On Existence of Equilibrium in Inventory Competition with Fixed Order Costs

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Abstract

We show that fixed order costs fundamentally change whether there exists an equilibrium in inventory competition: The single-period game always has an equilibrium, but the multiple-period game may not. On the latter point we offer a family of problem instances with two periods and deterministic demands – the simplest possible multiple-period setup – that lack equilibrium. The most intriguing dynamic that contributes to nonexistence of equilibrium is that a firm can improve its future sales by deliberately creating scarcity.

Keywords: inventory competition, fixed order cost, setup cost, equilibrium

1. Introduction

Inventory competition has been a topic of great interest in operations management. The retailer maxim "stock it and they will come" embodies the basic story. Two retailers carrying the same product at the same price would compete on service, for which product availability is arguably the most important dimension. More broadly, retailers enter into competition with one another by stocking products that are substitutes of each other. Study of equilibrium between retailers' inventory decisions for substitutable products is what concerns the inventory competition literature.

To the best of our knowledge, fixed order cost has never been considered in this literature, even though it has a prominent place in inventory theory [1]. Inventory com-

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petition has been studied in single-period settings [2, 3, 4, 5] as well as multiple-period settings [6, 7, 8, 9, 10]; with two firms [2, 3, 6, 8, 9, 10] or *n* firms [4, 7, 5, 9]; under deterministic substitution with lost sales [2, 6, 5, 9, 10] or various forms of backlogging [8] and more sophisticated market dynamics [11]; and under probabilistic substitution that allows for rich micromodels of consumer response to stockouts [3, 4]. Also relevant are monopolistic models of inventory management for substitutable products, e.g., [12, 13, 14]. For a more comprehensive review of the inventory competition literature, the reader is referred to [9, 8, 11].

In this paper, we consider two substitutable products, each carried by a separate firm that can replenish its inventory only at a positive fixed order cost. Each product attracts demand from two sources: customers who prefer to shop at the firm that carries it, and customers who ordinarily prefer to shop at the other firm but need to switch loyalties due to shortage of inventory there. This substitution behavior, modeled by a fixed percentage of a firm's excess demand switching to its competitor (a standard abstraction commonly employed in the literature, e.g., [2, 6, 5]), creates a strategic interaction between the firms' inventory replenishment decisions. We study the existence and nature of equilibrium in single- and multiple-period games resulting from this interaction. Our main point of departure from the literature is to allow a non-negligible fixed order cost for each firm.

We consider only pure Markov strategies. Although mixed and behavior strategies are commonly used in broad game theory literature, it is hard to imagine that a firm would use randomization to decide on inventory levels; hence we presume that firms in inventory competition use pure strategies only. Markov strategies are also quite natural in our context, because beginning inventory levels almost completely summarize the history of the game. In fact, an equilibrium that cannot be implemented in Markov strategies would be quite awkward in that it would result in different inventory decisions for the same state, which may have been reached via different histories.

The contribution of our paper is two-fold. We first characterize the set of single-period pure-strategy equilibria as a function of the firms' initial inventory levels and fixed order costs. In particular, we observe that a pure-strategy equilibrium always exists, and for some initial inventory levels multiple pure-strategy equilibria may exist.

We also relate these pure-strategy equilibria to the pure-strategy equilibria in zero-initial-inventory, zero-fixed-order-cost case (tying our results to the extant literature), and give necessary and sufficient conditions for when both firms, one firm, and neither firm order. We then explore the two-period game with deterministic demands, and establish sufficient conditions for nonexistence of subgame-perfect equilibrium in pure Markov strategies. The nonexistence of equilibrium even in the simplest possible multiple-period setup suggests that it extends to more general models with multiple periods and fixed order costs. It is interesting that demand uncertainty is not essential for nonexistence to occur, whereas fixed order costs and multiple periods are. In addition we offer a simple sufficient condition for existence of equilibrium in the two-period game with deterministic demands. Therefore, to the extent that fixed order costs are a significant driver of inventory decisions, the multiple-period inventory competition problem must be reexamined in light of our results.

We use the following notation throughout the paper: $-i$ denotes indices other than i; $I\{\cdot\}$ is the indicator function that equals 1 if the condition within brackets is true, and 0 otherwise; F_X (f_X) denotes the cumulative distribution function (density function) of a random variable X; $E[\cdot]$ denotes the expectation operator; $A \Rightarrow B$ means "A implies B"; $A \Leftrightarrow B$ means "A and B are equivalent statements"; $:=$ states a definition; and $(x)^+ := \max(0, x)$ for a real number x.

2. Models

In this section we introduce our models of single- and multiple-period inventory competition with fixed order costs. Both models involve two firms and a single product that they each carry. It is best to think of these two products as substitutes. They may or may not be differentiated, but some customers substitute one for the other. A natural special case of our models is applicable to two retailers carrying exactly the same product. For brevity, our models speak of a single product as in this special case, even though they apply to any two substitutable products each sold by a separate firm.

2.1. A Single-Period Inventory Competition Model with Fixed Order Costs

Consider firm $i \in \{1, 2\}$ with a given initial inventory level $x_i \geq 0$. Both firms simultaneously act on whether or not to place an inventory replenishment order, and how large an order to place. Orders are fulfilled at zero leadtime regardless of their size. As a result, firm i takes its inventory level from x_i to y_i ($y_i \geq x_i$), incurring a total cost of $c_i(y_i - x_i) + k_i I\{y_i > x_i\}$, where $c_i \geq 0$ is the variable order cost per unit, and $k_i \geq 0$ is the fixed order cost.

Then, firm i receives demand d_i from the customers for whom firm i is the first choice. We assume that (d_1, d_2) are exogenous random variables with nonnegative support (they can be correlated). If the first-choice demand d_i cannot be fully satisfied by firm i, i.e., $d_i > y_i$, then a fixed proportion $\alpha_{-i} \in [0, 1]$ of excess demand $(d_i - y_i)^+$ switches to firm $-i$. (The remainder of the first-choice demand is lost for both firms.) Therefore, the total demand faced by firm i is

$$\bar{d}_i(y_{-i}) := d_i + \alpha_i(d_{-i} - y_{-i})^+. \tag{1}$$

(We suppress the dependence of $\bar{d}_i$ on y_{-i} when there is no possibility of confusion.) After demand realizations, including the switching behavior, firm i collects a total revenue of $r_i \min(y_i, \bar{d}_i)$ and incurs inventory holding cost $h_i(y_i - \bar{d}_i)^+$, where $r_i \geq 0$ is the revenue per unit sales, and $h_i \geq 0$ is the holding cost per unit of inventory. The resulting expected profit for firm i is

$$g_i(y_i, y_{-i}; x_i) := E\big[r_i \min(y_i, \bar{d}_i) - h_i(y_i - \bar{d}_i)^+ - c_i(y_i - x_i) - k_i I\{y_i > x_i\}\big],$$

where the expectation is taken with respect to (d_1, d_2). We rewrite firm i's expected profit as

$$g_i(y_i, y_{-i}; x_i) \quad = \quad \bar{g}_i(y_i, y_{-i}) - k_i I\{y_i > x_i\} + c_i x_i, \tag{2}$$

where

$$\begin{aligned} \bar{g}_i(y_i, y_{-i}) &= E\left[r_i \min(y_i, \bar{d}_i) - h_i(y_i - \bar{d}_i)^+ - c_i y_i\right] \\ &= (r_i - c_i)y_i - (r_i + h_i)E\left[(y_i - \bar{d}_i)^+\right]. \end{aligned}$$

We denote the noncooperative game with payoffs $g_i(y_i, y_{-i}; x_i)$ and strategy sets $\{[x_i, \infty)\}_{i\in\{1,2\}}$ with $\Gamma^{x,k}$, where the dependence on initial inventory levels $x := (x_1, x_2)$ and fixed order costs $k := (k_1, k_2)$ is made explicit. Our model is standard in the inventory competition literature, except the fixed order cost part. We show that fixed order costs cause some fundamental changes in the nature of strategic interaction between firms competing on inventory.

2.2. A Multiple-Period Inventory Competition Model with Fixed Order Costs

Now consider the same game unfolding over multiple periods. Firm i has an initial inventory level $x_{i,1} \geq 0$. In each period $t \in \{1, \dots, T\}$ four events happen in the following sequence. At the beginning of period t firm i brings its inventory level from $x_{i,t}$ up to $y_{i,t} \geq x_{i,t}$. Firm i then receives the first-choice demand $d_{i,t}$, where $(d_{1,t}, d_{2,t})$ for $t = 1, \dots, T$ form an independently (but not necessarily identically) distributed exogenous random sequence. Inventory shortages, if any, cause a switching behavior that yields an effective demand of $\bar{d}_{i,t}(y_{-i,t}) = d_{i,t} + \alpha_i(d_{-i,t} - y_{-i,t})^+$ for firm i. Finally, firm i receives its period-t profit whose expected value, denoted by $g_{i,t}\big(y_{i,t}, y_{-i,t}; x_{i,t}\big)$, equals $g_i\big(y_{i,t}, y_{-i,t}; x_{i,t}\big)$ using (2) with the expectation taken with respect to $(d_{1,t}, d_{2,t})$.

In sum, every period firms make simultaneous inventory replenishment decisions to meet demands that reflect the customers' switching behavior. Firm i's inventory level evolves over time according to the equation $x_{i,t+1} = \big(y_{i,t} - \bar{d}_{i,t}(y_{-i,t})\big)^+$. As in the single-period model, the unmet demand after switching is lost.

