Math 431 - Real Analysis I
Test 2

Instructions: On a separate sheet of paper, write your solutions neatly and carefully. In your proofs, no "Discussion" is needed, but complete, well-written proofs are required for full credit. Unless otherwise stated, let S and T be metric spaces with metric functions d_S and d_T, respectively.

Question 1. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions and let $a, b \in \mathbb{R}$ be distinct real numbers. If $f(a) \leq g(a)$ and $g(b) \leq f(b)$, show that a some point $c \in [a, b]$, $f(c) = g(c)$.

Solution 1. Consider the function $h(x) = f(x) - g(x)$. Since f and g are continuous, then h is continuous. Notice that $h(c) = 0$ if and only if $f(c) = g(c)$. We wish to apply Bolzano's Theorem to h. Consider $h(a)$. Since $f(a) \leq g(a)$, then $h(a) = f(a) - g(a) \leq 0$. If $h(a) = 0$, then, $f(a) = g(a)$ and we are done. So, assume that $h(a) < 0$. Now consider that $h(b) = f(b) - g(b)$. Since $g(b) \leq f(b)$, we have that $h(b) = f(b) - g(b) \geq 0$. Again, if $h(b) = 0$, then $f(b) = g(b)$ and we are done. If not, then $h(b) > 0$. Thus, we are left with a case where $h(a) < 0$ and $h(b) > 0$. Thus, by Bolzano's Theorem, h has a root $c \in (a, b)$. Thus, $h(c) = 0$ and $f(c) = g(c)$, as desired.

Question 2. Consider the function

$$f(x) = \begin{cases} \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

Show that f is discontinuous at 0.

Solution 2. Consider the sequence $x_n = \dfrac{1}{2\pi n}$. Notice that $x_n \to 0$ and that $f(0) = 0$. Also, we have that

$$f(x_n) = \cos\left(\frac{1}{x_n}\right) = \cos\left(\frac{1}{1/2\pi n}\right) = \cos(2\pi n) = 1.$$

Thus, $f(x_n) \to 1 \neq 0 = f(0)$. So, f is not continuous at 0.

Question 3. In class, we proved the Squeeze Theorem for sequences. Here, you are asked to prove the Squeeze Theorem for functions. That is, let $f, g, h : \mathbb{R} \to \mathbb{R}$ with $f(x) \leq g(x) \leq h(x)$ for all x. Show that if

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x),$$

then $\lim\limits_{x \to a} g(x) = L$.

Solution 3. Let $\varepsilon > 0$. Since $\lim_{x \to a} f(x) = L$, there exists a δ_f such that whenever $0 < |x - a| < \delta_f$, then $|f(x) - L| < \varepsilon$. Thus, $-\varepsilon < f(x) - L < \varepsilon$. Similarly, since $\lim_{x \to a} h(x) = L$, there exists a δ_h such that whenever $0 < |x - a| < \delta_g$, then $|h(x) - L| < \varepsilon$. Thus, $-\varepsilon < h(x) - L < \varepsilon$. Choose $\delta = \min\{\delta_f, \delta_g\}$. We will show that whenever $0 < |x - a| < \delta$, then $|g(x) - L| < \varepsilon$. By the above, for x values satisfying, $0 < |x - a| < \delta$ we have that $-\varepsilon < f(x) - L$. Since $f(x) \leq g(x)$, we have that $-\varepsilon < f(x) - L \leq g(x) - L$. Also, by the above, for x satisfying $0 < |x - a| < \delta$, we have that $h(x) - L < \varepsilon$. Since $g(x) \leq h(x)$, then $g(x) - L \leq h(x) - L < \varepsilon$. THus, we have that $-\varepsilon < g(x) - L < \varepsilon$. So, $|g(x) - L| < \varepsilon$. So, $\lim_{x \to a} g(x) = L$.

Question 4. Let $a \in S$ be an *isolated* point of S. Show that any function $f : S \to T$ is continuous at a.

