Mean lag in general error correction models*

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Abstract

Most of the empirical literature inappropriately applies Hendry's (1995) mean lag formula—which he derived for first order autoregressive distributed lag models under the assumption of a homogeneous long-run equilibrium—to error correction models that have complex lag structures and lack long-run homogeneity. We derive an expression for the mean lag in general error correction models without imposing the assumption of a homogeneous equilibrium. In addition, we quantify the bias due to the incorrect use of Hendry's (1995) formula.

Keywords: mean lag; autoregressive distributed lag model; error correction model.

JEL codes: B41, C18, C22, C32, C50.

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1 Introduction

The mean lag is a summary measure of the lag structure of dynamic models. It can be used to estimate the average delay in the transmission of shocks, such as the passthrough of income shocks to consumption, oil price shocks to gas prices, or market interest rates to retail rates, among others. A large number of empirical studies have resorted to an explicit mean lag formula published by Hendry (1995, p. 215, eq. 6.53). He derived it for the first order autoregressive distributed lag ($ADL(1,1)$) model and the associated error correction (EC) model under the assumption of a homogeneous equilibrium relationship. However, the formula is invalid in cases when the lag structure is more complex or the long-run homogeneity assumption does not hold. Nonetheless, we found a number of studies that use the formula inappropriately under these more general conditions, including Charoenseang and Manakit (2007); Chong and Liu (2009); De Bondt (2005); De Graeve et al. (2007); Leibrecht and Scharler (2008, 2011); Leszkiewicz-Kedzior and Welfe (2014); Scholnick (1996), among others.

We fill a gap in the literature by deriving an expression for the mean lag in general EC models without imposing the assumption of a homogeneous equilibrium. In addition, we evaluate the bias of the mean lag estimate arising from inappropriately imposing long-run homogeneity.

2 General form of the mean lag

In this section we derive the mean lag in a general, non-homogeneous, relationship. A general autoregressive distributed lag, or $ADL(p,q;n)$, model can be written as

$$y_t = c + \sum_{i=1}^{p} \alpha_i y_{t-i} + \sum_{k=1}^{n} \sum_{j=0}^{q} \beta_{k,j} x_{k,t-j} + \epsilon_t \quad \text{or} \quad \alpha(L) y_t = c + \sum_{k=1}^{n} \beta_k(L) x_{k,t} + \epsilon_t \ , \ (1)$$

where $\epsilon_t \sim IID$, $\alpha(L) = 1 - \sum_{i=1}^{p} \alpha_i L^i$ and $\beta_k(L) = \sum_{j=0}^{q} \beta_{k,j} L^j$ are lag polynomials, and n is the number of exogenous variables in the model. Regressors with varying lag lengths can be readily accommodated at the cost of further notational complexity. Rearrange equation (1) to obtain the reduced form equation

$$y_t = \frac{c}{\alpha(L)} + \frac{1}{\alpha(L)} \sum_{k=1}^{n} \beta_k(L) x_{k,t} + \frac{\epsilon_t}{\alpha(L)} = c^* + \sum_{k=1}^{n} w_k(L) x_{k,t} + u_t \ , \tag{2}$$

where $w_k(L) = \dfrac{\beta_k(L)}{\alpha(L)} = \sum_{j=1}^{\infty} w_{k,j} L^j$. The "weight" associated with lag j of variable x_k, $w_{k,j} = \dfrac{\partial y_t}{\partial x_{k,t-j}}$, captures the effect of $x_{k,t-j}$ on y_t. Hendry (1995, p. 215) defined the mean lag as

$$\begin{aligned} \mu_k = \frac{\sum_{j=0}^{\infty} j w_{k,j}}{\sum_{j=0}^{\infty} w_{k,j}} &= \frac{1}{w_k(1)} \left[\frac{\partial w_k(L)}{\partial L} \right]_{L=1} = \frac{1}{w_k(1)} \left[\frac{\beta_k'(L)}{\alpha(L)} - \frac{\beta_k(L)\alpha'(L)}{\alpha(L)^2} \right]_{L=1} \\ &= \frac{1}{w_k(1)} \left[w_k(L) \left(\frac{\beta_k'(L)}{\beta_k(L)} - \frac{\alpha'(L)}{\alpha(L)} \right) \right]_{L=1} = \frac{\beta_k'(1)}{\beta_k(1)} - \frac{\alpha'(1)}{\alpha(1)} \ , \end{aligned} \tag{3}$$

where $z' = \dfrac{\partial z}{\partial L}$. Note, the mean lag associated with variable x_k does not depend on the coefficients of the other variables $x_l, l \neq k$.

