\mathrm{d}V\leqslant\sum_{i=1}^{N}\sum_{k=1}^{n_i}\Big\{-\beta_{ik}z_{ik}^2\,\mathrm{d}t-m_{4N_i^2}(\tilde{\eta}_{ik}^2)\tilde{\eta}_{ik}^2\,\mathrm{d}t$$

$$+\sigma_{ik}^{\mathrm{T}}\,\mathrm{d}w-\frac{n\mu}{4}|\sigma_{ik}|^2\,\mathrm{d}t\Big\}$$

$$-\sum_{i=1}^{N}z_{i1}^2\,\mathrm{d}t-\sum_{l=1}^{N}\kappa_lz_{l1}^2\,\mathrm{d}t$$

$$+\sum_{l=1}^{N}z_{l1}^2\sum_{i=1}^{N}[G_{i1l}^2+z_{l1}^2H_{i1l}^2+H_{i1l}]\,\mathrm{d}t$$

$$+\sum_{l=1}^{N}z_{l1}^2\sum_{i=1}^{N}\sum_{k=2}^{n_i}\sum_{m=1}^{k}[G_{iml}^2$$

$$+3kNz_{l1}^2H_{iml}^2/2]\,\mathrm{d}t+\sum_{i=1}^{N}\frac{(n_i-1/2)R}{n}\,\mathrm{d}t. \tag{11}$$

To cancel out the terms in (11) which are due to the interactions between the subsystems, we pick the functions κ_l as

$$\kappa_l(z_{l1}) = \sum_{i=1}^{N} [G_{i1l}^2 + z_{l1}^2 H_{i1l}^2 + H_{i1l}]$$

$$+ \sum_{i=1}^{N} \sum_{k=2}^{n_i} \sum_{m=1}^{k} [G_{iml}^2 + 3kNz_{l1}^2 H_{iml}^2/2].$$

With this choice of κ_l, we write the following from (11):

$$\mathrm{d}V \leqslant -\sum_{i=1}^{N} [z_{i1}^2 + L_i(\zeta_{in_i})]\,\mathrm{d}t + \sigma^{\mathrm{T}}\,\mathrm{d}w$$

$$-\frac{\mu}{4}|\sigma|^2\,\mathrm{d}t + R\,\mathrm{d}t,$$

where

$$L_i(\zeta_{in_i}) := \sum_{k=1}^{n_i} [\beta_{ik} z_{ik}^2 + m_{4N_i^2}(\tilde{\eta}_{ik}^2)\tilde{\eta}_{ik}^2],$$

$$\sigma := \sum_{i=1}^{N} \sum_{k=1}^{n_i} \sigma_{ik}. \tag{12}$$

We note that the cost functions L_i come as a byproduct of the controller design, and hence we do not have complete freedom in their selection. However, since the design functions β_{ik} are at our disposal, we still have substantial flexibility in shaping the cost functions L_i. This now leads to the following main result of this paper.

Theorem 4. *Consider the nonlinear system* (1) *under the Assumptions* 1–3. *If the design functions satisfy* $\beta_{ik} \geqslant k_\beta$ *for some* $k_\beta > 0$, $\forall k \in \{1, \ldots, n_i\}$, $\forall i \in \{1, \ldots, N\}$, *then for any given risk-sensitivity parameter* $\mu > 0$ *and desired average risk-sensitive cost* $R > 0$, *the controller* (10) *achieves*

1. $$\limsup_{T\to\infty} \frac{2}{\mu T} \ln E$$
   $$\times \exp\frac{\mu}{2}\left[\int_0^T \sum_{i=1}^{N} \{y_i^2 + L_i(\zeta_{in_i})\}\,\mathrm{d}t\right] \leqslant R,$$
   where the cost functions $L_i(\zeta_{in_i})$ *are as defined in* (12).
2. *The closed-loop signals are stochastically bounded in probability, i.e.*,
   $$\lim_{c\to\infty} \sup_{t\geqslant 0} \mathscr{P}\{|\zeta(t)| > c\} = 0,$$
   where $\zeta(t) := [\zeta_{1n_1}^{\mathrm{T}}(t), \ldots, \zeta_{Nn_N}^{\mathrm{T}}(t)]^{\mathrm{T}}$.

