Question 1. Let \( f(x) = k \) be a constant function for \( k \in \mathbb{R} \).

1. Show that \( f \) is integrable over any \([a, b]\) by using Cauchy’s \( \varepsilon - P \) condition for integrability.

2. Show that \( \int_a^b k \, dx = k(b - a) \).

Question 2. In class, we proved that if \( f \) is integrable on \([a, b]\), then \(|f|\) is also integrable. Show that the converse is not true by finding a function \( f \) that is not integrable on \([a, b]\) but that \(|f|\) is integrable on \([a, b]\).

Question 3. (a) Let \( x, y \in S \). Show that \(|f(x)| - |f(y)| \leq |f(x) - f(y)|\).

(b) Let \( x, y \in S \). Show that \(|f(x) - f(y)| \leq M(f, S) - m(f, S)\).

(c) Use (a) and (b) to show that \( M(|f|, S) - m(|f|, S) \leq M(f, S) - m(f, S)\). Hint: To do this, show that for any \( \varepsilon > 0 \),

\[
M(|f|, S) - m(|f|, S) \leq M(f, S) - m(f, S) + \varepsilon.
\]

Question 4. Let \( f \) and \( g \) be integrable functions on \([a, b]\).

(a) Show that \( 4fg = (f + g)^2 - (f - g)^2 \).

(b) Use (a) to show that \( fg \) is also integrable on \([a, b]\).

Question 5. Consider the function \( f \) on \([0, 1]\) given by

\[
f(x) = \begin{cases} 
  x & \text{if } x \in \mathbb{Q} \\
  0 & \text{if } x \notin \mathbb{Q}
\end{cases}
\]

(a) Let \( P = \{0 = t_0 < \cdots < t_n = 1\} \) be any partition of \([0, 1]\). Show that

\[U(f, P) = U(x, P).
\]

(b) Compute the Upper and Lower Darboux sums, \( U(f) \) and \( L(f) \), and use this to decide if \( f \) is integrable.

Question 6. Let \( f \) be a bounded function on \([a, b]\). Suppose that there exists a sequence of partitions \( P_n \) on \([a, b]\) such that

\[
\lim_{n \to \infty} U(f, P_n) = \lim_{n \to \infty} L(f, P_n).
\]

Show that \( f \) is integrable and that

\[
\int_a^b f \, dx = \lim_{n \to \infty} U(f, P_n) = \lim_{n \to \infty} L(f, P_n).
\]
Question 7. Let $f$ and $g$ be bounded functions on $[a, b]$. In what follows, we will show that $\max\{f, g\}$ and $\min\{f, g\}$ are integrable if we know that $f$ and $g$ are individually integrable. Define these functions as

$$\min\{f, g\}(x) = \min\{f(x), g(x)\},$$

and similarly for $\max\{f, g\}$.

(a) Let $a, b \in \mathbb{R}$. Show that

$$\min\{a, b\} = \frac{1}{2}(a + b) - \frac{1}{2}|a - b|.$$

(b) Use (a) to show that if $f$ and $g$ are integrable, then $\min\{f, g\}$ is also integrable.

(c) Find an expression similar to one in (a) for $\max\{a, b\}$. Prove that your expression is correct.

(d) Use (c) to show that if $f$ and $g$ are integrable, then $\max\{f, g\}$ is integrable.