Math 431 - Real Analysis
Constructing the Real Numbers

Define the *real numbers* $\mathbb{R}$ to be the non-empty set of objects with two binary operations $+$ (addition) and $\cdot$ (multiplication) such that the following ten axioms hold. In what follows, assume that $x, y, z \in \mathbb{R}$.

The first five axioms establish $\mathbb{R}$ as a *field*.

1. Commutativity: $x + y = y + x$ and $xy = yx$.
2. Associativity: $x + (y + z) = (x + y) + z$ and $(xy)z = x(yz)$.
3. Distributivity: $x(y + z) = xy + xz$.
4. Additive Inverses: For any $x, y \in \mathbb{R}$, there exists a $z \in \mathbb{R}$ such that $x + z = y$.
   (a) Such a z is denoted by $y - x$.
   (b) The real number $x - x$ is dented by 0. Note: the definition of 0 is independent of x.
   (c) $0 - x$ is denoted by $-x$.
5. Multiplicative Inverses:
   (a) There exists at least one real number $x \neq 0$.
   (b) If $x, y \in \mathbb{R}$ and $x \neq 0$, there exists $z \in \mathbb{R}$ such that $xz = y$.
   (c) Such a z is denoted by y/x.
   (d) The number x/x is denoted by 1. Note: the definition of 1 is independent of x.
   (e) For $x \neq 0$, $1/x$ is denoted b x^{-1}.

Furthermore, $\mathbb{R}$ is equipped with a relation $<$ such that the following four order axioms hold.

6. Total Order: Exactly one of the following relations hold: $x < y$, $y < x$, or $x = y$.
7. Translation Invariant: If $x < y$, then for every $z \in \mathbb{R}$, $x + z < y + z$.
8. Closure of Positivity under Multiplication: If $x > 0$ and $y > 0$, then $xy > 0$.
   (a) A real number x is called *positive* if $x > 0$ and *negative* is $x < 0$.
   (b) Denote by $\mathbb{R}^+$ the set of positive real numbers and $\mathbb{R}^-$ the set of negative real numbers.
9. Transitivity: If $x > y$ and $y > z$, then $x > z$.

The last (and most distinctive) axiom is the Completeness Axiom.

10. Completeness Axiom: Every non-empty set S of real numbers that is bounded above has a supremum. In other words, there exists a real number b such that $b = \sup S$.