Question 1. In this question, we will consider the $L^p$ metric on the space $C([0,1])$ of continuous (real-valued) functions on $[0,1]$. Here, the $L^p$ norm (for $1 \leq p \leq \infty$) is given by

$$||f||_p = \left( \int_0^1 |f(x)|^p \, dx \right)^{1/p}$$

and

$$||f||_\infty = \sup \{|f(x)| \mid x \in [0,1]\}.$$

Let $f, g \in C([0,1])$ be given by the following:

$f(x) = 1/2 - x$

$g(x) = |1/2 - x|$

(a) Compute $f + g$ and $f - g$. Write your answers in piecewise form.

(b) Compute $||f||_\infty$, $||g||_\infty$, $||f + g||_\infty$, and $||f - g||_\infty$. Does the parallelogram law hold for these vectors with the $L^\infty$ metric?

(c) Compute $||f||_p$, $||g||_p$, $||f + g||_p$, and $||f - g||_p$.

(d) For what values of $p$ does the parallelogram law hold for the vectors $f$ and $g$?

(e) Using the above, what can you conclude about which $L^p$ norms on $C[0,1]$ are not induced by inner products?

Question 2. (à la Julian) Let $V$ be an inner product space and $\{v_1, v_2, \ldots, v_m\}$ be an orthogonal collection of vectors. Show that this collection is linearly independent. Note: This is similar to a theorem we proved in class, but has weaker hypotheses (and thus more powerful than the class’ theorem). Also, recall that a collection of vectors is orthogonal if $\langle v_i, v_j \rangle = 0$ if and only if $i \neq j$.

Question 3. Consider the vector space

$C(\mathbb{T}) = \{f : [-\pi, \pi] \to \mathbb{C} \mid f \text{ is continuous and } f(-\pi) = f(\pi)\}$,

and the $L^2$ inner product given by

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t) \overline{g(t)} \, dt.$$

Consider the vectors given by

$$\left\{ \frac{e^{inx}}{\sqrt{2\pi}} \right\}_{n=-\infty}^{\infty} = \left\{ \ldots, e^{-2ix}/\sqrt{2\pi}, e^{-ix}/\sqrt{2\pi}, 1/\sqrt{2\pi}, e^{ix}/\sqrt{2\pi}, e^{2ix}/\sqrt{2\pi}, \ldots \right\}.$$

In what follows, we will prove that this collection of vectors in $C(\mathbb{T})$ is an orthonormal set.

(a) Using Euler’s formula ($e^{i\theta} = \cos \theta + i \sin \theta$) and properties of sin and cos to show the following:

(i) If $n \in \mathbb{Z}$ is even, then $e^{in\pi} = 1$.

(ii) If $n \in \mathbb{Z}$ is odd, then $e^{in\pi} = -1$.

(iii) $e^{inx} = e^{-inx}$
(b) Show that this collection of vectors forms an orthonormal set.

In class, we showed that Fourier Analysis (and some easy facts about orthonormal bases) imply that any \( f \in C(\mathbb{T}) \) can be written as

\[
f = \sum_{n=-\infty}^{\infty} \left\langle f, \frac{e^{inx}}{\sqrt{2\pi}} \right\rangle \frac{e^{inx}}{\sqrt{2\pi}} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \langle f, e^{inx} \rangle e^{inx}.
\]

After some a little bit of complex algebra, we were able to write it in a form that was more familiar to many of us:

\[
f = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)
\]

where the harmonics \( a_n \) and \( b_n \) are given by

\[
a_n = \frac{1}{\pi} \langle f, \cos(nx) \rangle \quad \text{and} \quad b_n = \frac{1}{\pi} \langle f, \sin(nx) \rangle.
\]

**Question 4.** For the following, we will consider the following *triangular wave function* \( f \in C(\mathbb{T}) \) given by \( f(x) = |x| \). This function is called a triangular wave because, when this function is extended by periodicity to \( \mathbb{R} \), it looks like a wave made up of several triangles.

(a) Show that \( |x| \sin(nx) \) is an odd function. Use this to quickly prove that \( b_n = 0 \) for all \( n \).

(b) Compute \( a_n \). Your answer will be in terms of \( n \).

(c) Using this, write out the Fourier expansion for \( f(x) \). Graph the Fourier sum (i.e., Fourier series where the sum only contains finitely many terms) and show that this graph does look roughly like a triangular wave. Using the sum up to \( n = 9 \) or so should suffice.