Math 432 - Real Analysis II
Test 1 Review Sheet

Logistics: This exam will be a 110-minute, closed book, closed notes exam. Successful test-takers will, at the least, know the below concepts well.

Major Concepts:

Power Series

- Understand the power series can be viewed as functions of the variable $x$ as long as the power series converges for that particular $x$.
- Compute the Domain of Convergence, Interval of Convergence, and Radius of Convergence for a given power series using the Root Test or the Ratio Test
- Give examples of power series with specific intervals of convergence

Pointwise vs. Uniform convergence

- Compute the pointwise limit of a function by computing $\lim_{n \to \infty} f_n(x)$ for each $x$.
- Understand that analytic properties of $f_n$ do not necessarily transfer over to the pointwise limit.
- Show that a series of functions uniformly converges to a limit function on some $S$. Understand the role of the set $S$ in the proofs of uniform convergence.
- Understand why if $f_n \to f$ uniformly on $S$, then $f_n \to f$ pointwise, but not necessarily vice versa.
- Use the fact that $f_n \to f$ uniformly on $S$ if and only if
  \[ \lim_{n \to \infty} \sup \{|f_n(x) - f(x)| \mid x \in S\} = 0. \]
- Know how to prove that $f_n \not\to f$ uniformly using a proof by contradiction.
- Understand the definition of $f_n$ being uniformly Cauchy on $S$ and that this condition is equivalent to $f_n$ converging to some uniform limit on $S$.

Analytic properties of uniform limits

- Understand the relationship between definite integration and uniform convergence
- Know the uniform Cauchy Criterion for series of functions and use it to prove that a series does or does not uniformly converge
- Use the Weierstrass $M$-test to show that a given series of functions uniformly converges.
- Show that a series of functions $\sum g_n(x)$ does not converge uniformly by showing that
  \[ \lim_{n \to \infty} \sup \{|g_n(x)| \mid x \in S\} \neq 0. \]
Differentiation and Integration of Power Series

- Use the fact that given a power series with radius of convergence $R$, it pointwise converges to a continuous function on $(-R, R)$.
- Use the fact that if a power series with radius of convergence $R$, it will uniformly converge on $[-R', R']$ for some $0 < R' < R$.
- Use the fact that you can differentiate a function term-by-term and that the radius of convergence of the integral will be identical (but not necessarily the interval of convergence).
- Differentiate a power series term-by-term and know that this derivative will have identical radius of convergence (but not necessarily the interval of convergence).

Taylor Series and Taylor’s Theorem

- Given a smooth function, compute its Taylor Series
- Know all of the well-known Taylor Series (like $e^x$, $\sin x$, $\cos x$, and $1/(1-x)$)
- Know the definition the remainder $R_n(x)$ for a Taylor Series
- Use Taylor’s Theorem to gain control of $R_n(x)$.
- Know what it means for a smooth function to be analytic; in particular, know how to use Taylor’s Theorem to prove analyticity.
- Use Taylor polynomials of analytic functions to approximate various constants in mathematics.
- Provide bounds on the error of various approximations, especially those coming from alternating series.
- Give examples of non-analytic functions

L’Hôpital’s Rule

- Use L’Hôpital’s Rule in relation to its various indeterminate forms to compute limits