Math 431 - Real Analysis I
Test 1 Review Sheet

Logistics: Our test will occur on Monday, October 15. It will be a 110 minute, no notes, no calculator test. Please bring blank paper on which you will write your solutions.

The successful test-taker will have mastered the following concepts.

The Axiomatic Foundation of the Real Line

- Know the 10 real number axioms: 5 field axioms, 4 order axioms, and 1 completeness axiom
- Use the 10 real number axioms to prove well-known facts about the real numbers and their ordering.

Properties of the Integers

- Definition of an inductive set of real numbers
- Definition of $\mathbb{Z}_+$ as an inductive set.
- Bezout's Identity for the gcd of two integers
- Euclid's Lemma
- The role of primes in divisibility statements (e.g., if $p|ab$, then $p|a$ or $p|b$).
- The unique factorization theorem for integers

Properties of Rational Numbers

- Use proof by contradiction to prove that a number is irrational.
- Use the fact that $\mathbb{Q}$ is a field (and is thus closed under the four operations)

Bounds, Suprema, and Infima

- Know the definition of bounded above, bounded below, bounded, maximum, minimum
- Know the definition of a supremum/infimum; use the Completeness Axiom to prove that they exist.
- Prove that a number is a supremum or infimum for a given set.
- Use the approximation theorem for suprema in proofs
- Use the additive and comparison properties for suprema

Applications of the Completeness Axiom

- Know the proof that $\mathbb{Z}_+$ is an unbounded set and how to use it in proofs.
- Know the Archimedean Property and how to use it in proofs.

Euclidean Space and its Metric

- Know the definition of $\mathbb{R}^n$ as well as the vector space and metric properties of $\mathbb{R}^n$.
- Know how to compute the distance between two points in $\mathbb{R}^n$.

- Know the Caucy-Schwarz inequality and how to apply it in proofs and to obtain new inequalities.
- Know the various metric properties of $\mathbb{R}^n$ (e.g., triangle inequality, positive-definiteness, symmetry).

The Topology of $\mathbb{R}^n$

- Definition of the open n-ball $B(\mathbf{x}; r)$.
- Definition of an interior point of a set; know how to show a point is or is not interior to a set.
- Definition of an open set; know how to show a set is or is not open.
- Basic topological properties of open sets (closed under unions and finite intersections)
- Definition of a closed set; showing a set is closed; topological properties of closed sets.

Adherent Points and Accumulation Points

- Know the definition of an adherent point; show that that a point does or does not adhere to a set.
- Know the definition of an accumulation point; show that a point is or is not an accumulation point of a set.
- Know the subtle but important difference between an accumulation point and an adherent point. (e.g., isolated points)
- Show that a set is or is not discrete.
- Understand the relationship between closed sets, adherent points, and accumulation points.
- Given a set, find its closure and its derived set.

The Analytic Properties of $\mathbb{R}$

- Know the Bolzano-Weierstrass Theorem and when to apply it
- Know the Cantor Intersection Theorem and when to apply it
- Understand what properties of $\mathbb{R}$ allow the above two theorems to hold

Open Covers and Compactness

- Show that a collection of sets $\mathcal{F}$ is an open cover for a set S. In particular, be comfortable with the nuanced notation
- Show that a cover has no finite subcover
- Use the Lindelöf Covering Theorem to reduce an open cover for a subset of $\mathbb{R}^n$ to a countable subcover.
- Know the definition of compactness; in particular, know how to show that a set is or is not compact using this definition
- Know the Heine-Borel Theorem and the theorem giving equivalent conditions for a subset of $\mathbb{R}^n$ to be compact.

Metric Spaces

- Know the three metric properties and use them to show that a set with a distance function is indeed a metric space.

- Know several examples of metric spaces. In particular, know several examples of different metrics on $\mathbb{R}^n$
- Know which concepts from the topology of $\mathbb{R}^n$ carry over to metric spaces
- Know which theorems from $\mathbb{R}^n$ are or are not true in general metric spaces.

Sequences

- Know the definition of convergence of a sequence in $\mathbb{R}$ and in a general metric space.
- Be able to show that a sequence in a metric space does/does not converge.
- Show that $x_n \to p$ in a metric space by showing that $d(x_n, p) \to 0$ in $\mathbb{R}$.
- Know various properties of convergent sequences in general metric spaces (e.g., are bounded, have unique limit)
- Know various properties of convergent sequences in $\mathbb{R}$