Math 431 - Real Analysis I
Solutions to Quest 1

Question 1. Below, you are given an *open* set S and a point $\mathbf{x} \in S$. Thus, by definition of openness, there exists an $\varepsilon > 0$ such that

$$B(\mathbf{x}; \varepsilon) \subset S.$$

Your job is to do the following:

(i) Provide such an $\varepsilon > 0$ that "works".

(ii) Show that your ε is actually positive.

Note: There is no need to *prove* that $B(\mathbf{x}; \varepsilon) \subset S ... that would take too long$!

(a) Let $x \in (-2, 8)$ [the open interval from -2 to 8]. What is an $\varepsilon > 0$ such that $B(x; \varepsilon) \subset (-2, 8)$?

(b) Let $(u, v) \in T$, where $T = \{(x, y) \mid x > 0, y < 0\}$ is the "open fourth quadrant" in $\mathbb{R}^2$. What is an $\varepsilon > 0$ such that $B((u, v); \varepsilon) \subset T$?

(c) Let $(u, v) \in (0, 1) \times (0, 1)$ [the "unit open square" in $\mathbb{R}^2$] . What is an $\varepsilon > 0$ such that $B((u, v); \varepsilon) \subset (0, 1) \times (0, 1)$?

Solution 1. Note: The answers given below are not unique; these solutions give merely one example of a correct ε.

(a) Let $\varepsilon = \min\{8 - x, x + 2\}$. Since $x \in (-2, 8)$, then $-2 < x < 8$. Thus, $8 - x > 0$and $x + 2 = x - (-2) > 0$. Since both $8 - x$ and $x + 2$ are positive, their minimum is positive. So, $\varepsilon > 0$.

(b) Let $\varepsilon = \min\{u, -v\}$. Since $(u, v) \in T$, then $u > 0$ and $v < 0$. Thus, $-v > 0$. Thus, the minimum of u and $-v$ is also positive. So, $\varepsilon > 0$.

(c) Let $\varepsilon = \min\{u, 1 - u, v, 1 - v\}$. Since $(u, v) \in (0, 1) \times (0, 1)$, we have that $0 < u < 1$ and $0 < v < 1$. Thus, $u > 0$, $1 - u > 0$, $v > 0$, and $1 - v > 0$. Since ε is the minimum of these four positive numbers, $\varepsilon > 0$.

Question 2. Consider the set

$$S = \{(x, y) \in \mathbb{R}^2 \mid x > 0\},$$

which is geometrically the right half-space in $\mathbb{R}^2$. Also, consider the set

$$T = \{(0, y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\},$$

which is geometrically the y-axis in $\mathbb{R}^2$.

(a) Use the definition of an adherent point to show that any $(0, y) \in T$ is adherent to S.

(b) Use (a) and a theorem from class to show that S is not closed.

Solution 2.

(a) Let $(0, y) \in T$. Let $\varepsilon > 0$. We will show that

$$B((0, y); \varepsilon) \cap S \neq \varnothing.$$

Consider $(\varepsilon/2, y)$. Note that

$$\|(\varepsilon/2, y) - (0, y)\| = \varepsilon/2 < \varepsilon.$$

Thus, $(\varepsilon/2, y) \in B((0, y); \varepsilon)$. Also, since $\varepsilon/2 > 0$, $(\varepsilon/2, y) \in S$. So,

$$B((0, y); \varepsilon) \cap S \neq \varnothing.$$

So, any $(0, y)$ is an adherent point.

(b) We proved that S is closed if and only if S contains all its adherent points. However, since every point in T is adherent to S and no point in T is in S, S does not contain all its adherent points. Thus, S is note closed.

Question 3. Let $S \subset \mathbb{R}$ be a non-empty, *closed* set that is bounded above.

(a) Show that $\sup S$ exists.

(b) Prove that $\sup S = \max S$ by showing that $\sup S \in S$. A proof by contradiction may be helpful.

(c) Conclude that every non-empty, closed set that is bounded above has a maximum.

Solution 3.

(a) S is a non-empty set that is bounded above. Thus, by the Completeness Axiom, $\sup S$ exists.

(b) By definition of $\sup S$, $x \leq \sup S$ for all $x \in S$. Next, we show that $\sup S \in S$. Assume, to the contrary, that $\sup S \notin S$. Thus, $\sup S \in \overline{S}$, which is open since S is closed. Thus, $\sup S$ is interior to $\overline{S}$. So, there exists an $\varepsilon > 0$ such that $B(x; \varepsilon) \subset \overline{S}$. By the Approximation Theorem, there exists a $z \in S$ such that $\sup S - \varepsilon/2 < z \leq \sup S$. Thus, $z \in B(x; \varepsilon) \subset \overline{S}$ and also $z \in S$, a contradiction. Thus, $\sup S \in S$. So, $\sup S = \max S$.

Question 4. [Extra Credit] Give a synonym for the following terms:

(a) Adherent Point

(b) Accumulation Point

(c) Bob Pelayo

Solution 4.

(a) Adherent Point = Closure Point

(b) Accumulation Point = Limit Point

(c) Bob Pelayo = ????