In equation (3), $w_k(1)$ represents the long run impact of x_k on y. Consequently, $y_t - c^* - \sum_{k=1}^{n} w_k(1) x_{k,t}$ captures a deviation from the long-run equilibrium between the dependent variable y and regressors $x_1 \ldots x_n$. Following the steps outlined in Section 2.1 of Banerjee et al. (1993), the $ADL(p, q, n)$ model in equation (1) can be transformed into an $EC(p-1, q-1; n)$ model

$$\begin{aligned} \Delta y_t = & \kappa \left[y_{t-1} - c^* - \sum_{k=1}^{n} \omega_k x_{k,t-1} \right] + \sum_{k=1}^{n} b_{k,0} \Delta x_{k,t} \\ & + a(L) \Delta y_t + \sum_{k=1}^{n} b_k(L) \Delta x_{k,t} + \epsilon_t \ , \end{aligned} \tag{4}$$

where $\kappa = -\alpha(1)$, $\omega_k = w_k(1)$, $b_{k,0} = \beta_{k,0}$, $a(L) = \sum_{j=1}^{p-1} a_j L^j$ with $a_j = -\sum_{i=j+1}^{p} \alpha_i$, $b_k(L) = \sum_{j=1}^{q-1} b_{k,j} L^j$ with $b_{k,j} = -\sum_{i=j+1}^{q} \beta_i$, and $p-1$ and $q-1$ stand for the maximum lag lengths of Δy and Δx, respectively. By convention, a term does not enter the summation if the lower limit exceeds the upper limit. The models described by equations (1) and (4) are isomorphic.

Example 1. Transformation of the $ADL(3,3;1)$ model

$$y_t = c + \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + \alpha_3 y_{t-3} + \beta_0 x_t + \beta_1 x_{t-1} + \beta_2 x_{t-2} + \beta_3 x_{t-3} + \epsilon_t \ , \tag{5}$$

yields the following $EC(2,2;1)$ model

$$\begin{aligned} \Delta y_t = & -(1-\alpha_1-\alpha_2-\alpha_3)\left[y_{t-1} - \frac{c}{1-\alpha_1-\alpha_2-\alpha_3} - \frac{\beta_0+\beta_1+\beta_2+\beta_3}{1-\alpha_1-\alpha_2-\alpha_3} x_{t-1}\right] \\ & + \beta_0 \Delta x_t - (\alpha_2+\alpha_3)\Delta y_{t-1} - \alpha_3 \Delta y_{t-2} - (\beta_2+\beta_3)\Delta x_{t-1} - \beta_3 \Delta x_{t-2} + \epsilon_t \ , \end{aligned} \tag{6}$$

which can be estimated in a simplified form

$$\begin{aligned} \Delta y_t = \quad & \kappa \left[y_{t-1} - c^* - \omega x_{t-1}\right] + b_0 \Delta x_t \\ & + a_1 \Delta y_{t-1} + a_2 \Delta y_{t-2} + b_1 \Delta x_{t-1} + b_2 \Delta x_{t-2} + \epsilon_t \ . \end{aligned} \tag{7}$$

The expression in brackets represents the equilibrium error. The coefficients estimated in (7) can be mapped back to the ones in (5) and (6) with $\beta_0 = b_0$, $\beta_1 = b_1 - b_0 - \kappa\omega$, $\beta_2 = b_2 - b_1$, $\beta_3 = -b_2$, $\alpha_1 = 1 + \kappa + a_1$, $\alpha_2 = a_2 - a_1$, $\alpha_3 = -a_2$. Hence, for the $EC(2,2;1)$ model, the mean lag defined in equation (3) takes the following form

$$\begin{aligned} \mu &= \frac{\beta_1 + 2\beta_2 + 3\beta_3}{\beta_0+\beta_1+\beta_2+\beta_3} + \frac{\alpha_1 + 2\alpha_2 + 3\alpha_3}{1-\alpha_1-\alpha_2-\alpha_3} \\ &= \frac{\kappa\omega + b_0 + b_1 + b_2}{\kappa\omega} - \frac{1+\kappa-a_1-a_2}{\kappa} \\ &= \frac{\omega(a_1+a_2-1) + b_0 + b_1 + b_2}{\kappa\omega} \ . \end{aligned} \tag{8}$$