We wish to study the equilibria of the ensuing dynamic noncooperative game with each firm i making its order-up-to decisions according to $y_{i,t} = \eta_{i,t}(x_{i,t}, x_{-i,t})$, where $\eta_i := (\eta_{i,1}, \dots, \eta_{i,T})$ is the pure Markov strategy employed by firm i satisfying

$$\eta_{i,t}(x_{i,t}, x_{-i,t}) \geq x_{i,t}, \qquad \text{for all } t \in \{1, \dots, T\}. \tag{3}$$

Under a strategy profile (η_i, η_{-i}), firm i's total expected profit is given as

$$J_i(\eta_i, \eta_{-i}) = E\left[\sum_{t=1}^{T} \delta_i^{t-1} g_{i,t}(y_{i,t}, y_{-i,t}; x_{i,t})\right]$$

where $y_{i,t} = \eta_{i,t}(x_{i,t}, x_{-i,t})$, $x_{i,t+1} = \left(y_{i,t} - \bar{d}_{i,t}\right)^+$, and $\delta_i \in [0,1]$ is firm i's discount factor.

A strategy profile (η_1^*, η_2^*) is called an equilibrium if

$$J_i(\eta_i^*, \eta_{-i}^*) = \max_{\eta_i} J_i(\eta_i, \eta_{-i}^*)$$

for all i, where the maximization is subject to (3). We are interested in a refinement called subgame perfection. An equilibrium strategy profile (η_1^*, η_2^*) is called subgame perfect if, for any $\tilde{t} \in \{1, \ldots, T\}$, the truncated strategy profile

$$\left((\eta_{1,\tilde{t}}^*, \ldots, \eta_{1,T}^*), (\eta_{2,\tilde{t}}^*, \ldots, \eta_{2,T}^*)\right)$$

is an equilibrium of all subgames starting in period $\tilde{t}$ with *any* nonnegative inventory levels $(\tilde{x}_{1,\tilde{t}}, \tilde{x}_{2,\tilde{t}})$ and unfolding over the periods $\{\tilde{t}, \ldots, T\}$. Note that subgame perfection requires equilibrium behavior in any subgame starting with any nonnegative inventory levels $(\tilde{x}_{1,\tilde{t}}, \tilde{x}_{2,\tilde{t}})$ regardless of whether or not $(\tilde{x}_{1,\tilde{t}}, \tilde{x}_{2,\tilde{t}})$ can be reached in the original game under (η_1^*, η_2^*). Henceforth, we use the term "equilibrium" to mean "equilibrium in pure strategies" in the single-period case and "subgame-perfect equilibrium in pure Markov strategies" in the multiple-period case (sometimes referred in the literature as pure-strategy Markov Perfect Equilibrium), except when we use the longer forms for emphasis.

We impose the following assumption throughout the paper to avoid trivial cases.

Assumption 1. *For $i \in \{1,2\}$, (a) $x_{i,1},\ c_i,\ h_i,\ k_i \geq 0$ and $\alpha_i,\ \delta_i \in [0,1]$; (b) $r_i > c_i$; (c) $c_i + h_i > 0$; (d) $\delta_i \alpha_i \alpha_{-i} < 1$; (e) $d_{i,t}$ have nonnegative support and finite mean for all t.*

Part (a) of Assumption 1 ensures sensible parameter values. If part (b) does not hold, i.e., $r_i \leq c_i$, then placing no orders would be optimal for firm i regardless of the actions taken by firm $-i$, which renders the entire game trivial. If part (c) fails to hold, i.e., $c_i = h_i = 0$, then ordering an unbounded amount could be optimal for firm i, in which case there would be no inventory competition as customers would have no reason to switch between firms. Part (d) is a technical assumption made for expositional clarity; our results can be extended to the case with $\delta_i = \alpha_i = \alpha_{-i} = 1$. Part (e) rules out the possibility of observing negative demand or unbounded optimal order quantity.

3. Equilibrium in Inventory Competition with Fixed Order Costs

We now present our analyses of existence and nature of equilibrium in single- and multiple-period inventory competition with fixed order costs.

3.1. Single-Period Analysis

First, we use the supermodular games framework to establish the existence of equilibrium in the single-period game (§2.1). We then characterize the equilibrium. In particular, we relate the equilibria of the general model with any set of initial inventories and fixed order costs to the equilibria in zero-initial-inventory, zero-fixed-order-cost case. We also give necessary and sufficient conditions for when equilibria with both firms, one firm, and neither firm placing orders occur. (The proofs are in Appendix A.)

We first define the best response correspondence for firm i taking part in the game $\Gamma^{x,k}$ as follows:

$$BR_i^{x,k}(y_{-i}) := \text{argmax}_{y_i \in [x_i, \infty)} \{\bar{g}_i(y_i, y_{-i}) - k_i I\{y_i > x_i\} + c_i x_i\}$$

where the dependence on initial inventories $x = (x_1, x_2)$ and fixed costs $k = (k_1, k_2)$ is made explicit. Note that $BR_i^{0,0}(y_{-i}) = \text{argmax}_{y_i \in [0,\infty)} \bar{g}_i(y_i, y_{-i})$.

Firm i has two options: place an order, in which case firm i cannot gain an expected profit any higher than $\{\max_{y_i \geq 0} \bar{g}_i(y_i, y_{-i}) - k_i + c_i x_i\}$; place no orders and gain $\{\bar{g}_i(x_i, y_{-i}) + c_i x_i\}$. Therefore, as commonly observed in inventory replenishment problems with fixed order cost, firm i will want to place an order only if its inventory is above a threshold. We define that threshold as follows:

$$s_i(y_{-i}) := \max\left\{z_i \leq \min BR_i^{0,0}(y_{-i}) \ : \ \bar{g}_i(z_i, y_{-i}) \leq \max_{y_i \geq 0} \bar{g}_i(y_i, y_{-i}) - k_i\right\}. \tag{4}$$

The constraint $z_i \leq \min BR_i^{0,0}(y_{-i})$ is needed for there may exist multiple optima for the subproblem $\max_{y_i \geq 0} \bar{g}_i(y_i, y_{-i})$. Firm i must then use the best response

$$BR_i^{x,k}(y_{-i}) = \begin{cases} BR_i^{0,0}(y_{-i}), & \text{if } x_i < s_i(y_{-i}) \\ \{x_i\} \cup BR_i^{0,0}(y_{-i}), & \text{if } x_i = s_i(y_{-i}) \\ x_i, & \text{if } x_i > s_i(y_{-i}) \end{cases} \tag{5}$$

Clearly, a profile $y^* := (y_1^*, y_2^*)$ of inventory replenishment decisions is an equilibrium of $\Gamma^{x,k}$ if and only if $y_i^* \in BR_i^{x,k}(y_{-i}^*)$ for all i. We call (y_1^*, y_2^*) an *interior* equilibrium of $\Gamma^{x,k}$ when both firms place orders in equilibrium, i.e., $y_i^* > x_i$ and $y_i^* \in BR_i^{x,k}(y_{-i}^*)$ for all i.

Next, we observe that the expected profit of each firm obeys certain properties.

Lemma 1. *Consider any game $\Gamma^{x,k}$ arising from the single-period inventory competition model with fixed order costs. The expected profit $g_i(y_i, y_{-i}; x_i)$ for each $i \in \{1, 2\}$ possesses the following properties.*

1. $g_i(y_i, y_{-i}; x_i)$ *is upper semi-continuous in* y_i.
2. *For any $y_i \geq \bar{y}_i \geq 0$, $g_i(y_i, y_{-i}; x_i) - g_i(\bar{y}_i, y_{-i}; x_i)$ is nonincreasing in y_{-i}.*
3. *Given an arbitrary constant $\rho \in (\frac{r_i - c_i}{r_i + h_i}, 1)$, there exists a finite bound $Y_i^U := \inf\{\xi \geq 0 : F_{d_i + \alpha_i d_{-i}}(\xi) \geq \rho\}$ such that, for all $x_i, y_{-i} \geq 0$,*

$$\bar{y}_i \in \textit{argmax}_{y_i \in [x_i, \infty)} g_i(y_i, y_{-i}; x_i) \qquad \Rightarrow \qquad \bar{y}_i \in [x_i, \max(x_i, Y_i^U)].$$

We are now ready to state our main result regarding the existence and nature of equilibrium in single-period inventory competition with fixed order costs. Lemma 1 implies that $\Gamma^{x,k}$ is a supermodular game, where each firm's strategy belongs to a compact interval and each firm's expected profit is upper semi-continuous with respect to its own order-up-to level. It is well-known that such games have equilibria [15]. We state this and show several characteristics of the equilibria.

Theorem 1. *Consider any game $\Gamma^{x,k}$ arising from the single-period inventory competition model with fixed order costs.*

1. ***[Existence]*** *$\Gamma^{x,k}$ has a nonempty equilibrium set.*
2. ***[An Equilibrium with Both Firms Ordering]***
   (a) *An interior equilibrium of $\Gamma^{x,k}$, if it exists, is an equilibrium of $\Gamma^{0,0}$.*
   (b) *Let y^* be an equilibrium of $\Gamma^{0,0}$. Then, y^* is an interior equilibrium of $\Gamma^{x,k}$ if and only if $x_i \leq s_i\big(y_{-i}^*\big)$ and $x_i < y_i^*$ for all i.*
3. ***[An Equilibrium with One Firm Ordering]*** *For any firm i, $y^* = \big(x_i, y_{-i}^*\big) \neq x$ is an equilibrium of $\Gamma^{x,k}$ if and only if $x_i \geq s_i\big(y_{-i}^*\big)$, $x_{-i} \leq s_{-i}(x_i)$, and $x_{-i} < y_{-i}^* \in BR_{-i}^{0,0}(x_i)$.*

4. ***[An Equilibrium with Neither Firm Ordering]*** *$y^* = x$ is an equilibrium of $\Gamma^{x,k}$ if and only if $x_i \geq s_i\big(x_{-i}\big)$ for all i.*

For an interior equilibrium with both firms placing orders to occur, their initial inventories must be below their order thresholds. If both initial inventories exceed the threshold, then neither firm orders in equilibrium. The remaining two (hybrid) situations result in an asymmetric equilibrium with only one of the firms placing an order - the one with below-threshold initial inventory. Example 1 illustrates these equilibria in the space of initial inventory levels.

