QUESTIONS

1. Consider the following matrix $A$ and let $\bar{A}$ denote a Jordan canonical form representation of $A$. Find the distinct eigenvalues of $A$ and their algebraic and geometric multiplicities. How many Jordan blocks would $\bar{A}$ have for each distinct eigenvalue of $(A \text{ or } \bar{A})$? Find chains of generalized eigenvectors (with maximum number of linearly independent generalized eigenvectors in each chain) for each distinct eigenvalue. What is an appropriate similarity transformation that will give us $\bar{A}$? How does $\bar{A}$ look like? Hint: You can use the MATLAB commands rank and null (however, keep in mind that some MATLAB commands are not always reliable).

$$A = \begin{bmatrix} -1 & 0 & -1 & 1 & 1 & 3 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & -1 & -1 & -6 & 0 \\ -2 & 0 & -1 & 2 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 & 2 & 4 & 1 \end{bmatrix}$$

2. Consider the system $\dot{x} = Ax$ where

$$A = \begin{bmatrix} 3 & -3 & 3 \\ 0 & -3 & 0 \\ 6 & -3 & 0 \end{bmatrix}.$$  

(i) Find the Jordan canonical form representation of $A$ by a real similarity transformation.
(ii) Using your answer in part (i), find the values of $e^{At}$ at $t = 0.1$ and $t = 1$.
(iii) Verify your answer in part (ii) with MATLAB’s “expm” function.
(iv) Let $x(t)$ be the solution to $\dot{x} = Ax$ with $x(0) = [1 \ 0 \ 0]^T$. Find $x(0.1)$ and $x(1)$.
(v) Let $z(t)$ be the solution to $\dot{z} = Az$ with $z(1) = [0 \ 1 \ 0]^T$. Find $z(1.1)$ and $z(2)$.
(vi) Find all possible initial states $x(0)$ such that $x(t) \to 0$.

3. Consider the system $\dot{x} = Ax$ where

$$A = \begin{bmatrix} -1 & 1 & 1 \\ -1 & -1 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$  

(i) Find the real Jordan canonical form representation of $A$ by a real similarity transformation, i.e., find a real $Q$ such that

$$Q^{-1}AQ = \begin{bmatrix} S_1 \\ \vdots \\ S_q \\ \lambda_{2q+1} \\ \vdots \\ \lambda_n \end{bmatrix},$$

where $S_i = \begin{bmatrix} \sigma_i & w_i \\ -w_i & \sigma_i \end{bmatrix}$, $i = 1, ..., q$, are real matrices, $\sigma_1 + jw_1, \sigma_1 - jw_1, ..., \sigma_q + jw_q, \sigma_q - jw_q$ are complex eigenvalues of $A$ and $\lambda_{2q+1}, ..., \lambda_n$ are real eigenvalues of $A$.
(ii) Find a one dimensional subspace invariant under $\dot{x} = Ax$. Find a two dimensional subspace invariant under $\dot{x} = Ax$.

(iii) Find all possible initial states $x(0)$ such that $x(t) \to 0$.

4. Consider the continuous-time dynamical system

$$
\dot{x}(t) = \begin{bmatrix} -2 & 6 \\ -6 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)
$$

$$
y(t) = [1 \ 0] x(t),
$$

with initial state

$$
x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
$$

(i) Find the state transition matrix $\Phi(t, t_0)$ using the Laplace transformation method.

(ii) Find the output $y(t)$ for $t \geq 0$ in closed form (not a numerical solution) due to the input $u(t) = e^{-t}$ for $t \geq 0$.

(iii) Verify your answer in part (ii) by simulating the system in MATLAB using the lsim function with a time vector generated by $T = [0 : 0.01 : 3]$. Typing in help command-name (for instance help lsim) gives information about a particular command. Provide a plot with three subplots using the subplot command where the first subplot shows the output obtained by the lsim command versus time, the second subplot shows the closed-form output obtained in part (ii) versus time, and the third subplot shows the difference between the two versus time.