Math 432 - Real Analysis II
Solutions to Homework due March 22

Question 1. For the following integrable function $f$ defined on $[a, b]$, compute

$$F(x) = \int_a^x f(t) \, dt.$$ 

Be sure to discuss the continuity and differentiability of the functions $F(x)$ and its relation to the continuity of $f(x)$.

(a) $f(x) = |x|$ on $[-1, 1]$

(b) $f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$ on $[-1, 1]$

Solution 1.

(a) For $f(x) = |x|$, we get that

$$F(x) = \int_{-1}^x |t| \, dt = \begin{cases} 1/2 - x^2/2 & \text{if } -1 \leq x \leq 0 \\ 1/2 + x^2/2 & \text{if } 0 < x \leq 1 \end{cases}$$

Clearly $F$ is differentiable at all $x$ in its domain, except for possible at $x = 0$. However, a closer evaluation shows that

$$\lim_{x \to 0^-} F'(x) = \lim_{x \to 0^+} F'(x) = 1/2.$$ 

Thus, $F$ is differentiable, as predicted by the FTC.

(b) For $f(x)$, we get that

$$F(x) = \int_{-1}^x f(t) \, dt = \begin{cases} -x - 1 & \text{if } -1 \leq x \leq 0 \\ -1 + x & \text{if } 0 \leq x \leq 1 \end{cases}$$

This function is continuous on its entire domain; however, it is not differentiable at 0. For discontinuous $f(x)$ functions, the continuity of $F(x)$ is guaranteed, but the differentiability is not.

In class, we showed that if $g$ is an injective function on $[a, b]$, then we have the following relationship between the integral of $g$ and that of its inverse $g^{-1}$:

$$\int_{g(a)}^{g(b)} g^{-1}(u) \, du = b \cdot g(b) - a \cdot g(a) - \int_a^b g(x) \, dx.$$ 

Question 2. Let $g : [0, 1] \to [0, 1]$ be an increasing, bijective, continuous function. Since $g$ is bijective, it is invertible. Assume $g^{-1}$ is also continuous. [FYI - such a $g$ is called a homeomorphism of $[0, 1]$.

(a) Show that

$$\int_0^1 g(x) \, dx + \int_0^1 g^{-1}(x) \, dx = 1.$$ 

(b) Explain geometrically (using areas, etc) why the above equation makes sense.

Solution 2.
(a) Notice that since $g$ is increasing and onto, then $g(0) = 0$ and $g(1) = 1$. Thus, we have that
\[ \int_0^1 g^{-1}(u) \, du + \int_0^1 g(x) \, dx = 1 \cdot g(1) - 0 \cdot g(0) = 1. \]

(b) This makes geometric sense because the integral $\int_0^1 g(x) \, dx$ represents the area between the graph of $g$ and the $x$-axis between 0 and 1. Likewise, $\int_0^1 g^{-1}(u) \, du$ represents the area between the graph of $g$ and the $y$-axis between $y = 0$ and $y = 1$. These two complementary pieces make up the entire square given by $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Thus, the sum of their areas should be equal to 0.

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**Question 3.** In this problem, we will compute an antiderivative of $\arcsin x$.

(a) By differentiating, check that
\[ x \cdot \arcsin x + \sqrt{1 - x^2} \]

is an antiderivative for $\arcsin x$.

(b) Use the equation relating the integral of $g$ and $g^{-1}$ given above Question 2 to find
\[ \int_0^x \arcsin t \, dt. \]

Be sure to simplify as much as possible. Do you get the same antiderivative as the one presented in (a)?

**Solution 3.**

(a) Differentiating, we get that
\[ \frac{d}{dx} \arcsin x + \sqrt{1 - x^2} = \arcsin x + x \cdot \frac{1}{\sqrt{1 - x^2}} - \frac{2x}{2\sqrt{1 - x^2}} = \arcsin x. \]

(b) Let $g(x) = \sin x$, $a = 0$, $b = \arcsin x$. Thus, $g(0) = 0$ and $g(b) = \sin(\arcsin x) = x$. Using the equation above, we have that
\[ \int_0^x \arcsin t \, dt = x \arcsin x - 0 - \int_0^{\arcsin x} \sin t \, dt = x \arcsin x + \cos t \big|_0^{\arcsin x} = x \arcsin x + \cos(\arcsin x) - 1. \]

Since $\cos x = \sqrt{1 - \sin^2 x}$, we can simplify the penultimate term as $\sqrt{1 - x^2}$. Thus, we are left with
\[ x \arcsin x + \sqrt{1 - x^2} - 1. \]

We do not get the same answer as the antiderivative presented in (a); the two answers differ by a constant and thus are both valid antiderivatives.

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**Question 4.** Using the relation before Question 2, find
\[ \int_0^x \arctan t \, dt. \]

Be sure to simplify your answer as much as possible; compare your answer to those found online or in the back of your calculus textbook.

**Solution 4.**
(a) Proceeding as in 3(b) with \( g(x) = \tan x \), \( a = 0 \), \( b = \arctan x \), \( g(0) = 0 \), and \( g(b) = \tan(\arctan x) = x \), we get

\[
\int_0^x \arctan x \, dx = x \arctan x - 0 \cdot 0 - \int_0^{\arctan x} \tan t \, dt = x \arctan x + \ln|\sec(\arctan x)|.
\]

Since \( \sec x = \sqrt{1 + \tan^2 x} \), our answer simplifies to

\[
x \arctan x + \ln\sqrt{1 + x^2} = x \arctan x + \frac{1}{2}\ln(1 + x^2).
\]

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**Question 5.** In class, we learned of the function space \( C_c(\mathbb{R}) \), which is equal to the set of all continuous functions \( f : \mathbb{R} \to \mathbb{R} \) that have compact support. Recall that the support of a function \( f \) (denoted \( \text{supp}(f) \)) is defined as the closure of the set

\[
\{x \in \mathbb{R} \mid f(x) \neq 0\}.
\]

In this question, we will investigate some of the nuances of this space.

(a) First, we seek to understand why the definition of the support of a function requires us to take a closure. Show that if \( f \) is continuous, then \( \{x \in \mathbb{R} \mid f(x) \neq 0\} \) is an open set. Hint: Think about the definition of a continuous function in terms of open sets.

(b) Show that if \( f \) is a continuous function on \( \mathbb{R} \) such that \( \{x \in \mathbb{R} \mid f(x) \neq 0\} \) is a bounded set, then \( f \in C_c(\mathbb{R}) \).

(c) Give an example of a function \( f \) where \( \text{supp}(f) = [a, b] \), where \( a < b \).

**Solution 5.**

(a) Note that in \( \mathbb{R} \), the set of all non-zero real numbers is open (since \( \{x\} \) is closed in \( \mathbb{R} \)). Since \( f \) is continuous, the pre-image of any open set is open. Since \( \{x \mid f(x) \neq 0\} \) is exactly the pre-image of the set of non-zero real numbers, it must be open.

(b) If \( \{x \mid f(x) \neq 0\} \) is a bounded set, then taking its closure produces another bounded set. This set is closed and bounded. Thus, by the Heine-Borel Theorem, is a compact set. Thus, \( \text{supp}(f) \) is compact and \( f \in C_c(\mathbb{R}) \).

(c) Consider the function

\[
f(x) = \begin{cases} 
0 & \text{if } x \leq a \\
(x-a)(x-b) & \text{if } a \leq x \leq b \\
0 & \text{if } x > b
\end{cases}
\]

Note that \( \{x \mid f(x) \neq 0\} = (a, b) \). The closure of this set is exactly \([a, b]\). Thus, \( \text{supp}(f) = [a, b] \).