Example 1. *Consider a problem instance with parameters $k_1 = k_2 = 0.2$, $\alpha_1 = \alpha_2 = 0.5$, $h_1 = h_2 = c_1 = c_2 = 1$, $r_1 = r_2 = 3$, and exponential first-choice demands with mean 1, i.e., d_1 and d_2 are independent and identically distributed with density $e^{-\xi} I\{\xi \geq 0\}$. In this case, $BR_1^{0,0}$ and $BR_2^{0,0}$ are single valued. Moreover, $y^{0,0}$ is a singleton, the unique point at which $BR_1^{0,0}$ and $BR_2^{0,0}$ intersect. The top graph in Figure 1 shows two regions, one representing the set of x for which $y^{0,0}$ is an equilibrium, and the other the set of x for which x is an equilibrium, i.e., neither firm orders. The middle graph shows the region for which $\big(x_1, BR_2^{0,0}(x_1)\big) \neq x$ is an equilibrium, i.e., firm 1 does not order but firm 2 orders up to level $BR_2^{0,0}(x_1) > x_2$. The bottom graph shows the reverse possibility in which only firm 1 orders, up to level $BR_1^{0,0}(x_2) > x_1$. Note that these regions overlap; for certain x, there exist multiple equilibria (up to three) that belong to $\Big\{y^{0,0},\ \big(x_1, BR_2^{0,0}(x_1)\big),\ \big(BR_1^{0,0}(x_2), x_2\big)\Big\}$.*

3.2. Multiple-Period Analysis

In this subsection, we show that no equilibrium may exist in the multiple-period game with fixed order costs (§2.2). To this end we choose the simplest possible setup: two periods and deterministic demands. The nonexistence result hinges on a series of conditions that create conflicting incentives on when to order. Its building blocks (Claims 1-6) and a formal proof that puts them all together is presented in Appendix B. Here we first state the result and then give an intuitive account of why it is true.

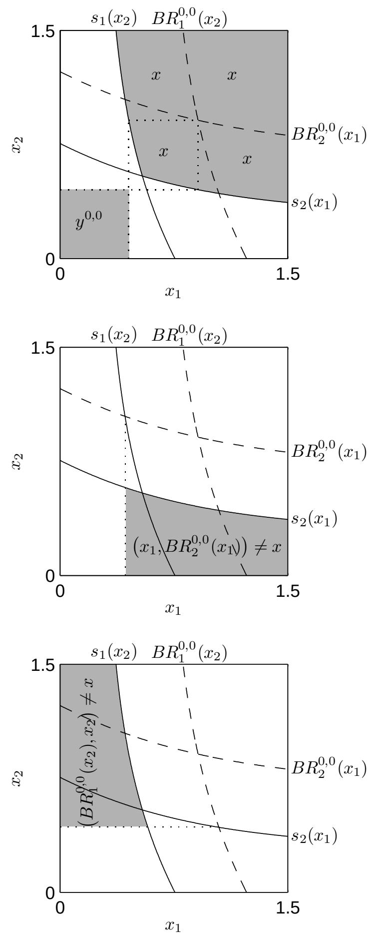

Figure 1: The equilibrium set as a function of x in Example 1.

Proposition 1. *The two-period inventory competition game with fixed order costs does not have a subgame-perfect equilibrium in pure Markov strategies if the following conditions are satisfied:*

1. $d_{i,t}$ *are deterministic for all* i, t*;*
2. $d_{i,1} - \frac{k_i}{r_i - c_i} > x_{i,1}$ *and* $d_{i,2} - \frac{k_i}{r_i - c_i} > 0$ *for all* i*;*
3. $\delta_i r_i > c_i + h_i$ *for all* i*;*
4. $(c_i + h_i)d_{i,2} < \delta_i(k_i + c_i d_{i,2})$ *for all* i*;*
5. *For some* i

(a) $\alpha_{-i}(d_{i,1} - x_{i,1}) < \frac{k_{-i}}{r_{-i}-c_{-i}}$, *and*

(b) $(c_i(1 - \delta_i) + h_i)d_{i,2} + (1 - \delta_i)k_i > (r_i - c_i)(1 - \delta_i\alpha_i\alpha_{-i})(d_{i,1} - x_{i,1})$.

In words, the five conditions, which define a family of problem instances, say that:

1. Demands are deterministic;
2. In both periods, satisfying first-choice demand is enough to cover the fixed order cost;
3. Saving first-period inventory for the second period is worth the cost;
4. If already placing an order, it is less costly to order some more in period 1 to meet second-period demand, than to place another order in period 2 at the expense of a fixed order cost;
5. For at least one of the firms, if it were to place no orders in period 1, then (a) profit potential from switchers to the other firm is not enough to cover fixed order cost for the other firm in period 2, and (b) inventory holding cost savings that come from not having to carry inventory from period 1 to period 2 outweigh the profits lost in period 1.

The proof of Proposition 1 advances a sequence of five major arguments regarding an equilibrium of the two-period game – should one exist – will need to contend with:

- In period 2, firms either do not order, or order up to their second-period demand regardless of their inventory levels at the beginning of period 2. This is simply due to subgame perfection .

- In period 1, firms choose one of three options: (a) order up to total two-period demand, (b) order up to period 1 demand, or (c) do not order, i.e.,

$$y_{i,1} \in \{\bar{d}_{i,1}(y_{-i,1}) + \bar{d}_{i,2}(y_{-i,2}), \bar{d}_{i,1}(y_{-i,1}), x_{i,1}\}.$$

  Although intuitively appealing due to piecewise linearity of expected profits in each period, this is by no means obvious, because a firm's decision in period 1 has some bearing on the total demand that it faces in period 2. In fact, we use Conditions (1)-(3) to prove it.

- Firm i cannot take option (b) in equilibrium, because by Condition (4) it would rather place a bigger order in period 1 to cover for second-period demand as well.

- Option (c) cannot occur in equilibrium either, because Condition (2) creates enough incentive for each firm to order in period 1.

- The only remaining candidate for equilibrium, both firms taking option (a), can also be ruled out. By Condition (5) at least one of the firms has an incentive to deviate from this strategy.

The most interesting dynamic that contributes to nonexistence occurs when both firms attempt to order their total two-period demand in period 1 (the last item above). This strategy is not an equilibrium, because at least one of the firms can profitably deviate from it by creating a deliberate scarcity – by not ordering at all in period 1. The reason is subtle. Suppose firm i deviates. By placing no orders in period 1, firm i would give up some of its demand and lose profits. However, it *can* fully recover this loss in period 2. Some of the demand unmet by firm i in period 1 would switch to firm $-i$. As a result, firm $-i$ would have less inventory, namely $[d_{-i,2} - \alpha_{-i}(d_{i,1} - x_{i,1})]$ as opposed to $d_{-i,2}$, at the beginning of period 2. This would result in firm $-i$ failing to fully meet its demand in period 2, as it would not want to place another order in period 2 by Condition (5a). Hence, some of the demand unmet by firm $-i$ in period 2, namely $\alpha_i \alpha_{-i}(d_{i,1} - x_{i,1})$, would switch to firm i. Also to firm i's advantage, it would avoid the cost of holding inventory for meeting second-period demand $d_{i,2}$. Condition (5b) ensures that inventory holding cost savings outweigh the profits given up. In sum, the deviating firm (firm i) is able to generate a strategic advantage out of scarcity that it deliberately introduces in period 1. The unfortunate byproduct of this strategic behavior is that both firms ordering their total two-period demand in period 1 cannot be an equilibrium.

A few final observations about the robustness of Proposition 1 is in order. First, the family of problem instances for which nonexistence of equilibrium is assured is quite large. To illustrate this, we offer a special case of the two-period model as an example, and graph the regions of parameter values that guarantee nonexistence (Figure 2). The result is robust to parameter perturbations, including deterministic perturbations to demands. Furthermore, the nonexistence can also be formally established for ϵ-equilibrium (the details are available from the authors). The ϵ-equilibrium is a simple extension of Nash that requires no player gaining more than ϵ by unilaterally deviating from the Nash strategy. Hence, the nonexistence of ϵ-equilibrium is a further indication that Proposition 1 is a robust result. It in fact suggests that a close cousin of Proposition 1 may hold true under stochastic perturbations in demand.