Proof (Sketch). From (12), it follows that

$$\frac{2}{\mu T} \ln E \exp\frac{\mu}{2}\left[\int_0^T \left[\sum_{i=1}^{N} z_{i1}^2 + L_i(\zeta_{in_i})\right]\mathrm{d}t\right]$$

$$\leqslant \frac{V(0)}{T} + R$$

$$+\frac{2}{\mu T} \ln E \exp\frac{\mu}{2} \int_0^T [\sigma^{\mathrm{T}}\,\mathrm{d}w - \frac{\mu}{4}|\sigma|^2\,\mathrm{d}t],$$

for all $T > 0$. Theorem 11 of [15] shows that the last term on the right-hand side of the above inequality is nonpositive. Hence letting $T \uparrow \infty$ proves the first part. To prove the second part, we note that the function V is radially unbounded and it satisfies

$$\mathscr{L}V \leqslant -c_1 V + c_2$$

for some positive constants c_1 and c_2. This, from Theorem 2 on p. 330 of [8], proves the second part. □

4. Simulation results

We consider the following two interconnected systems.

$$\mathrm{d}x_{11} = x_{12}\,\mathrm{d}t$$

$$\mathrm{d}x_{12} = [10u_1 - 0.1x_{12}^2 \sin(x_{11})$$
$$+y_1 a_1 + (y_2 - y_1 + 1)a_2]\,\mathrm{d}t$$
$$+y_1\,\mathrm{d}w_1 + y_2\,\mathrm{d}w_2$$

$$y_1 = x_{11}$$

$$\mathrm{d}x_{21} = x_{22}\,\mathrm{d}t$$

$$\mathrm{d}x_{22} = [10u_2 - 0.1x_{22}^2 \sin(x_{21})$$
$$+y_2 a_1 + (y_1 - y_2 - 1)a_2]\,\mathrm{d}t$$
$$+y_2\,\mathrm{d}w_1 + y_1\,\mathrm{d}w_2$$

$$y_2 = x_{21},$$

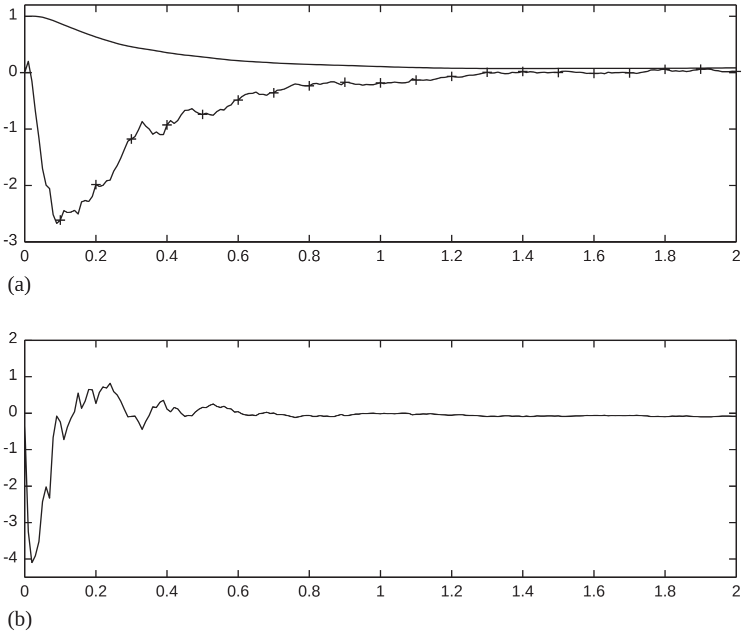

Fig. 1. ($\mu = 0, R = 10$): (a) $x_{11}(t)$ (solid) and $x_{12}(t)$ (+) (b) $u_1(t)$.