Solution 4. Let $a \in S$ be an isolated point. Then there exists some $\delta > 0$ such that $B(a;\delta) \cap S = \{a\}$. Let $\varepsilon > 0$. Notice that for all $x \in B_S(a;\delta) = \{a\}$, then $x = a$. Thus, $d_T(f(x), f(a)) = d_T(f(a), f(a) = 0 < \varepsilon$. Thus, f is continuous at a.

Question 5. Consider the recursive sequence given by $x_1 = 2$ and

$$x_{n+1} = 2 - \frac{1}{x_n}.$$

(a) Compute x_1, x_2, x_3, x_4, and x_5. Keep your answers as fractions.

(b) Show that $x_n \geq 1$ for all n.

(c) Show that x_n is a decreasing sequence.

(d) Show that x_n converges.

Solution 5.

(a) When $n = 1$, $x_1 = 2$. When $n = 2$, $x_2 = 3/2$. When $n = 3$,, $x_3 = 4/3$. When $n = 4$, $x_4 = 5/4$. When $n = 5$, $x_5 = 6/5$.

(b) Let $P(n)$ be the statement that $x_n \geq 1$. We will show that $P(n)$ is true for all $n \geq 1$. For the base case, consider $P(1)$. Notice that $x_1 = 2 > 1$. Now, assume that $P(k)$ is true for some $k \geq 1$. We will show that $P(k+1)$ is true. Thus, we know that $x_k \geq 1$. Thus, $\frac{1}{x_k} \leq 1$. Thus,

$$x_{k+1} = 2 - \frac{1}{x_k} \geq 2 - 1 = 1.$$

(c) Let $P(n)$ be the statement that $x_n \geq x_{n+1}$. We will show that $P(n)$ is true for all $n \geq 1$. For the base case, consider $P(1)$. Note that $x_1 = 2$ and $x_2 = 1.5$. Thus, $x_1 \geq x_2$. Now, assume that $P(k)$ is true for some $k \geq 1$. We will show that $P(k+1)$ is true. Since $P(k)$ is true, then we know that $x_k \geq x_{k+1}$. Since $x_n > 0$ for all n, then $\frac{1}{x_{k+1}} \leq \frac{1}{x_k}$. Thus, we have that

$$x_{k+2} = 2 - \frac{1}{x_{k+1}} \geq 2 - \frac{1}{x_k} = x_{k+1}.$$

Thus, by induction $x_{n+1} \geq x_n$ for all $n \geq 1$.

(d) By the above, x_n is a decreasing sequence that is bounded below. Thus, it converges.

Question 6. Let x_n and y_n be real sequences. Assume that $y_n \to 0$ and that x_n is bounded. Thus, there exists some number M such that $|x_n| < M$ for all n. Show that the product sequence $x_n \cdot y_n$ also converges to 0.

Solution 6. Let $\varepsilon > 0$. Since $y_n \to 0$, there exists an N such that for all $n > N$, $|x_n - 0| < \frac{\varepsilon}{M}$ Thus, $M|x_n| < \varepsilon$. For this N, for $n > N$, we have that

$$|x_n \cdot y_n - 0| = |x_n y_n| = |x_n| \cdot |y_n| < M|x_n| < \varepsilon.$$

Thus, $x_n \cdot y_n \to 0$

Extra Credit Question. Find the *closed form* equation for the x_n given recursively in Question 5. A complete answer will have the correct equation "$x_n = \ldots$" that is not expressed recursively, as well as a proof (try induction) that your equation is correct.

Extra Credit Solution. The closed form for x_n is

$$x_n = \frac{n+1}{n}$$

for $n \geq 1$. Let $A(n)$ be the statement that $x_n = \dfrac{n+1}{n}$. For the base case, notice that $x_1 = 2 = \dfrac{1+1}{1}$. Now, assume that $A(k)$ is true for some $k \geq 1$. Then,

$$x_{k+1} = 2 - \frac{1}{x_n} = 2 - \frac{1}{\frac{n+1}{n}} = 2 - \frac{n}{n+1} = \frac{2n+2}{n+1} - \frac{n}{n+1} = \frac{n+2}{n+1} = \frac{(n+1)+1}{n+1}.$$

Thus, $A(k+1)$ holds.