Example 2. *Consider the case with no initial inventories, $x_{i,1} = 0$, no discounting for period 2, $\delta_i = 1$, and constant demand, $d_{i,t} = d$, for all i, t and for some constant $d > 0$. Conditions (2)-(4) of Proposition 1 can then be reduced to*

$$1 > \frac{k_i}{(r_i - c_i)d} > \frac{h_i}{r_i - c_i}, \qquad \textit{for all } i \tag{6}$$

and Conditions (5a)-(5b) of Proposition 1 can be reduced to

$$\left(\frac{k_{-i}}{(r_{-i} - c_{-i})d} > \alpha_{-i} \qquad \textit{and} \qquad \frac{h_i}{r_i - c_i} > 1 - \alpha_i \alpha_{-i} \right), \qquad \textit{for some } i. \tag{7}$$

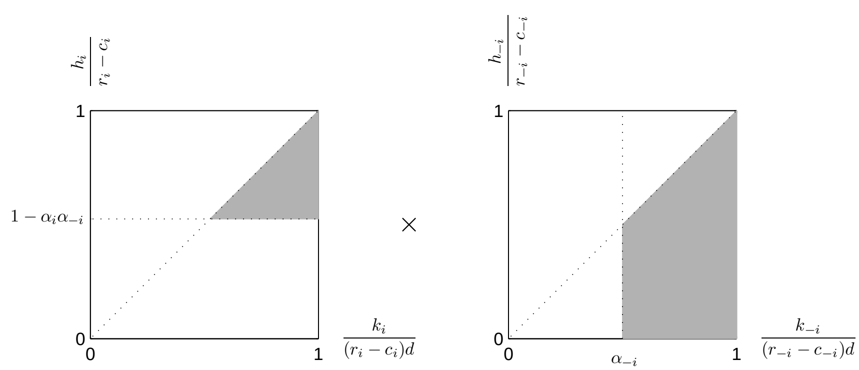

Figure 2: The range of parameters (shaded) that ensure nonexistence of equilibrium in Example 2.

Figure 2 illustrates the range of parameters for which (6) and (7) are satisfied. The x-axis is the fixed order cost per period normalized by profit potential per period (i.e., profit margin times demand). The y-axis is the unit inventory holding cost normalized by profit margin. Loosely speaking, for nonexistence fixed order costs need to be high relative to inventory holding costs (in both graphs, the shaded regions are below the

45-degree line). Fixed order costs cannot be too high though (as indicated by the upper bound of 1 in horizontal axes), to the point that they cannot be recovered by the profit potential produced by first-choice demands. Some asymmetry between the firms is also necessary, and it stems from Conditions (5a)-(5b), which make "deviation by scarcity" strategy possible (as we elaborate above). Again loosely speaking, one of the firms (firm $-i$) is allowed to have a much lower holding cost than the other; this seduces firm $-i$ to order everything in period 1, but firm i deviates by creating scarcity for all the reasons explained earlier.

We now provide a set of simple sufficient conditions for the equilibrium to exist.

Proposition 2. *The multiple-period inventory competition game with fixed order costs does possess an equilibrium in pure Markov strategies (which is not subgame perfect) if the following conditions are satisfied:*

1. $d_{i,t}$ *are deterministic for all* i, t*; and*
2. $d_{i,1} - \frac{k_i}{r_i - c_i} > x_{i,1}$ *and* $d_{i,t} - \frac{k_i}{r_i - c_i} > 0$ *for all* i, t*.*

A proof is provided in Appendix C. The equilibrium is constructed based on the inventory replenishment policy that would have been optimal for each firm i had it assumed no switchers, as if firm $-i$ did not exist (this is reminiscent of [10]). The main idea is that, given deterministic demands, if any firm i blindly brings its inventory in each period to the same level as the one prescribed by an optimal policy for the single-firm problem, then firm i would meet all of its demand and prevent any demand switching to firm $-i$ regardless of firm $-i$'s policy. Thus, both parties blindly mimicking an optimal policy for the single-firm problem would lead to mutually optimal responses in two-firm inventory competition. Note that this idea does not extend to stochastic demands in a straightforward manner.

4. Concluding Remarks

The main insight that our paper offers is a cautionary one: Multiple-period inventory competition may not have an equilibrium under fixed order costs. This does not even require stochastic demands, which suggests that nonexistence extends to more general multiple-period settings with fixed order costs.

It is apparent that the presence of fixed order costs fundamentally changes the nature of inventory competition. The biggest open question for future research would then be: What would absolutely guarantee existence of equilibrium over multiple periods when fixed costs are significant? Aspects of inventory competition that we believe may contain the answer are: allowing incomplete and imperfect information structures, and restricting the inventory replenishment policies of each firm to a certain class.

Appendix A. Equilibrium in Single-Period Inventory Competition with Fixed Order Costs

Proof of Lemma 1.

1. *[Upper Semi-Continuity]* The only discontinuity in $g_i(y_i, y_{-i}; x_i)$ is due to the fixed order cost $-k_i I\{y_i > x_i\}$, which is upper semi-continuous in y_i.
2. *[Decreasing Differences]* This property can be justified by

$$\begin{aligned}
&\big(g_i(y_i, y_{-i}; x_i) - g_i(\bar{y}_i, y_{-i}; x_i)\big) - \big(g_i(y_i, \bar{y}_{-i}; x_i) - g_i(\bar{y}_i, \bar{y}_{-i}; x_i)\big) \\
&\quad = -(r_i + h_i)E\big[(y_i - \bar{d}_i(y_{-i}))^+ - (\bar{y}_i - \bar{d}_i(y_{-i}))^+ \\
&\qquad\qquad -(y_i - \bar{d}_i(\bar{y}_{-i}))^+ + (\bar{y}_i - \bar{d}_i(\bar{y}_{-i}))^+\big] \le 0
\end{aligned}$$

   for any $y_i \ge \bar{y}_i \ge 0$ and $y_{-i} \ge \bar{y}_{-i} \ge 0$ [15, p. 489].
3. *[Compact Strategy Space]* For a fixed $y_{-i} \ge 0$, $\max_{y_i \in [0,\infty)} \bar{g}_i(y_i, y_{-i})$ is the classic newsvendor problem [16, pp. 7-16] that has an optimal solution

$$\breve{y}_i = \inf\left\{\xi \ge 0 : F_{\bar{d}_i(y_{-i})}(\xi) \ge \frac{r_i - c_i}{r_i + h_i}\right\}. \tag{A.1}$$

   Pick some arbitrary constant $\rho \in \left(\frac{r_i - c_i}{r_i + h_i}, 1\right)$, and define

$$Y_i^U := \inf\{\xi \ge 0 : F_{d_i + \alpha_i d_{-i}}(\xi) \ge \rho\}.$$

   Note that $Y_i^U < \infty$. We first argue by contradiction that $Y_i^U \ge \breve{y}_i$ for all $y_{-i} \ge 0$. Suppose $Y_i^U < \breve{y}_i$. Then, there must exist y_0 such that $Y_i^U < y_0 < \breve{y}_i$,

$$F_{d_i + \alpha_i d_{-i}}(y_0) \ge \rho, \quad \text{and} \quad F_{\bar{d}_i(y_{-i})}(y_0) < \frac{r_i - c_i}{r_i + h_i}.$$

The last two inequalities contradict each other, because $d_i + \alpha_i d_{-i} \geq \bar{d}_i(y_{-i})$ for all $y_{-i} \geq 0$, and $\rho > \frac{r_i - c_i}{r_i + h_i}$.

Next, we establish the following two implications:

$$\begin{aligned} y_i > Y_i^U \qquad &\Rightarrow \qquad y_i > \breve{y}_i \\ y_i > Y_i^U \qquad &\Rightarrow \qquad F_{\bar{d}_i(y_{-i})}(y_i) > \frac{r_i - c_i}{r_i + h_i} \end{aligned}$$

The first one follows from $Y_i^U \geq \breve{y}_i$. The second follows from the definition of Y_i^U, and the facts that $F_{\bar{d}_i(y_{-i})}(y_i) \geq F_{d_i + \alpha_i d_{-i}}(y_i)$ for all $y_{-i} \geq 0$, and $\rho > \frac{r_i - c_i}{r_i + h_i}$. Using these two implications and the proof of (A.1), it can be shown (we omit further details) that $y_i > Y_i^U \;\Rightarrow\; \bar{g}_i(y_i, y_{-i}) < \bar{g}_i(\breve{y}_i, y_{-i})$. This implies that the following is also true: $\tilde{y}_i \in \text{argmax}_{y_i[0,\infty)} \bar{g}_i(y_i, y_{-i}) \;\Rightarrow\; \tilde{y}_i \in [0, Y_i^U]$. An optimal solution to the general problem $\max_{y_i \in [x_i, \infty)} g_i(y_i, y_{-i}; x_i)$ easily follows from (A.1). It is either $\breve{y}_i$ if $\breve{y}_i > x_i$ (ordering), or x_i otherwise (not ordering). Therefore, we conclude that:

$$\bar{y}_i \in \text{argmax}_{y_i \in [x_i, \infty)} g_i(y_i, y_{-i}; x_i) \quad \Rightarrow \quad \bar{y}_i \in [x_i, \max(x_i, Y_i^U)].$$

$\square$

Proof of Theorem 1.