If $\kappa \neq 0, \omega_k \neq 0$, then consistent estimation of the parameters $\theta = (\kappa, \omega, a_1, a_2, b_0, b_1, b_2)'$ in equation (7) allows us to obtain a consistent estimate of the mean lag, $\hat{\mu} = \mu(\hat{\theta})$, and its variance $Var(\hat{\mu}) = \frac{\partial \mu(\hat{\theta})}{\partial \theta'} Var(\hat{\theta}) \frac{\partial \mu(\hat{\theta})}{\partial \theta}$, where $Var(\hat{\theta})$ is the covariance matrix of coefficients estimated in equation (7).

Before generalizing this result to an $EC(p-1, q-1; n)$ model

$$\Delta y_t = \kappa \left[y_{t-1} - c^* - \sum_{k=1}^{n} \omega_k x_{k,t-1} \right] + \sum_{i=1}^{p-1} a_i \Delta y_{t-i} + \sum_{k=1}^{n} \sum_{j=0}^{q-1} b_{k,j} \Delta x_{k,t-j} + \epsilon_t \ , \qquad (9)$$

we make the following set of assumptions:

Assumption 1. *The variables $y, x_1 \ldots, x_n$ entering models (1) and (9) are either jointly stationary, or cointegrated with a stationary equilibrium error $y_t - c^* - \sum_{k=1}^{n} \omega_k x_{k,t}$.*

Assumption 2. *The error in equations (1) and (9), ϵ_t, is independently and identically distributed and is independent of the variables $x_1 \ldots, x_n$.*

Assumption 3. *The parameters in equation (9), $\theta = (\kappa, \omega_1 \ldots \omega_n, a_1 \ldots a_{p-1}, b_{1,0} \ldots \ldots b_{1,q-1} \ldots b_{n,0} \ldots b_{n,q-1})'$, are estimated consistently with an estimator that has an asymptotically normal distribution $\sqrt{T}(\hat{\theta} - \theta) \xrightarrow{d} N\left(0, Var(\theta)\right)$.*

Assumption 4. *$\kappa \neq 0$, $\omega_k \neq 0$ for $k \in \{1 \ldots n\}$. The mean lag, $\mu_k(\theta)$, is a continuous function of θ and is continuously differentiable with respect to θ.*

Proposition 1. *Under Assumptions 1-4 the mean lag estimator*

$$\hat{\mu}_k = \mu_k(\hat{\theta}) = \frac{\hat{\omega}_k (\sum_{i=1}^{p-1} \hat{a}_i - 1) + \sum_{j=0}^{q-1} \hat{b}_{k,j}}{\hat{\kappa} \hat{\omega}_k} \ , \qquad (10)$$

is consistent and asymptotically normally distributed

$$\sqrt{T}(\mu_k(\hat{\theta}) - \mu_k(\theta)) \xrightarrow{d} N \left(0, \frac{\partial \mu_k(\theta)}{\partial \theta'} Var(\theta) \frac{\partial \mu_k(\theta)}{\partial \theta} \right) \ . \qquad (11)$$

Proposition 1 extends the results obtained in Example 1 to a general $EC(p-1, q-1; n)$ model. The details of the proof are provided in an Online Supplement on the first author's homepage.

Remark 1.1. *If the variables $y, x_1 \ldots, x_n$ are cointegrated, then the elements of the cointegrating vector, $\omega = (\omega_1 \ldots \omega_n)'$, are estimated super-consistently: $\sqrt{T}(\hat{\omega} - \omega) = o_p(1)$. As a result, the $Var(\theta)$ components associated with the cointegrating vector, ω, converge to zero and do not contribute to the asymptotic variance of the mean lag estimator in (11). The Supplement contains a more detailed exposition of this issue.*

Perhaps due to previous unavailability of the formula presented in equation (10), some researchers have ignored the lag structure of their EC models when estimating the mean lag (see for example De Graeve et al., 2007; Leibrecht and Scharler, 2011).

