Math 431 - Real Analysis
Solutions to Homework due September 17

In class, we learned of the famous *Cauchy-Schwarz Inequality.* Given two n-vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the Cauchy-Schwarz inequality relates the dot product with the norms of the individual vectors:

$$(\mathbf{x} \cdot \mathbf{y})^2 \leq \|\mathbf{x}\|^2 \, \|\mathbf{y}\|^2.$$

Written component-wise with

$$\mathbf{x} = (x_1, x_2, \ldots, x_n) \text{ and } \mathbf{y} = (y_1, y_2, \cdots, y_n),$$

the Cauchy-Schwarz inequality is

$$\left(\sum_{k=1}^{n} x_k y_k\right)^2 \leq \left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right).$$

Question 1. Many times, the Cauchy-Schwarz Inequality can be used to obtain some interesting inequalities by simply choosing an appropriate vector $\mathbf{x}$ and $\mathbf{y}$.

(a) Let $a, b, c \in \mathbb{R}$. Show that

$$(a+b+c)^2 \leq 3(a^2+b^2+c^2).$$

(b) Let $a, b, c \in \mathbb{R}_+$. Show that

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9.$$

(c) Let $a_1, a_2, \cdots a_n \in \mathbb{R}$. Show the *Sum of Squares inequality:*

$$\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^2 \leq \frac{1}{n}\sum_{k=1}^{n} a_k^2.$$

Solution 1.

(a) Consider the vectors $\mathbf{x} = (a, b, c)$ and $\mathbf{y} = (1, 1, 1)$. Using the Cauchy-Schwarz Inequality, we get that

$$(a+b+c)^2 = (1 \cdot a + 1 \cdot b + 1 \cdot c)^2 \leq (1+1+1)(a^2+b^2+c^2) = 3(a^2+b^2+c^2).$$

$\square$

(b) Consider the vectors

$$\mathbf{x} = \left(\sqrt{a}, \sqrt{b}, \sqrt{c}\right) \text{ and } \mathbf{y} = \left(\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}\right).$$

Using the Cauchy-Schwarz Inequality, we get that

$$9 = (1+1+1)^2 = \left(\sqrt{a} \cdot \frac{1}{\sqrt{a}} + \sqrt{b} \cdot \frac{1}{\sqrt{b}} + c \cdot \frac{1}{\sqrt{c}}\right)^2 \leq$$

$$\left(\sqrt{a}^2 + \sqrt{b}^2 + \sqrt{c}^2\right)\left(\left(\frac{1}{\sqrt{a}}\right)^2 + \left(\frac{1}{\sqrt{b}}\right)^2 + \left(\frac{1}{\sqrt{c}}\right)^2\right) = (a+b+c)\left(\frac{1}{a}+\frac{1}{n}+\frac{1}{c}\right).$$

$\square$

(c) Let $\mathbf{x} = (1/n, 1/n, \ldots 1/n)$ and $\mathbf{y} = (a_1, a_2, \ldots a_n)$. Using the Cauchy-Schwarz inequality, we get that

$$\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^2 = \left(\sum_{k=1}^{n} \frac{1}{n} a_k\right)^2 \leq \left(\sum_{k=1}^{n}\left(\frac{1}{n}\right)^2\right)\left(\sum_{k=1}^{n} a_k^2\right) = \frac{1}{n}\sum_{k=1}^{n} a_k^2.$$

$\square$

Question 2. It is often easier to prove that a given set S is *not open*. To do so, one needs to find a point $\mathbf{x} \in S$ such that for no $r > 0$, $B(\mathbf{x}; r) \subset S$. In other words, one needs to find a $\mathbf{x} \in S$ such that for all $r > 0$, there exists some $y \in B(\mathbf{x}; r)$ such that $y \in B(\mathbf{x}; r)$ but $y \notin S$. Show that the following subsets $S \subset \mathbb{R}^n$ are *not open*.