1. *[Existence]* The properties shown in Lemma 1 lead to a supermodular game, which possesses a pure-strategy equilibrium [15, pp. 491-492].
2. *[An Equilibrium with Both Firms Ordering]*
   (a) In view of (5), an interior equilibrium (y_1^*, y_2^*) of $\Gamma^{x,k}$ satisfies

   $$y_i^* \in BR_i^{0,0}(y_{-i}^*) \qquad \text{for all } i.$$

   (b) We have $y_i^* \in BR_i^{0,0}(y_{-i}^*)$ for all i. Then, in view of (5),

   $$\begin{gathered} \left(x_i \leq s_i(y_{-i}^*) \text{ and } x_i < y_i^* \text{ for all } i\right) \\ \Updownarrow \\ \left(x_i < y_i^* \in BR_i^{x,k}(y_{-i}^*) \text{ for all } i\right). \end{gathered}$$

3. *[An Equilibrium with One Firm Ordering]* In view of (5), we have

$$x_i \geq s_i(y^*_{-i}) \quad \Leftrightarrow \quad x_i \in BR_i^{x,k}(y^*_{-i})$$

$$\Big(x_{-i} \leq s_{-i}(x_i) \text{ and } x_{-i} < y^*_{-i} \in BR_{-i}^{0,0}(x_i)\Big)$$
$$\Updownarrow$$
$$\Big(x_{-i} < y^*_{-i} \in BR_{-i}^{x,k}(x_i)\Big).$$

4. *[An Equilibrium with Neither Firm Ordering]* It follows from (4)-(5) that

$$\big(x_i \geq s_i(x_{-i}) \text{ for all } i\big) \quad \Leftrightarrow \quad \big(x_i \in BR_i^{x,k}(x_{-i}) \text{ for all } i\big).$$

□

Appendix B. Nonexistence of Equilibrium in Multiple-Period Inventory Competition with Fixed Order Costs

Proof of Proposition 1. Assume that there exists a subgame-perfect equilibrium (η_1, η_2) in pure Markov strategies where $\eta_1 = (\eta_{1,1}, \eta_{1,2})$ and $\eta_2 = (\eta_{2,1}, \eta_{2,2})$. For all i, let $y_{i,1} := \eta_{i,1}(x_{i,1}, x_{-i,1})$, $x_{i,2} := (y_{i,1} - \bar{d}_{i,1}(y_{-i,1}))^+$, and $y_{i,2} := \eta_{i,2}(x_{i,2}, x_{-i,2})$. It follows from subgame perfection and deterministic demands that, for any pair of nonnegative beginning inventory levels $(\tilde{x}_{1,2}, \tilde{x}_{2,2})$ in period 2,

$$(\tilde{y}_{1,2}, \tilde{y}_{2,2}) := \big(\eta_{1,2}(\tilde{x}_{1,2}, \tilde{x}_{2,2}), \eta_{2,2}(\tilde{x}_{2,2}, \tilde{x}_{1,2})\big)$$

must be an equilibrium, i.e.,

$$\tilde{y}_{i,2} \in BR_i^{\tilde{x},k}(\tilde{y}_{-i,2}) = \begin{cases} \bar{d}_{i,2}(\tilde{y}_{-i,2}), & \tilde{x}_{i,2} < \bar{d}_{i,2}(\tilde{y}_{-i,2}) - \frac{k_i}{r_i - c_i} \\ \{\bar{d}_{i,2}(\tilde{y}_{-i,2}), \tilde{x}_{i,2}\}, & \tilde{x}_{i,2} = \bar{d}_{i,2}(\tilde{y}_{-i,2}) - \frac{k_i}{r_i - c_i} \\ \tilde{x}_{i,2}, & \tilde{x}_{i,2} > \bar{d}_{i,2}(\tilde{y}_{-i,2}) - \frac{k_i}{r_i - c_i} \end{cases}$$

for all i, where $\tilde{x} := (\tilde{x}_{1,2}, \tilde{x}_{2,2})$ and $k := (k_1, k_2)$. At equilibrium (η_1, η_2) in period 2, each firm either places no orders or orders up to its second-period demand regardless of the beginning inventory levels.

The proof hinges on six claims stated and proven in the sequel. Claims 1 and 2 are proven under the following assumption whose validity is established in Claim 3.

Assumption 2. *Consider a specific firm $i \in \{1,2\}$. Firm $-i$ using its equilibrium strategy η_{-i} is not indifferent between ordering and not ordering in period 2, if no customers switch from firm i to firm $-i$ in both periods, i.e., $y_{-i,1} \neq d_{-i,1} + d_{-i,2} - \frac{k_{-i}}{r_{-i}-c_{-i}}$.*

If $y_{i,2} = x_{i,2}$, then by Claim 1 it follows that $y_{i,1} = \bar{d}_{i,1}(y_{-i,1}) + \bar{d}_{i,2}(y_{-i,2})$ (conditional on Assumption 2). Else, if $y_{i,2} = \bar{d}_{i,2}(y_{-i,2}) > x_{i,2}$, then it follows from Claim 2 that $y_{i,1} \in \{x_{i,1}, \bar{d}_{i,1}(y_{-i,1})\}$ (again, conditional on Assumption 2). Claim 3 shows that Assumption 2 can be made without loss of generality, that it is always valid. Therefore, collectively, Claims 1-3 imply for all i that

$$y_{i,1} \in \{x_{i,1}, \bar{d}_{i,1}(y_{-i,1}), \bar{d}_{i,1}(y_{-i,1}) + \bar{d}_{i,2}(y_{-i,2})\}. \tag{B.1}$$

In words, at equilibrium (η_1, η_2) in period 1, each firm chooses one of three options: do not order, order up to first-period demand, or order up to cumulative demand over two periods. Claims 4 and 5 rule out the first two: $y_{i,1} = x_{i,1}$ and $y_{i,1} = \bar{d}_{i,1}(y_{-i,1})$, respectively, for each firm i. Claim 6 rules out the only remaining possibility for equilibrium, $(y_{1,1}, y_{2,1}) = (d_{1,1} + d_{1,2}, d_{2,1} + d_{2,2})$. We thus conclude that this two-period game has no subgame-perfect equilibrium in pure Markov strategies. □

Claim 1. *Suppose Assumption 2 is valid for firm i. At equilibrium (η_1, η_2), firm i does not order in period 2 only if it orders up to its cumulative two-period demand in period 1, i.e., $y_{i,2} = x_{i,2}$ implies $y_{i,1} = \bar{d}_{i,1}(y_{-i,1}) + \bar{d}_{i,2}(y_{-i,2})$.*

Proof of Claim 1. To have $y_{i,2} = x_{i,2}$, we must have $y_{i,1} > \bar{d}_{i,1}(y_{-i,1})$, since otherwise we would have $x_{i,2} = 0 < d_{i,2} - \frac{k_i}{r_i - c_i}$ (see Condition 2), which would imply $y_{i,2} > x_{i,2}$. Therefore,

$$\begin{aligned} J_i(\eta_i, \eta_{-i}) &= (r_i - c_i)\bar{d}_{i,1}(y_{-i,1}) - k_i + c_i x_{i,1} + (\delta_i r_i - c_i - h_i)x_{i,2} \\ &\quad -\delta_i(r_i + h_i)(x_{i,2} - \bar{d}_{i,2}(y_{-i,2}))^+. \end{aligned}$$

Define $\tilde{x}_{-i,2} := (y_{-i,1} - d_{-i,1})^+$ and

$$\tilde{y}_{-i,2} := \begin{cases} d_{-i,2}, & \tilde{x}_{-i,2} < d_{-i,2} - \frac{k_{-i}}{r_{-i}-c_{-i}} \\ \tilde{x}_{-i,2}, & \tilde{x}_{-i,2} > d_{-i,2} - \frac{k_{-i}}{r_{-i}-c_{-i}} \end{cases}.$$

Note that $\tilde{x}_{-i,2} \neq d_{-i,2} - \frac{k_{-i}}{r_{-i}-c_{-i}}$ due to Assumption 2. Since $y_{i,1} > \bar{d}_{i,1}(y_{-i,1})$, we have $x_{-i,2} = \tilde{x}_{-i,2}$. Hence, the following implication is true:

$$y_{i,2} \geq d_{i,2} \qquad \Rightarrow \qquad y_{-i,2} = \tilde{y}_{-i,2}. \tag{B.2}$$

Let $\hat{\eta}_i = (\hat{\eta}_{i,1}, \hat{\eta}_{i,2})$ be an alternative strategy for firm i such that

$$\begin{aligned} \hat{\eta}_{i,1}(x_{i,1}, x_{-i,1}) &= \bar{d}_{i,1}(y_{-i,1}) + \bar{d}_{i,2}(\tilde{y}_{-i,2}) \\ \hat{\eta}_{i,2}(\bar{d}_{i,2}(\tilde{y}_{-i,2}), \tilde{x}_{-i,2}) &= \bar{d}_{i,2}(\tilde{y}_{-i,2}). \end{aligned}$$

Let $(\hat{x}_{i,2}, \hat{x}_{-i,2})$ and $(\hat{y}_{i,2}, \hat{y}_{-i,2})$ denote the inventory levels at the beginning and at the end of period 2 under $(\hat{\eta}_i, \eta_{-i})$, respectively. We have

$$(\hat{x}_{i,2}, \hat{x}_{-i,2}) = (\bar{d}_{i,2}(\tilde{y}_{-i,2}), \tilde{x}_{-i,2}) \quad \text{and} \quad (\hat{y}_{i,2}, \hat{y}_{-i,2}) = (\bar{d}_{i,2}(\tilde{y}_{-i,2}), \tilde{y}_{-i,2})$$

where $\hat{y}_{-i,2} = \tilde{y}_{-i,2}$ is due to subgame perfection. Therefore,

$$J_i(\hat{\eta}_i, \eta_{-i}) = (r_i - c_i)\bar{d}_{i,1}(y_{-i,1}) - k_i + c_i x_{i,1} + (\delta_i r_i - c_i - h_i)\bar{d}_{i,2}(\tilde{y}_{-i,2}).$$

From $J_i(\eta_i, \eta_{-i}) \geq J_i(\hat{\eta}_i, \eta_{-i})$ and $\delta_i r_i - c_i - h_i > 0$ (see Condition 3), we conclude $x_{i,2} \geq \bar{d}_{i,2}(\tilde{y}_{-i,2})$. This together with (B.2) allows us to also conclude $y_{-i,2} = \tilde{y}_{-i,2}$. In view of this, $J_i(\eta_i, \eta_{-i}) \geq J_i(\hat{\eta}_i, \eta_{-i})$ now leads to $x_{i,2} = \bar{d}_{i,2}(y_{-i,2})$, which in turn leads to $y_{i,1} = \bar{d}_{i,1}(y_{-i,1}) + \bar{d}_{i,2}(y_{-i,2})$. □