Corollary 1.1. *For the frequently used $ADL(1, 1; n)$ and the associated $EC(0, 0; n)$ model, expression (10) simplifies to*

$$\mu_k(\hat{\theta}) = \frac{\hat{b}_{k,0} - \hat{\omega}_k}{\hat{\kappa}\hat{\omega}_k} \ . \tag{12}$$

3 Mean lag under long-run homogeneity

An error correction model is considered to be homogeneous if, for each k, the x_k and y variables move one-for-one in equilibrium. Homogeneity implies $\omega_k = w_k(1) = 1$ or $\alpha(1) = \beta_k(1)$, leading to a special form of the mean lag

$$\bar{\mu}_k = \frac{\beta_k'(1) - \alpha'(1)}{\alpha(1)} = \frac{\beta_k'(1) - \alpha'(1)}{\beta_k(1)} \ . \tag{13}$$

Table 1: Impact of Imposing Long-Run Homogeneity in the Literature

Study	Item	$\bar{\mu}(\hat{\theta})$	$\mu(\hat{\theta})$	Relative Bias
A	Overnight	1.58	1.24	27%
	Maturity over 2 years	0.97	0.50	94%
B	IB	0.26	0.14	78%
	TD6-12	7.09	2.63	169%
C	Footnote 8	2.90	2.20	39%

Note: Illustration of bias in the mean lag estimate due to an inappropriate assumption of long-run homogeneity. Panels (A)-(C) refer to the following published results: (A) Table 8 in De Bondt (2005), (B) Table 6 in Charoenseang and Manakit (2007), (C) page 501 in Leibrecht and Scharler (2008). Each of these studies incorrectly used $\bar{\mu}(\hat{\theta})$ defined in equation (15) in place of $\mu(\hat{\theta})$ defined in equation (12) to estimate the mean lag.

Corollary 1.2. *The mean lag estimator associated with variable x_k in a homogeneous $EC(p-1, q-1; n)$ model takes the form*

$$\bar{\mu}_k(\hat{\theta}) = \frac{\sum_{i=1}^{p-1} \hat{a}_i - 1 + \sum_{j=0}^{q-1} \hat{b}_{k,j}}{\hat{\kappa}} , \tag{14}$$

which simplifies to

$$\bar{\mu}_k(\hat{\theta}) = \frac{\hat{b}_{k,0} - 1}{\hat{\kappa}} . \tag{15}$$

for a homogeneous $EC(0, 0; n)$ model.

Expression (15) is equivalent to the formula derived by Hendry (1995, p. 215, eq. 6.53). Although this formula does not hold for general, non-homogeneous, relationships between x_k and y, it has been inappropriately used in place of equation (12) by several researchers. The list of studies that relied on equation (15) despite non-unity ω_k coefficients includes Charoenseang and Manakit (2007); Chong and Liu (2009); De Bondt (2005); De Graeve et al. (2007); Leibrecht and Scharler (2008, 2011); Leszkiewicz-Kedzior and Welfe (2014); Scholnick (1996), among others. In Table 1, we illustrate

