EE 650 Homework #4 due Wednesday, 10/8/08

QUESTIONS

1. Consider the equation $Ax = b$ where $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

   (i) Let S be the subspace spanned by the columns of A. Find $S_\perp$.

   (ii) Find the projections $\Pi_S b$ and $\Pi_{S_\perp} b$ of b onto S and $S_\perp$ respectively.

   (iii) Find an approximate solution x^* that minimizes $||Ax^* - b||_2^2$ by solving $Ax^* = \Pi_S b$.

   (iv) Find an approximate solution x^* that minimizes $||Ax^* - b||_2^2$ by first computing the pseudo-inverse of A.

2. Let M be a positive semi-definite matrix.

   (i) Show that
   $$\lambda_{\min}(M)||x||_2^2 \leq x^* M x \leq \lambda_{\max}(M)||x||_2^2, \text{ for all } x,$$
   where $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ are the minimum and maximum eigenvalues of M respectively. Hint: Use the fact that M can be diagonalized by a unitary matrix.

   (ii) Show that the upper and lower bounds above are tight by finding vectors that achieve these bounds.

3. Determine which of the following are legitimate norms for the space of continuous functions on $[0, 1]$.

   (i) $||f|| := \max_{t\in[0,1]} f(t)$

   (ii) $||f|| := \max_{t\in[0,1]} |e^t f(t)|$

   (iii) $||f|| := \max_{t\in[0,1/2]} |f(t)|$

   (iv) $||f|| := \int_0^1 |f(t)|^2 dt$

4. Given a positive semi-definite matrix A, another positive semi-definite matrix B is said to be the square root of A, if all eigenvalues of B are square roots of the eigenvalues of A, and $BB = A$. Such a matrix is unique and denoted by $A^{1/2}$. A method to compute the unique square root of a given positive semi-definite matrix is devised as follows: Given a positive semi-definite matrix A, there exists a unitary matrix Q (i.e., $Q^* = Q^{-1}$) such that $Q^* A Q = \Lambda$ where $\Lambda := diag\{\lambda_1, \lambda_2, ..., \lambda_n\}$. If A is positive semi-definite, then $\lambda_i \geq 0$, $i = 1, 2, ..., n$. Therefore, we can define $\Lambda^{1/2} := diag\{\sqrt{\lambda_1}, \sqrt{\lambda_2}, ..., \sqrt{\lambda_n}\}$, and $B := Q\Lambda^{1/2}Q^*$.

   (i) Use the method given above to compute $A^{1/2}$ where $A = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$.

   (i) Verify that your solution is indeed the square root of A.

5. True or False? If a matrix M is positive semi-definite and $M \neq 0$, then M is positive definite. Justify your answer.

6. True or False? If a matrix M is positive semi-definite and nonsingular, then M is positive definite. Justify your answer.

7. Consider a vector space V equipped with an inner product $<,>$. Let S be a subspace of V, and let $S^\perp := \{x \in V :< x, y >= 0, \ \forall y \in S\}$. Show that any x can be written as $x = x_S + x_{S^\perp}$ where $x \in S$ and $x \in S^\perp$. Hint: Assume $\{v^1, ..., v^n\}$ is an orthonormal basis for S. Let $x_S := \sum_{i=1}^n < v^i, x > v_i$ and $x_{S^\perp} := x - x_S$. Clearly, $x_S \in S$ and $x_S + x_{S^\perp} = x$. Show $x_{S^\perp} \in S^\perp$.

8. Find a linear approximation $x(t)$ to $\sin(t)$ on $[-1, 1]$ that minimizes $\int_{-1}^{1} (\sin(t) - x(t))^2 dt$.
9. Textbook Problem 3.33.
10. Textbook Problem 3.34.