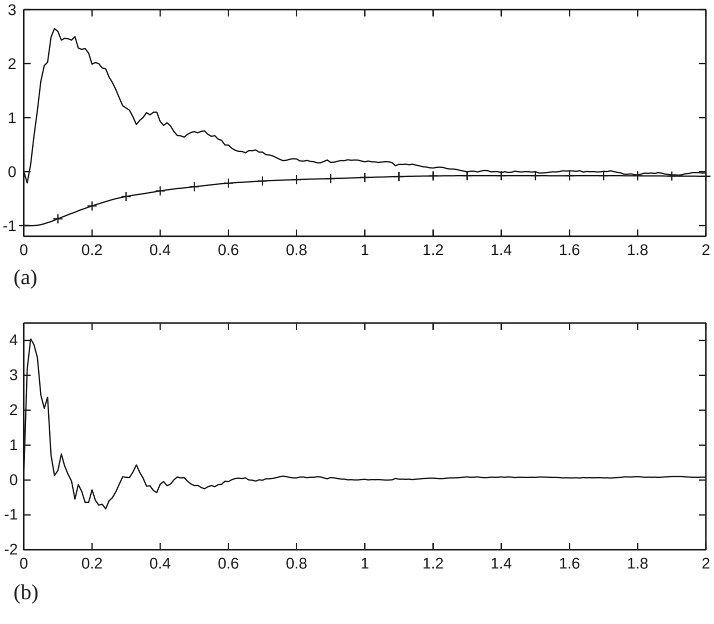

Fig. 2. ($\mu = 0, R = 10$): (a) $x_{21}(t)$ (solid) and $x_{22}(t)$ (+) (b) $u_2(t)$.

where w_1 and w_2 are independent standard Wiener processes, and $a_1 = 1$, $a_2 = 1$ are some parameters whose values are not available to the controller designer. This system models two inverted pendulums mounted on two carts which are connected by a spring; see [10] for details. The design functions are picked as

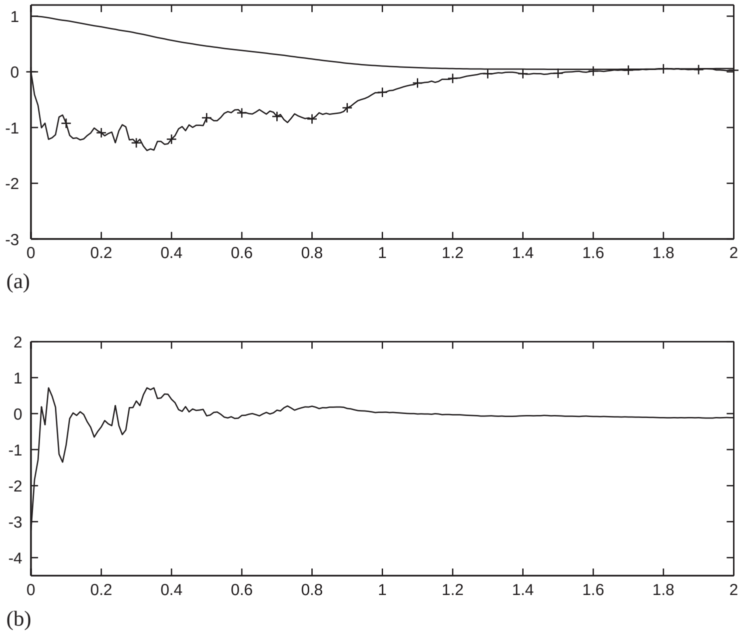

Fig. 3. ($\mu = 1, R = 0.5$): (a) $x_{11}(t)$ (solid) and $x_{12}(t)$ (+) (b) $u_1(t)$.