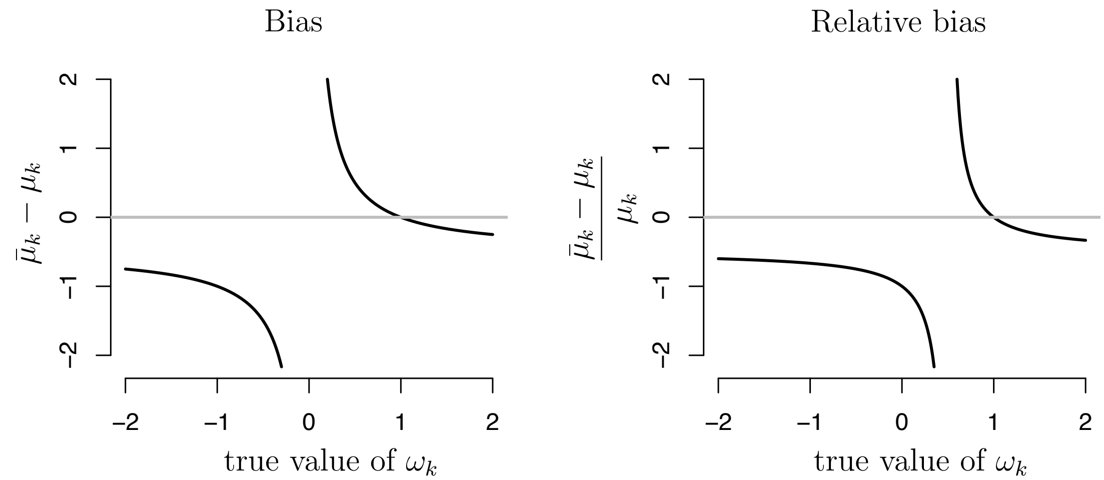

Figure 1: Bias and relative bias of the mean lag estimate arising from an inappropriate assumption of homogeneity in an $EC(p-1, q-1; n)$ model with $\sum_{i=1}^{p-1} a_i = 0.6$, $\sum_{j=0}^{q-1} b_{k,j} = 0.2$ and $\kappa = -0.4$.

the impact of imposing long-run homogeneity in some of these studies.

Expressions (10) and (14) allow us to quantify the bias in the mean lag estimate arising from an inappropriate assumption of homogeneity. The bias

$$\bar{\mu}_k(\theta) - \mu_k(\theta) = \frac{(\omega_k - 1)\sum_{j=0}^{q-1} b_{k,j}}{\kappa \omega_k} \tag{16}$$

and the relative bias

$$\frac{\bar{\mu}_k(\theta) - \mu_k(\theta)}{\mu_k(\theta)} = \frac{(\omega_k - 1)\sum_{j=0}^{q-1} b_{k,j}}{\omega_k(\sum_{i=1}^{p-1} a_i - 1) + \sum_{j=0}^{q-1} b_{k,j}} \tag{17}$$

vanish as $\omega_k \to 1$, but diverge otherwise. Figure 1 illustrates the magnitude of the bias and relative bias for $\omega_k \in (-2, 2)$, $\sum_{i=1}^{p-1} a_i = 0.6$, $\sum_{j=0}^{q-1} b_{k,j} = 0.2$, and $\kappa = -0.4$.

4 Concluding remarks

Rsearchers have been inappropriately using Hendry's (1995) mean lag formula—which he derived for homogeneous $EC(0,0,1)$ models—in more complex settings. We fill a gap in the literature by deriving an expression for the mean lag in general EC models, and show that using the incorrect formula can have a sizable impact on the estimated delay in the transmission of shocks. The ADL and EC models discussed above can be viewed as components of vector autoregressive (VAR) and vector error correction (VEC) models, respectively. Consequently, the presented results are also valid for the mean lags of variables in individual equations of VAR and VEC models.

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Online Supplement

A.1 Proof of Proposition 1

We first outline the steps required to obtain the mean lag estimator defined in equation (10) and then derive its asymptotic distribution.

To transform the $ADL(p, q; n)$ model in equation (1) into the $EC(p-1, q-1; n)$ model in equation (9), carry out the following operations:

- subtract y_{t-1} from both sides of equation (1),
- for each $k \in \{1 \dots n\}$ add and subtract $\beta_{k,0}x_{k,t-1}$ on the right hand side of equation (1),
- add and subtract $\sum_{i=1}^{p-1}(\sum_{j=i+1}^{p} \alpha_j)y_{t-i}$ on the right hand side of equation (1),
- for each $k \in \{1 \dots n\}$ add and subtract $\sum_{i=1}^{q-1}(\sum_{j=i+1}^{q} \beta_{k,j})x_{k,t-i}$ on the right hand side of equation (1).

