Math 432 - Real Analysis II
Homework due March 22

Question 1. For the following integrable function \( f \) defined on \([a, b]\), compute
\[
F(x) = \int_a^x f(t) \, dt.
\]
Be sure to discuss the continuity and differentiability of the functions \( F(x) \) and its relation to the continuity of \( f(x) \).

(a) \( f(x) = |x| \) on \([-1,1]\)

(b) \( f(x) = \begin{cases} 
-1 & \text{if } x < 0 \\
1 & \text{if } x \geq 0
\end{cases} \) on \([-1,1]\)

In class, we showed that if \( g \) is an injective function on \([a, b]\), then we have the following relationship between the integral of \( g \) and that of its inverse \( g^{-1} \):
\[
\int_{g(a)}^{g(b)} g^{-1}(u) \, du = b \cdot g(b) - a \cdot g(a) - \int_a^b g(x) \, dx.
\]

Question 2. Let \( g : [0, 1] \to [0, 1] \) be an increasing, bijective, continuous function. Since \( g \) is bijective, it is invertible. Assume \( g^{-1} \) is also continuous. [FYI - such a \( g \) is called a homeomorphism of \([0, 1]\).]

(a) Show that
\[
\int_0^1 g(x) \, dx + \int_0^1 g^{-1}(x) \, dx = 1.
\]

(b) Explain geometrically (using areas, etc) why the above equation makes sense.

Question 3. In this problem, we will compute an antiderivative of \( \arcsin x \).

(a) By differentiating, check that
\[
x \cdot \arcsin x + \sqrt{1 - x^2}
\]
is an antiderivative for \( \arcsin x \).

(b) Use the equation relating the integral of \( g \) and \( g^{-1} \) given above Question 2 to find
\[
\int_0^x \arcsin t \, dt.
\]
Be sure to simplify as much as possible. Do you get the same antiderivative as the one presented in (a)?

Question 4. Using the relation before Question 2, find
\[
\int_0^x \arctan t \, dt.
\]
Be sure to simplify your answer as much as possible; compare your answer to those found online or in the back of your calculus textbook.

Question 5. In class, we learned of the function space \( C_c(\mathbb{R}) \), which is equal to the set of all continuous functions \( f : \mathbb{R} \to \mathbb{R} \) that have compact support. Recall that the support of a function \( f \) (denoted \( \text{supp}(f) \)) is defined as the closure of the set
\[
\{ x \in \mathbb{R} \mid f(x) \neq 0 \}.
\]
In this question, we will investigate some of the nuances of this space.
(a) First, we seek to understand why the definition of the support of a function requires us to take a closure. Show that if $f$ is continuous, then $\{ x \in \mathbb{R} \mid f(x) \neq 0 \}$ is an open set. Hint: Think about the definition of a continuous function in terms of open sets.

(b) Show that if $f$ is a continuous function on $\mathbb{R}$ such that $\{ x \in \mathbb{R} \mid f(x) \neq 0 \}$ is a bounded set, then $f \in C_c(\mathbb{R})$.

(c) Give an example of a function $f$ where $\text{supp}(f) = [a, b]$, where $a < b$. 