Question 1. For each of the following power series, find the radius of convergence and determine the exact interval of convergence (including endpoints).

(a) \[ \sum_{k=1}^{\infty} k^2 x^k \]
(b) \[ \sum_{k=1}^{\infty} \frac{2^k}{k^2} x^k \]
(c) \[ \sum_{k=1}^{\infty} \frac{2^k}{k!} x^k \]
(d) \[ \sum_{k=1}^{\infty} \frac{3^k}{k^4} x^k \]

In the next set of questions, we will focus on (pointwise) limits of sequences of continuous functions, which are not always continuous themselves.

Question 2. For each \( n \geq 1 \), consider \( f_n(x) = (\cos x)^n \).

Notice that each \( f_n \) is a continuous function. Compute \( \lim_{n \to \infty} f_n(x) \) for each \( x \in \mathbb{R} \). When doing so, you may want to consider separate cases for \( x \). In some cases, the limit may not even exist.

Question 3. For each \( n \geq 1 \), consider \( f_n(x) = \frac{1}{n} \sin(nx) \).

Notice that each \( f_n \) is differentiable.

(a) For each \( x \in \mathbb{R} \), show that \( \lim_{n \to \infty} f_n(x) = 0 \). Thus, \( \frac{d}{dx} \lim_{n \to \infty} f_n(x) = 0 \).

(b) Now, show that \( \lim_{n \to \infty} \frac{d}{dx} f_n(x) \) at \( x = \pi \) does not exist. Thus, even if each \( f_n \) is differentiable, we cannot “swap” the limit and the derivative in general.

Question 4. Consider the sequence of continuous functions given by

\[ f_n(x) = \frac{1 + 2 \cos^2(nx)}{\sqrt{n}}. \]

Show that \( f_n(x) \) converges uniformly to the constant 0 function on \( \mathbb{R} \).

Question 5. Consider the sequence of continuous functions

\[ f_n(x) = \frac{x}{n}. \]

1. Compute \( f(x) = \lim_{n \to \infty} f_n(x) \). That is, compute the pointwise limit.

2. Determine whether \( f_n \) converges to \( f \) uniformly on \([0, 1]\). Prove your claim.

3. Determine whether \( f_n \) converges to \( f \) uniformly on \( \mathbb{R} \). Prove your claim.
Question 6. Consider the sequence of continuous functions

\[ f_n(x) = \frac{nx}{1 + n^2x^2}. \]

(a) Compute \( f(x) = \lim_{n \to \infty} f_n(x) \), the pointwise limit.

(b) Consider \( f_n(x) \) on \([-1, 1]\). Decide if \( f_n \) converges uniformly to \( f(x) \) on \([-1, 1]\). It might be helpful to compute

\[ \lim_{n \to \infty} \sup \{|f_n(x) - f(x)| \mid x \in [-1, 1]\}. \]

(c) Consider \( f_n(x) \) on \([1, \infty)\). Decide if \( f_n \) converges uniformly to \( f(x) \) on \([1, \infty)\). It might be helpful to compute

\[ \lim_{n \to \infty} \sup \{|f_n(x) - f(x)| \mid x \in [1, \infty)\}. \]