In the resulting equation

$$
\begin{array}{cccccccc}
y_t - y_{t-1} = & c & -y_{t-1} & +\alpha_1 y_{t-1} & +\alpha_2 y_{t-1} & +\alpha_3 y_{t-1} & \dots & +\alpha_p y_{t-1} \\
 & & & & -\alpha_2 y_{t-1} & -\alpha_3 y_{t-1} & \dots & -\alpha_p y_{t-1} \\
 & & & & +\alpha_2 y_{t-2} & +\alpha_3 y_{t-2} & \dots & +\alpha_p y_{t-2} \\
 & & & & & -\alpha_3 y_{t-2} & \dots & -\alpha_p y_{t-2} \\
 & & & & & & & \dots \\
 & & & & & +\alpha_{p-1} y_{t-(p-1)} & & +\alpha_p y_{t-(p-1)} \\
 & & & & & & & -\alpha_p y_{t-(p-1)} \\
 & & & & & & & +\alpha_p y_{t-p} \\
 & +\beta_{1,0} x_{1,t} & +\beta_{1,1} x_{1,t-1} & +\beta_{1,0} x_{1,t-1} & +\beta_{1,2} x_{1,t-1} & +\beta_{1,3} x_{1,t-1} & \dots & +\beta_{1,p} x_{1,t-1} \\
 & & & & -\beta_{1,2} x_{1,t-1} & -\beta_{1,3} x_{1,t-1} & \dots & -\beta_{1,p} x_{1,t-1} \\
 & & & & +\beta_{1,2} x_{1,t-2} & +\beta_{1,3} x_{1,t-2} & \dots & +\beta_{1,p} x_{1,t-2} \\
 & & & & & -\beta_{1,3} x_{1,t-2} & \dots & -\beta_{1,p} x_{1,t-2} \\
 & & & & & & & \dots \\
 & & & & & +\beta_{1,p-1} x_{1,t-(p-1)} & & +\beta_{1,p} x_{1,t-(p-1)} \\
 & & & & & & & -\beta_{1,p} x_{1,t-(p-1)} \\
 & & & & & & & +\beta_{1,p} x_{1,t-p} \\
 & & & & & & & \dots \\
 & & & & & +\beta_{n,p-1} x_{n,t-(p-1)} & & +\beta_{n,p} x_{n,t-(p-1)} \\
 & & & & & & & -\beta_{n,p} x_{n,t-(p-1)} \\
 & & & & & & & +\beta_{n,p} x_{n,t-p} \\
 & & & & & & & +\epsilon_t
\end{array}
$$

substitute Δy_t and Δx_t for $y_t - y_{t-1}$ and $x_t - x_{t-1}$, respectively, and group items to obtain equation (9) with

- $a_j = -\sum_{i=j+1}^{p} \alpha_i$ for $j \in \{1 \dots p-1\}$,
- $b_0 = \beta_0$,
- $b_{k,j} = -\sum_{i=j+1}^{q} \beta_{k,i}$ for $k \in \{1 \dots n\}$ and $j \in \{1 \dots q-1\}$,
- $\kappa = -(1 - \sum_{i=1}^{p} \alpha_i)$,
- $\omega_k = \dfrac{\sum_{i=0}^{q} \beta_{k,i}}{1 - \sum_{i=1}^{p} \alpha_i} = -\dfrac{\sum_{i=0}^{q} \beta_{k,i}}{\kappa}$ for $k \in \{1 \dots n\}$,
- $c^* = \dfrac{c}{1 - \sum_{i=1}^{p} \alpha_i} = -\dfrac{c}{\kappa}$.

The parameters of the $EC(p-1, q-1; n)$ model in equation (9) can be mapped back to the parameters of the $ADL(p, q; n)$ model in equation (1) using

- $\alpha_1 = 1 + \kappa + a_1$,
- $\alpha_i = a_i - a_{i-1}$ for $i \in \{2 \dots q-1\}$,
- $\alpha_p = -a_{p-1}$,
- $\beta_{k,0} = b_0$ for $k \in \{1 \dots n\}$,
- $\beta_{k,1} = -\omega\kappa + b_1 - b_0$ for $k \in \{1 \dots n\}$,
- $\beta_{k,i} = b_{k,i} - b_{k,i-1}$ for $k \in \{1 \dots n\}$ and $i \in \{2 \dots q-1\}$,
- $\beta_{k,q} = -b_{k,q-1}$ for $k \in \{1 \dots n\}$.