Claim 2. *Suppose Assumption 2 is valid for firm i. At equilibrium (η_1, η_2), firm i orders in period 2 only if it either places no orders in period 1 or orders up to its first-period demand, i.e., $y_{i,2} > x_{i,2}$ implies $y_{i,1} \in \{x_{i,1}, \bar{d}_{i,1}(y_{-i,1})\}$.*

Proof of Claim 2. To have $y_{i,2} > x_{i,2}$, we must have $y_{i,2} = \bar{d}_{i,2}(y_{-i,2})$. This leads to

$$\begin{aligned} J_i(\eta_i, \eta_{-i}) \;\; = \;\; & (r_i - c_i)\min(y_{i,1}, \bar{d}_{i,1}(y_{-i,1})) - ((1-\delta_i)c_i + h_i)x_{i,2} \\ & -k_i I\{y_{i,1} > x_{i,1}\} + c_i x_{i,1} + \delta_i(r_i - c_i)\bar{d}_{i,2}(y_{-i,2}) - \delta_i k_i. \end{aligned}$$

Define $\tilde{x}_{-i,2} := (y_{-i,1} - d_{-i,1})^+$ and

$$\tilde{y}_{-i,2} \;\; := \;\; \begin{cases} d_{-i,2}, & \tilde{x}_{-i,2} < d_{-i,2} - \frac{k_{-i}}{r_{-i}-c_{-i}} \\ \tilde{x}_{-i,2}, & \tilde{x}_{-i,2} > d_{-i,2} - \frac{k_{-i}}{r_{-i}-c_{-i}} \end{cases}.$$

Note that $\tilde{x}_{-i,2} \neq d_{-i,2} - \frac{k_{-i}}{r_{-i}-c_{-i}}$ due to Assumption 2. Since $y_{i,2} = \bar{d}_{i,2}(y_{-i,2})$, we have $\bar{d}_{-i,2}(y_{i,2}) = d_{-i,2}$. Hence, the following implications are true:

$$y_{i,1} \geq d_{i,1} \qquad \Rightarrow \qquad x_{-i,2} = \tilde{x}_{-i,2} \qquad \Rightarrow \qquad y_{-i,2} = \tilde{y}_{-i,2}. \tag{B.3}$$

Let $\hat{\eta}_i = (\hat{\eta}_{i,1}, \hat{\eta}_{i,2})$ be an alternative strategy for firm i such that

$$\begin{aligned} \hat{\eta}_{i,1}(x_{i,1}, x_{-i,1}) &= \bar{d}_{i,1}(y_{-i,1}) \\ \hat{\eta}_{i,2}(0, \tilde{x}_{-i,2}) &= \bar{d}_{i,2}(\tilde{y}_{-i,2}). \end{aligned}$$

Let $(\hat{x}_{i,2}, \hat{x}_{-i,2})$ and $(\hat{y}_{i,2}, \hat{y}_{-i,2})$ denote the inventory levels at the beginning and at the end of period 2 under $(\hat{\eta}_i, \eta_{-i})$, respectively. We have $(\hat{x}_{i,2}, \hat{x}_{-i,2}) = (0, \tilde{x}_{-i,2})$ and $(\hat{y}_{i,2}, \hat{y}_{-i,2}) = (\bar{d}_{i,2}(\tilde{y}_{-i,2}), \tilde{y}_{-i,2})$, where $\hat{y}_{-i,2} = \tilde{y}_{-i,2}$ is due to subgame perfection. Therefore,

$$J_i(\hat{\eta}_i, \eta_{-i}) = (r_i - c_i)\bar{d}_{i,1}(y_{-i,1}) - k_i + c_i x_{i,1} + \delta_i(r_i - c_i)\bar{d}_{i,2}(\tilde{y}_{-i,2}) - \delta_i k_i.$$

We must have $y_{i,1} \leq \bar{d}_{i,1}(y_{-i,1})$, since otherwise we would have $((1-\delta_i)c_i + h_i)x_{i,2} > 0$ and $\bar{d}_{i,2}(y_{-i,2}) = \bar{d}_{i,2}(\tilde{y}_{-i,2})$ (see (B.3)), which would result in $J_i(\hat{\eta}_i, \eta_{-i}) > J_i(\eta_i, \eta_{-i})$. This implies that

$$J_i(\eta_i, \eta_{-i}) = (r_i - c_i)y_{i,1} - k_i I\{y_{i,1} > x_{i,1}\} + c_i x_{i,1} + \delta_i(r_i - c_i)\bar{d}_{i,2}(y_{-i,2}) - \delta_i k_i.$$

Let us now compare $J_i(\eta_i, \eta_{-i})$ with $J_i(\hat{\eta}_i, \eta_{-i})$. If $y_{i,1} > x_{i,1}$, then

$$\frac{J_i(\eta_i, \eta_{-i}) - J_i(\hat{\eta}_i, \eta_{-i})}{r_i - c_i} = y_{i,1} - \bar{d}_{i,1}(y_{-i,1}) + \delta_i(\bar{d}_{i,2}(y_{-i,2}) - \bar{d}_{i,2}(\tilde{y}_{-i,2})).$$

We note that

$$\begin{aligned} \bar{d}_{i,2}(y_{-i,2}) - \bar{d}_{i,2}(\tilde{y}_{-i,2}) &= \alpha_i \left(d_{-i,2} - y_{-i,2}\right)^+ - \alpha_i \left(d_{-i,2} - \tilde{y}_{-i,2}\right)^+ \\ &\leq \alpha_i \left(\tilde{y}_{-i,2} - y_{-i,2}\right)^+ \\ &\leq \alpha_i \left(\tilde{x}_{-i,2} - x_{-i,2}\right) \\ &= \alpha_i \left(y_{-i,1} - d_{-i,1}\right)^+ - \alpha_i \left(y_{-i,1} - \bar{d}_{-i,1}(y_{i,1})\right)^+ \\ &\leq \alpha_i \alpha_{-i}(d_{i,1} - y_{i,1})^+ \\ &\leq \alpha_i \alpha_{-i}(\bar{d}_{i,1}(y_{-i,1}) - y_{i,1}) \end{aligned}$$

where the second inequality follows from $\tilde{x}_{-i,2} \geq x_{-i,2}$, the definition of $\tilde{y}_{-i,2}$, and

$$y_{-i,2} \quad \in \quad \begin{cases} d_{-i,2}, & x_{-i,2} < d_{-i,2} - \frac{k_{-i}}{r_{-i}-c_{-i}} \\ \{d_{-i,2}, x_{-i,2}\}, & x_{-i,2} = d_{-i,2} - \frac{k_{-i}}{r_{-i}-c_{-i}} \\ x_{-i,2}, & x_{-i,2} > d_{-i,2} - \frac{k_{-i}}{r_{-i}-c_{-i}} \end{cases} .$$

Therefore, if $y_{i,1} > x_{i,1}$, then

$$\frac{J_i(\eta_i, \eta_{-i}) - J_i(\hat{\eta}_i, \eta_{-i})}{r_i - c_i} \leq (y_{i,1} - \bar{d}_{i,1}(y_{-i,1}))(1 - \delta_i \alpha_i \alpha_{-i}).$$

This implies the desired result, because $1 - \delta_i \alpha_i \alpha_{-i} > 0$. $\square$

Claim 3. *At equilibrium* (η_1, η_2), *we must have* $y_{i,1} \neq d_{i,1} + d_{i,2} - \frac{k_i}{r_i - c_i}$ *for all* i.

Proof of Claim 3. Suppose, for some i, $y_{i,1} \neq d_{i,1} + d_{i,2} - \frac{k_i}{r_i - c_i}$ and $y_{-i,1} = d_{-i,1} + d_{-i,2} - \frac{k_{-i}}{r_{-i} - c_{-i}}$. Claims 1 and 2 would then imply for firm $-i$ that $y_{-i,1} \in \{x_{-i,1}, \bar{d}_{-i,1}(y_{i,1}), \bar{d}_{-i,1}(y_{i,1}) + \bar{d}_{-i,2}(y_{i,2})\}$. Since $x_{-i,1} < y_{-i,1} < d_{-i,1} + d_{-i,2}$ (see Condition 2), we must have

$$y_{-i,1} = \bar{d}_{-i,1}(y_{i,1}) \quad \text{and} \quad y_{i,1} < d_{i,1}. \tag{B.4}$$

From (B.4), we have $(x_{i,2}, x_{-i,2}) = (0, 0)$ and $(y_{i,2}, y_{-i,2}) = (d_{i,2}, d_{-i,2})$. Therefore,

$$J_i(\eta_i, \eta_{-i}) = (r_i - c_i)y_{i,1} - k_i I\{y_{i,1} > x_{i,1}\} + c_i x_{i,1} + \delta_i(r_i - c_i)d_{i,2} - \delta_i k_i.$$

Let $\hat{\eta}_i = (\hat{\eta}_{i,1}, \hat{\eta}_{i,2})$ be an alternative strategy for firm i such that

$$\begin{aligned} \hat{\eta}_{i,1}(x_{i,1}, x_{-i,1}) &= d_{i,1} \\ \hat{\eta}_{i,2}(0, (y_{-i,1} - d_{-i,1})^+) &= d_{i,2}. \end{aligned}$$

Let $(\hat{x}_{i,2}, \hat{x}_{-i,2})$ and $(\hat{y}_{i,2}, \hat{y}_{-i,2})$ denote the inventory levels at the beginning and at the end of period 2 under $(\hat{\eta}_i, \eta_{-i})$, respectively. We have $(\hat{x}_{i,2}, \hat{x}_{-i,2}) = (0, (y_{-i,1} - d_{-i,1})^+)$ and $\hat{y}_{i,2} = d_{i,2}$. Therefore,

$$J_i(\hat{\eta}_i, \eta_{-i}) = (r_i - c_i)d_{i,1} - k_i + c_i x_{i,1} + \delta_i(r_i - c_i)d_{i,2} - \delta_i k_i.$$

In view of $J_i(\eta_i, \eta_{-i}) \geq J_i(\hat{\eta}_i, \eta_{-i})$ and $x_{i,1} < d_{i,1} - \frac{k_i}{r_i - c_i}$ (see Condition 4), we must have $y_{i,1} = d_{i,1}$ which contradicts (B.4).

