QUESTIONS

1. Investigate the controllability properties of the linear time-invariant system \( \dot{x} = Ax + Bu \), for the three pairs of \((A, B)\) matrices given below. For the case(s) when the system is not completely controllable, determine the controllable subspace(s) and bring the system(s) to a controllability canonical form. (Do not use MATLAB in this question)

(i) \( A = \begin{bmatrix} -5 & 1 \\ 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \).

(ii) \( A = \begin{bmatrix} 3 & 3 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \).

(iii) \( A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \).

For the system corresponding to the \((A, B)\) pair given in part (i) obtain a non-impulsive input signal, if possible, that drives the system from \( x(0) = [1 \ 0]^T \) at \( t = 0 \) to \( x(1) = [0 \ 1]^T \) at \( t = 1 \).

2. Consider the linear time-invariant system \( \dot{x} = Ax + Bu \) where

\[
A = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}.
\]

(i) Check the controllability of this system using the controllability matrix. (You may use \texttt{ctrb} and \texttt{rank} commands of MATLAB)

(ii) Check the controllability of this system using the PBH test. (You may use \texttt{eig} command of MATLAB)

(iii) Check the controllability of this system using the limiting controllability gramian which may be obtained by solving the controllability Lyapunov matrix equation (You may use \texttt{lyap} or \texttt{gram} commands of MATLAB).

(iv) Identify the controllable and uncontrollable modes of the system and convert the system to a controllability canonical form.

3. Consider a linear time-invariant system \( \dot{x} = Ax + Bu \) where \((A, B)\) is controllable. Prove or disprove the following statements. (If the statement is false, then producing a counterexample will suffice. A piecewise-continuous signal means a signal that is made up of a finite number of continuous pieces.)

(i) It is always possible to find a piecewise-continuous input signal, defined on \([0, \infty)\), that will bring the state from \( x(0) = x_0 \neq 0 \) to rest at \( t = 1 \), that is \( x(t) = 0 \) for all \( t \geq 1 \).

(ii) It is always possible to find a piecewise-continuous input signal, defined on \([0, \infty)\), that will bring the state from \( x(0) = 0 \) to any \( \bar{x} \neq 0 \) at \( t = 1 \), that is \( x(1) = \bar{x} \).

(iii) It is always possible to find a piecewise-continuous input signal, defined on \([0, \infty)\), that will bring the state from \( x(0) = 0 \) to any \( \bar{x} \neq 0 \) at \( t = 1 \) and maintain the state at \( \bar{x} \) for all \( t \geq 1 \), that is \( x(t) = \bar{x} \) for all \( t \geq 1 \).

4. Consider the linear time-invariant system \( \dot{x} = Ax + Bu \) where

\[
A = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.
\]
(i) Determine if it is possible to find a non-impulsive input signal that will transfer the state from $x(0) = [1 \ 1]^T$ to $x(1) = [1 + \frac{e^{-2}}{2} \ 2]^T$.

(ii) If your answer in part (i) was yes, find such a non-impulsive input signal that will transfer the state from $x(0) = [1 \ 1]^T$ to $x(1) = [1 + \frac{e^{-2}}{2} \ 2]^T$. You may transform your system to a controllability canonical form and accomplish the desired state transfer in the transformed coordinates.

(iii) If your answer in part (i) was yes, verify your answer in part (ii) by simulating the system in MATLAB using the `lsim` command. Provide a plot with two subplots where the first subplot shows the state trajectory and the second subplot shows the non-impulsive input signal. Clearly label the states in the first subplot and make sure that the achievement of the desired state transfer is clearly illustrated.

5. Consider the linear time-invariant system $\dot{x} = Ax + Bu$ where

$$A = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. $$

(i) Find a target state $x_f$ subject to $||x_f||_2 = 1$ such that the state can be transferred from $x(0) = 0$ to $x(t_f) = x_f$ using an input signal with minimum possible energy with no limitation on $t_f$.

(ii) Find the minimum energy required to achieve the state transfer described in part (i) with the target state $x_f$ you obtained in part (i).