Substituting these expressions into the definition of the mean lag in equation (3) yields the following formula for the mean lag in $EC(p-1, q-1; n)$ models

$$\mu_k = \frac{\sum_{i=0}^{q} i\beta_{k,i}}{\sum_{i=0}^{q} \beta_{k,i}} + \frac{\sum_{i=1}^{p} i\alpha_i}{\sum_{i=1}^{p} \alpha_i} = \frac{\omega_k(\sum_{i=1}^{p-1} a_i - 1) + \sum_{j=0}^{q-1} b_{k,j}}{\kappa\omega_k} \ .$$

Under Assumptions 3 and 4, the continuous mapping theorem implies that the mean lag estimator in equation (10) is consistent: $\mu_k(\hat{\theta}) \xrightarrow{p} \mu_k(\theta)$. To show asymptotic normality, we apply the mean value theorem to a Taylor expansion of μ_k

$$\mu_k(\hat{\theta}) = \mu_k(\theta) + \frac{\partial \mu_k(\bar{\theta})}{\partial \theta'}(\hat{\theta} - \theta) \ ,$$

where $\bar{\theta} = \lambda\hat{\theta} + (1-\lambda)\theta$ with $0 \leq \lambda \leq 1$. Multiplying both sides by $\sqrt{T}$ and rearranging yields

$$\sqrt{T}(\mu_k(\hat{\theta}) - \mu_k(\theta)) = \frac{\partial \mu_k(\bar{\theta})}{\partial \theta'}\sqrt{T}(\hat{\theta} - \theta) \ .$$

Since $\bar{\theta}$ lies on the line segment between $\hat{\theta}$ and θ, and since $\hat{\theta} \xrightarrow{p} \theta$, we have $\bar{\theta} \xrightarrow{p} \theta$. Consequently, under Assumption 4, the continuous mapping theorem implies $\frac{\partial \mu_k(\bar{\theta})}{\partial \theta'} \xrightarrow{p} \frac{\partial \mu_k(\theta)}{\partial \theta'}$. Finally, in conjunction with Assumption 3, the continuous mapping theorem implies

$$\sqrt{T}(\mu_k(\hat{\theta}) - \mu_k(\theta)) \xrightarrow{d} \frac{\partial \mu_k(\theta)}{\partial \theta'}\sqrt{T}(\hat{\theta} - \theta) \sim N\left(0, \frac{\partial \mu_k(\theta)}{\partial \theta'} Var(\theta) \frac{\partial \mu_k(\theta)}{\partial \theta}\right) \ . \quad \square$$

A.2 Details underlying Remark 1.1

Engle and Granger (1987) showed that if the variables $y, x_1 \ldots, x_n$ are $I(1)$ with an $I(0)$ equilibrium error $y_t - c^* - \sum_{k=1}^{n} \omega_k x_{k,t}$, then the elements of the cointegrating vector, $\omega = (\omega_1 \ldots \omega_n)'$, are estimated super-consistently: $\sqrt{T}(\hat{\omega} - \omega) = o_p(1)$. As a result, the mean value expansion of μ_k around θ consists of two components converging at different rates

$$\sqrt{T}(\mu_k(\hat{\theta}) - \mu_k(\theta)) = \left.\frac{\partial \mu_k}{\partial \theta'_{\setminus\omega}}\right|_{\bar{\theta}} \sqrt{T}(\hat{\theta}_{\setminus\omega} - \theta_{\setminus\omega}) + \left.\frac{\partial \mu_k}{\partial \omega'}\right|_{\bar{\theta}} \sqrt{T}(\hat{\omega} - \omega) = O_p(1) + o_p(1) \ ,$$

where $\theta_{\setminus\omega}$ refers to the complement of ω in θ, and $\bar{\theta} = \lambda\hat{\theta} + (1-\lambda)\theta$ with $0 \leq \lambda \leq 1$. Therefore, under Assumption 4, the continuous mapping theorem implies that the asymptotic distribution of the mean lag estimator is not affected by those components of $Var(\theta)$ that are associated with the cointegrating vector, ω:

$$\sqrt{T}(\mu_k(\hat{\theta}) - \mu_k(\theta)) \xrightarrow{d} \frac{\partial \mu_k(\theta)}{\partial \theta'_{\setminus\omega}} \sqrt{T}(\hat{\theta}_{\setminus\omega} - \theta_{\setminus\omega}) \sim N\left(0, \frac{\partial \mu_k(\theta)}{\partial \theta'_{\setminus\omega}} Var(\theta_{\setminus\omega}) \frac{\partial \mu_k(\theta)}{\partial \theta_{\setminus\omega}}\right) .$$