(a) $\{a\} \subset \mathbb{R}$

(b) $\{(x, 0) \in \mathbb{R}^2 \,|\, x \in \mathbb{R}\} \subset \mathbb{R}^2$

(c) $\{(x, y) \in \mathbb{R}^2 \,|\, x \geq 0 \text{ and } y \geq 0\} \subset \mathbb{R}^2$

Solution 2.

(a) We will show that a is not an interior point. Let $\varepsilon > 0$. We will show that $B(a; \varepsilon) \not\subset \{a\}$. Notice that $a + \varepsilon/2 \in B(a; \varepsilon)$ since

$$|a + \varepsilon/2 - a| = \varepsilon/2 < \varepsilon.$$

Since $\{a\}$ contains only the element a, then $a + \varepsilon/2 \notin \{a\}$. Thus, $B(a; \varepsilon) \not\subset \{a\}$ and therefore a is not an interior point. So, our set is not open. □

(b) Let $S = \{(x, 0) \in \mathbb{R}^2 \,|\, x \in \mathbb{R}\}$ and consider $(0, 0) \in S$, which we will show is not interior. Let $\varepsilon > 0$; we will show that $B((0, 0); \varepsilon) \not\subset S$. Notice that $(0, \varepsilon/2) \in B((0, 0); \varepsilon)$ since

$$\|(0, \varepsilon/2) - (0, 0)\| = \varepsilon/2 < \varepsilon.$$

However, since $\varepsilon/2 \neq 0$, then $(0, 0) \notin S$. Thus, $B((0, 0); \varepsilon) \not\subset S$. So, $(0, 0)$ is non-interior and S is not open. [Note: actually, any point in S is non-interior.] □

(c) Let $T = \{(x, y) \in \mathbb{R}^2 \,|\, x \geq 0 \text{ and } y \geq 0\}$. We will show that $(0, 0)$ is non-interior. Let $\varepsilon > 0$; we will show that $B((0, 0); \varepsilon) \not\subset T$. Notice that $(0, -\varepsilon/2) \notin T$ because

$$\|(0, -\varepsilon/2) - (0, 0)\| = \varepsilon/2 < \varepsilon.$$

However, since $-\varepsilon/2 < 0$, $(0, -\varepsilon/2) \notin T$. Thus, $(0, 0)$ is not an interior point and T is not open. □

Question 3. In what follows, we will demonstrate an important topological property of $\mathbb{Q} \subset \mathbb{R}$.

(a) Let $a \in \mathbb{Q}$. Show that $a + \frac{\sqrt{2}}{n}$ is irrational for all $n \in \mathbb{Z}$.

(b) Use (a) to show that $\mathbb{Q}$ is not an open subset of $\mathbb{R}$.

Solution 3.

(a) Assume, to the contrary, that $a + \sqrt{2}/n \in \mathbb{Q}$. Then, since $a \in \mathbb{Q}$, then $-a \in \mathbb{Q}$. So,

$$\sqrt{2}/n = a + \sqrt{2}/n - a \in \mathbb{Q}.$$

Furthermore, since $n \in \mathbb{Q}$, then

$$\sqrt{2} = n \cdot \frac{\sqrt{2}}{n} \in \mathbb{Q}.$$

However, it is known that $\sqrt{2} \notin \mathbb{Q}$, which is a contradiction. Thus, $a + \sqrt{2}/n \notin \mathbb{Q}$.