Now suppose $y_{i,1} = d_{i,1} + d_{i,2} - \frac{k_i}{r_i - c_i}$ for all i. There are three possibilities for $(y_{i,2}, y_{-i,2})$.

The case in which $(y_{i,2}, y_{-i,2}) = (d_{i,2}, d_{-i,2})$*:* In this case, we have

$$J_i(\eta_i, \eta_{-i}) = (r_i - c_i)d_{i,1} - ((1-\delta_i)c_i + h_i)x_{i,2} - k_i + c_i x_{i,1} + \delta_i(r_i - c_i)d_{i,2} - \delta_i k_i.$$

Let $\hat{\eta}_i = (\hat{\eta}_{i,1}, \hat{\eta}_{i,2})$ be an alternative strategy for firm i such that

$$\begin{aligned} \hat{\eta}_{i,1}(x_{i,1}, x_{-i,1}) &= d_{i,1} \\ \hat{\eta}_{i,2}(0, (y_{-i,1} - d_{-i,1})^+) &= d_{i,2}. \end{aligned}$$

Let $(\hat{x}_{i,2}, \hat{x}_{-i,2})$ and $(\hat{y}_{i,2}, \hat{y}_{-i,2})$ denote the inventory levels at the beginning and at the end of period 2 under $(\hat{\eta}_i, \eta_{-i})$, respectively. We have $(\hat{x}_{i,2}, \hat{x}_{-i,2}) = (0, (y_{-i,1} - d_{-i,1})^+)$ and $\hat{y}_{i,2} = d_{i,2}$. Therefore,

$$J_i(\hat{\eta}_i, \eta_{-i}) = (r_i - c_i)d_{i,1} - k_i + c_i x_{i,1} + \delta_i(r_i - c_i)d_{i,2} - \delta_i k_i.$$

This leads to $J_i(\hat{\eta}_i, \eta_{-i}) > J_i(\eta_i, \eta_{-i})$ due to $((1-\delta_i)c_i + h_i)x_{i,2} > 0$ (see Condition 2). Therefore, this case cannot occur.

The case in which $(y_{i,2}, y_{-i,2}) = (x_{i,2}, \bar{d}_{-i,2}(x_{i,2}))$ *for some* i*:* We have

$$J_i(\eta_i, \eta_{-i}) = (r_i - c_i)d_{i,1} - k_i + c_i x_{i,1} + (\delta_i r_i - c_i - h_i)x_{i,2}.$$

Let $\hat{\eta}_i = (\hat{\eta}_{i,1}, \hat{\eta}_{i,2})$ be an alternative strategy for firm i such that

$$\begin{aligned} \hat{\eta}_{i,1}(x_{i,1}, x_{-i,1}) &= d_{i,1} + d_{i,2} \\ \hat{\eta}_{i,2}(d_{i,2}, (y_{-i,1} - d_{-i,1})^+) &= d_{i,2}. \end{aligned}$$

Let $(\hat{x}_{i,2}, \hat{x}_{-i,2})$ and $(\hat{y}_{i,2}, \hat{y}_{-i,2})$ denote the inventory levels at the beginning and at the end of period 2 under $(\hat{\eta}_i, \eta_{-i})$, respectively. We have $(\hat{x}_{i,2}, \hat{x}_{-i,2}) = (d_{i,2}, (y_{-i,1} - d_{-i,1})^+)$ and $\hat{y}_{i,2} = d_{i,2}$. Therefore,

$$J_i(\hat{\eta}_i, \eta_{-i}) = (r_i - c_i)d_{i,1} - k_i + c_i x_{i,1} + (\delta_i r_i - c_i - h_i)d_{i,2}.$$

This leads to $J_i(\hat{\eta}_i, \eta_{-i}) > J_i(\eta_i, \eta_{-i})$ due to $x_{i,2} < d_{i,2}$ and $(\delta_i r_i - c_i - h_i) > 0$ (see Conditions 2 and 3). Therefore, this case cannot occur either. ☐

Claim 4. *At equilibrium* (η_1, η_2) *we must have* $y_{i,1} \neq \bar{d}_{i,1}(y_{-i,1})$ *for all* i*.*

Proof of Claim 4. Claims 1-3 imply that at (η_1, η_2) we must have, for all i,

$$y_{i,1} \in \{x_{i,1}, \bar{d}_{i,1}(y_{-i,1}), \bar{d}_{i,1}(y_{-i,1}) + \bar{d}_{i,2}(y_{-i,2})\}. \tag{B.5}$$

One consequence of this is that, for all i,

$$x_{i,2} = \begin{cases} 0, & y_{i,1} \in \{x_{i,1}, \bar{d}_{i,1}(y_{-i,1})\} \\ \bar{d}_{i,2}(y_{-i,2}), & y_{i,1} = \bar{d}_{i,1}(y_{-i,1}) + \bar{d}_{i,2}(y_{-i,2}) \end{cases} \tag{B.6}$$

which in turn implies that

$$(y_{i,2}, y_{-i,2}) = (d_{i,2}, d_{-i,2}). \tag{B.7}$$

Suppose that $y_{i,1} = \bar{d}_{i,1}(y_{-i,1})$. Then, using (B.5)-(B.7), we have

$$J_i(\eta_i, \eta_{-i}) = (r_i - c_i)\bar{d}_{i,1}(y_{-i,1}) - k_i + c_i x_{i,1} + \delta_i((r_i - c_i)d_{i,2} - k_i).$$

Let $\hat{\eta}_i = (\hat{\eta}_{i,1}, \hat{\eta}_{i,2})$ be an alternative strategy for firm i such that

$$\begin{aligned} \hat{\eta}_{i,1}(x_{i,1}, x_{-i,1}) &= \bar{d}_{i,1}(y_{-i,1}) + d_{i,2} \\ \hat{\eta}_{i,2}(d_{i,2}, (y_{-i,1} - d_{-i,1})^+) &= d_{i,2}. \end{aligned}$$

Let $(\hat{x}_{i,2}, \hat{x}_{-i,2})$ and $(\hat{y}_{i,2}, \hat{y}_{-i,2})$ denote the inventory levels at the beginning and at the end of period 2 under $(\hat{\eta}_i, \eta_{-i})$, respectively. We have $(\hat{x}_{i,2}, \hat{x}_{-i,2}) = (d_{i,2}, (y_{-i,1} - d_{-i,1})^+)$ and $\hat{y}_{i,2} = d_{i,2}$. Therefore,

$$\begin{aligned} J_i(\hat{\eta}_i, \eta_{-i}) &= (r_i - c_i)\bar{d}_{i,1}(y_{-i,1}) - k_i + c_i x_{i,1} \\ &\quad + \delta_i(r_i - c_i)d_{i,2} \underbrace{-(c_i(1 - \delta_i) + h_i)d_{i,2}}_{> -\delta_i k_i} \\ &> J_i(\eta_i, \eta_{-i}) \end{aligned}$$

where the inequality is due to Condition (4). Hence, if $y_{i,1} = \bar{d}_{i,1}(y_{-i,1})$, firm i can do strictly better by unilaterally deviating to $\hat{\eta}_i$. $\square$

Claim 5. *At equilibrium (η_1, η_2), we must have $y_{i,1} \neq x_{i,1}$ for all i.*

Proof of Claim 5. Suppose that $y_{i,1} = x_{i,1}$. Then, using (B.5)-(B.7), we have

$$J_i(\eta_i, \eta_{-i}) = (r_i - c_i)x_{i,1} + c_i x_{i,1} + \delta_i((r_i - c_i)d_{i,2} - k_i).$$

Let $\hat{\eta}_i = (\hat{\eta}_{i,1}, \hat{\eta}_{i,2})$ be an alternative strategy for firm i such that

$$\begin{aligned} \hat{\eta}_{i,1}(x_{i,1}, x_{-i,1}) &= \bar{d}_{i,1}(y_{-i,1}) \\ \hat{\eta}_{i,2}(0, (y_{-i,1} - d_{-i,1})^+) &= d_{i,2}. \end{aligned}$$

Let $(\hat{x}_{i,2}, \hat{x}_{-i,2})$ and $(\hat{y}_{i,2}, \hat{y}_{-i,2})$ denote the inventory levels at the beginning and at the end of period 2 under $(\hat{\eta}_i, \eta_{-i})$, respectively. We have $(\hat{x}_{i,2}, \hat{x}_{-i,2}) = (0, (y_{-i,1} - d_{-i,1})^+)$ and $\hat{y}_{i,2} = d_{i,2}$. Therefore,

$$\begin{aligned} J_i(\hat{\eta}_i, \eta_{-i}) &= \underbrace{(r_i - c_i)\bar{d}_{i,1}(y_{-i,1}) - k_i}_{>(r_i - c_i)x_{i,1}} + c_i x_{i,1} + \delta_i((r_i - c_i)d_{i,2} - k_i) \\ &> J_i(\eta_i, \eta_{-i}) \end{aligned}$$

where the inequality is due to Condition (2). Therefore, if $y_{i,1} = x_{i,1}$, firm i can do strictly better by unilaterally deviating to $\hat{\eta}_i$. □

Claim 6. *At equilibrium* (η_1, η_2)*, we must have* $(y_{1,1}, y_{2,1}) \neq (d_{1,1} + d_{1,2}, d_{2,1} + d_{2,2})$.