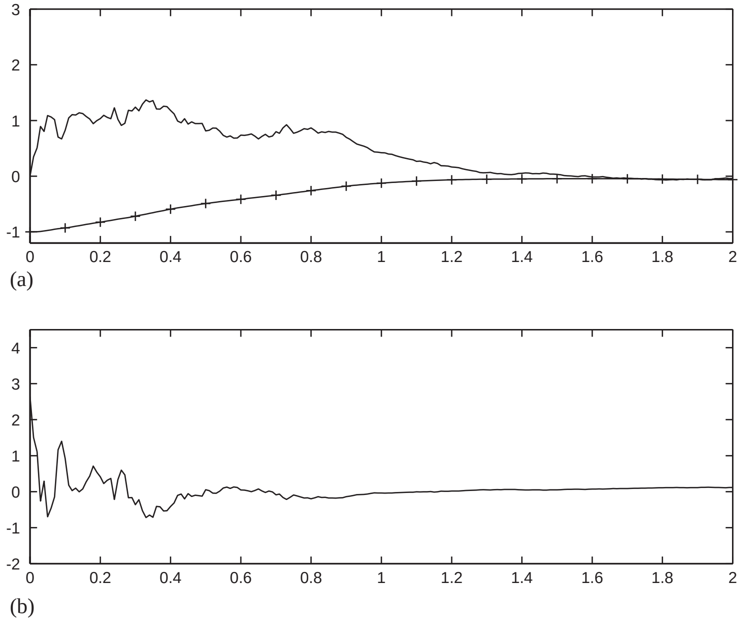

Fig. 4. ($\mu = 1, R = 0.5$): (a) $x_{21}(t)$ (solid) and $x_{22}(t)$ (+) (b) $u_2(t)$.

$\beta_{11} = \beta_{12} = 0.01$, and $\beta_{21} = \beta_{22} = 0.1$. The simulation is performed for two different sets of values of (μ, R): $(\mu = 0, R = 10)$ and $(\mu = 1.0, R = 0.5)$ corresponding to risk neutral and risk sensitive cases, respectively. The state variables and the control actions for the risk-neutral case are shown in Figs. 1 and 2, whereas the risk-sensitive case is illustrated in Figs. 3 and 4. These figures clearly show that the risk-sensitive decentralized controller achieves a superior performance over the risk-neutral decentralized controller.

5. Generalizations

In this section, we will discuss relaxation of various assumptions made and some extensions of the results presented in this paper. We made two fundamental assumptions in this paper: Assumptions 1 and 2. These assumptions essentially convert the structural uncertainty in the interconnection terms to linearly parameterized uncertainty, where the unknown parameters are bounded by some known numbers and the known functions (multiplied by the unknown parameters) are in the form of sums of nonlinear functions each of which depends on the output of only one subsystem. We can contemplate on relaxing these two assumptions in several ways. First relaxation would be to assume that no known upper bound on the unknown parameters are available. Second, we can consider the case where the known functions in the linearly parameterized uncertainty structure are general nonlinear functions. This second relaxation would be a substantial one and require significant additional research. The last relaxation could be the total elimination of Assumptions 1 and 2, which would be even more challenging and essentially lead to a risk-sensitive decentralized control problem for strict-feedback systems with structurally unknown interconnection terms.

6. Conclusions

In this paper the design of decentralized risk-sensitive controllers for a set of interconnected strict-feedback systems perturbed by external random disturbances is studied. This topic is motivated by some practical applications such as power system control. The nonlinear functions modeling the interconnections between the subsystems are taken unknown, but assumed to be bounded by some known functions of the outputs of the subsystems multiplied by some unknown parameters. The design procedure presented in this paper shows that the controller design for such an interconnected system can be completely decentralized. The resulting decentralized control scheme achieves an arbitrarily small average risk-sensitive cost for the overall system, and all closed-loop signals remain bounded in probability.

Some possible topics for a future study are: Decentralized risk-sensitive (DRS) asymptotic tracking; DRS design with partial state measurements and with measurement noise; DRS design for systems with (unstable) zero dynamics and with (unknown) virtual control coefficients; and DRS control of a set of pure-feedback systems.

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