Proof of Claim 6. Claims 1-5 imply that, at equilibrium (η_1, η_2), we must have

$$\begin{aligned} (y_{1,1}, y_{2,1}) &= (d_{1,1} + d_{1,2}, d_{2,1} + d_{2,2}) && \text{(B.8)} \\ (x_{1,2}, y_{2,2}) &= (d_{1,2}, d_{2,2}) && \text{(B.9)} \\ (y_{1,2}, y_{2,2}) &= (d_{1,2}, d_{2,2}). && \text{(B.10)} \end{aligned}$$

Consider the specific firm i that meets Conditions (5a)-(5b). Then, using (B.8)-(B.10), we have

$$J_i(\eta_i, \eta_{-i}) = (r_i - c_i)(d_{i,1} + d_{i,2}) - (r_i + h_i)d_{i,2} - k_i + c_i x_{i,1} + \delta_i((r_i - c_i)d_{i,2} + c_i d_{i,2}).$$

Let $\hat{\eta}_i = (\hat{\eta}_{i,1}, \hat{\eta}_{i,2})$ be an alternative strategy for firm i such that

$$\begin{aligned} \hat{\eta}_{i,1}(x_{i,1}, x_{-i,1}) &= x_{i,1} \\ \hat{\eta}_{i,2}(0, \tilde{x}_{-i,2}) &= \bar{d}_{i,2}(\tilde{x}_{-i,2}) \end{aligned}$$

where $\tilde{x}_{-i,2} := d_{-i,2} - \alpha_{-i}(d_{i,1} - x_{i,1})$. Note that, due to Condition (5a), we have

$$\tilde{x}_{-i,2} > d_{-i,2} - \frac{k_{-i}}{r_{-i} - c_{-i}}.$$

Let $(\hat{x}_{i,2}, \hat{x}_{-i,2})$ and $(\hat{y}_{i,2}, \hat{y}_{-i,2})$ denote the inventory levels at the beginning and at the end of period 2 under $(\hat{\eta}_i, \eta_{-i})$, respectively. We have $(\hat{x}_{i,2}, \hat{x}_{-i,2}) = (0, \tilde{x}_{-i,2})$ and $(\hat{y}_{i,2}, \hat{y}_{-i,2}) = (\bar{d}_{i,2}(\hat{x}_{-i,2}), \hat{x}_{-i,2})$, where $\hat{y}_{-i,2} = \hat{x}_{-i,2}$ is due to subgame perfection. Therefore,

$$\begin{aligned} J_i(\hat{\eta}_i, \eta_{-i}) &= (r_i - c_i)x_{i,1} + c_i x_{i,1} \\ &\quad + \delta_i((r_i - c_i)(d_{i,2} + \alpha_i \alpha_{-i}(d_{i,1} - x_{i,1}))) - \delta_i k_i \\ &> J_i(\eta_i, \eta_{-i}) \end{aligned}$$

where the inequality is due to Condition (5b). □

Appendix C. Existence of Equilibrium in Multiple-Period Inventory Competition

Proof of Proposition 2. Consider the single-firm inventory control problem in which firm i makes inventory decisions to meet its own first-choice demands $d_{i,1}, \ldots, d_{i,T}$ assuming that firm $-i$ and its first-choice demands $d_{-i,1}, \ldots, d_{-i,T}$ do not exists. Let $\mu_i = (\mu_{i,1}, \ldots, \mu_{i,T})$ be an optimal policy for firm i, which is known to exist. Let $(x_{i,t}, y_{i,t})$ be firm i's inventory levels at the beginning and at the end of period t, generated recursively by

$$y_{i,t} = \mu_{i,t}(x_{i,t}) \tag{C.1}$$

$$x_{i,t+1} = (y_{i,t} - d_{i,t})^+. \tag{C.2}$$

We first argue that firm i meets its demand in all periods, i.e., $y_{i,t} \geq d_{i,t}$ for all t. To see this, assume that we have $y_{i,\tilde{t}} < d_{i,\tilde{t}}$ in period $\tilde{t} \in \{1, \ldots, T\}$ for the first time. It must be that $\tilde{t} \geq 2$ and $y_{i,\tilde{t}} = x_{i,\tilde{t}} \geq d_{i,\tilde{t}} - \frac{k_i}{r_i - c_i} > 0$; otherwise, firm i could strictly improve its total profit by ordering up to $d_{i,\tilde{t}}$ in period $\tilde{t}$. Let $\hat{t} \in \{1, \ldots, \tilde{t} - 1\}$ be the last period (before period $\tilde{t}$) in which firm i places an order, i.e., $y_{i,\hat{t}} > x_{i,\hat{t}}$. We must have $y_{i,\hat{t}} = d_{i,\hat{t}} + \cdots + d_{i,\tilde{t}-1} + x_{i,\tilde{t}}$ and $x_{i,\hat{t}} < d_{i,\hat{t}}$; otherwise, firm i can reduce its holding cost by delaying its ordering decision to a later period in which the inventory carried from the previous period is not sufficient to meet the demand in that period.

Let μ_i^A be a policy for firm i such that

$$\begin{aligned}
\mu_{i,t}^A(x_{i,t}) &= y_{i,t}, \qquad \text{for } t = 1, \ldots, \hat{t} - 1 \\
\mu_{i,\hat{t}}^A(x_{i,\hat{t}}) &= d_{i,\hat{t}} + \cdots + d_{i,\tilde{t}-1} \\
\mu_{i,t}^A(d_{i,t} + \cdots + d_{i,\tilde{t}-1}) &= d_{i,t} + \cdots + d_{i,\tilde{t}-1}, \qquad \text{for } t = \hat{t}+1, \ldots, \tilde{t} - 1 \\
\mu_{i,\tilde{t}}^A(0) &= 0 \\
\mu_{i,t}^A(x_{i,t}) &= y_{i,t}, \qquad \text{for } t = \tilde{t}+1, \ldots, T.
\end{aligned}$$

Let $J_i(\mu_i)$ and $J_i(\mu_i^A)$ denote the total profits corresponding to the policies μ_i and μ_i^A, respectively. Since μ_i^A is not an optimal policy, we have

$$J_i(\mu_i) - J_i(\mu_i^A) = \left(r_i \delta_i^{\tilde{t}-1} - c_i \delta_i^{\hat{t}-1} - h_i \sum_{t=\hat{t}}^{\tilde{t}-1} \delta_i^{t-1} \right) \tilde{x}_{i,\tilde{t}} > 0$$

which leads to $r_i \delta_i^{\tilde{t}-1} - c_i \delta_i^{\hat{t}-1} - h_i \sum_{t=\hat{t}}^{\tilde{t}-1} \delta_i^{t-1} > 0$. Let μ_i^B be another policy for firm i such that

$$\begin{aligned}
\mu_{i,t}^B(x_{i,t}) &= y_{i,t}, \qquad \text{for } t = 1, \ldots, \hat{t} - 1 \\
\mu_{i,\hat{t}}^B(x_{i,\hat{t}}) &= d_{i,\hat{t}} + \cdots + d_{i,\tilde{t}} \\
\mu_{i,t}^B(d_{i,t} + \cdots + d_{i,\tilde{t}}) &= d_{i,t} + \cdots + d_{i,\tilde{t}}, \qquad \text{for } t = \hat{t}+1, \ldots, \tilde{t} \\
\mu_{i,t}^B(x_{i,t}) &= y_{i,t}, \qquad \text{for } t = \tilde{t}+1, \ldots, T.
\end{aligned}$$

Let $J_i(\mu_i^B)$ denote the total profit corresponding to the policy μ_i^B. We have

$$J_i(\mu_i^B) - J_i(\mu_i) = \left(r_i \delta_i^{\tilde{t}-1} - c_i \delta_i^{\hat{t}-1} - h_i \sum_{t=\hat{t}}^{\tilde{t}-1} \delta_i^{t-1} \right) (d_{i,\tilde{t}} - \tilde{x}_{i,\tilde{t}}) > 0$$

which contradicts the optimality of μ_i. Therefore, $y_{i,t} \geq d_{i,t}$ for all t.

Now, consider the two-firm multiple-period inventory competition model. For all $i \in \{1, 2\}$, let $\eta_i = (\eta_{i,1}, \ldots, \eta_{i,T})$ be such that $\eta_{i,t}(\tilde{x}_{i,t}, \tilde{x}_{-i,t}) = \max\{\tilde{x}_{i,t}, y_{i,t}\}$ for all t and all $\tilde{x}_{i,t}, \tilde{x}_{-i,t} \geq 0$, where $y_{i,t}$ is generated by (C.1)-(C.2). It follows that if firm i uses η_i, then firm i's inventory decision at the end of any period t will be $y_{i,t}$ and no demand will switch from firm i to firm $-i$ regardless of the strategy used by firm $-i$. As a result, if firm i uses η_i, firm $-i$ cannot possibly improve its total profit (in a strict sense) by unilaterally switching from η_{-i} to an alternative strategy. □

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