(b) We will show that 0 is not an interior point of $\mathbb{Q}$ (in fact, any $a \in \mathbb{Q}$ will not be interior). Let $\varepsilon > 0$. We will show that $B(0, \varepsilon) \not\subset \mathbb{Q}$. Since $\varepsilon > 0$, by the Archimedean Principle, there exists an $n \in \mathbb{Z}_+$ such that $\sqrt{2} < \varepsilon n$. Thus,

$$0 < \frac{\sqrt{2}}{n} < \varepsilon.$$

By (a), we know that $\sqrt{2}/n \notin \mathbb{Q}$. However, $\sqrt{2}/n \in B(0; \varepsilon)$. Thus, $B(0; \varepsilon) \not\subset \mathbb{Q}$. So, 0 is not an interior point and $\mathbb{Q}$ is not open in $\mathbb{R}$. $\square$

Given a set $S \subset \mathbb{R}^n$, a point $\mathbf{x} \in \mathbb{S}$ is called an *isolated point of* S if there exists an $\varepsilon > 0$ such that $B(\mathbf{x}; \varepsilon) \cap S = \{x\}$. In other words, $\mathbf{x}$ is *isolated* in S if there is a small enough $\varepsilon > 0$ such that $B(\mathbf{x}; \varepsilon)$ intersects S only at $\mathbf{x}$ itself. A set is called *discrete* if every point in S is isolated.

Question 4. Show that the following sets are or are not discrete.

(a) Show that $\mathbb{Z}$ is a discrete subset of $\mathbb{R}$

(b) Show that ever finite subset of $\mathbb{R}$ is a discrete subset of $\mathbb{R}$.

(c) Show that $S = \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{Z}_+ \right\}$ is a discrete subset of $\mathbb{R}$

(d) Show that $T = \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{Z}_+ \right\} \cup \{0\}$ is not a discrete subset of $\mathbb{R}$.

Solution 4.

(a) Let $n \in \mathbb{Z}$. We will show that n is an isolated point of $\mathbb{Z}$. Let $\varepsilon = 1/2$. Then, $B(n; 1/2) \cap \mathbb{Z} = \{n\}$. Thus, n is isolated and $\mathbb{Z}$ is discrete. $\square$

(b) Let $V = \{a_1, a_2, \ldots, a_n\}$ be a finite subset of $\mathbb{R}$. Consider all the possible distances between every pair of points in V and take the minimum. In other words, let $\varepsilon = \min\{|a_i - a_j| \mid a_i \neq a_j \in V\}$. Since there are only finitely many pairs of a_i's, a minimum exists. Furthermore, since each $|a_i - a_j| > 0$, $\varepsilon > 0$. We will now show that a_i is an isolated point in V. Consider $B(a_i; \varepsilon)$. Since $\varepsilon \leq |a_i - a_j|$ for every $a_j \neq a_i$, then the only element of V that is distance less than ε from a_i is a_i itself. Therefore, $B(a_i; \varepsilon) \cap V = \{a_i\}$. Therefore, a_i is an isolated point. Thus, every point is isolated and V is discrete. $\square$

(c) We will show that every $1/n \in S$ is an isolated point. Let

$$\varepsilon = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}.$$

Since $n < n+1$, we know that $\varepsilon > 0$. We will show that $B(1/n; \varepsilon) \cap S = \{1/n\}$. Let $m \neq n$. We will show that $1/m \notin B(1/n; \varepsilon)$. First, we take the case that $m > n$. Thus, $m \geq n+1$. So,

$$\frac{1}{m} \leq \frac{1}{n+1}$$

and thus

$$\left| \frac{1}{n} - \frac{1}{m} \right| = \frac{1}{n} - \frac{1}{m} \geq \frac{1}{n} - \frac{1}{n+1} = \varepsilon.$$

Thus, $1/m \notin B(1/n; \varepsilon)$. Next, consider the case where $m < n$. Thus, $m \leq n-1$ and so

$$0 < \frac{1}{n-1} \leq \frac{1}{m}.$$

Notice that since $n-1 < n+1$, then $n(n-1) < n(n+1)$ and so

$$\varepsilon = \frac{1}{n(n+1)} < \frac{1}{n(n-1)}.$$

Furthermore,

$$\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}.$$

Thus, since

$$\frac{1}{n-1} \le \frac{1}{m},$$

we know that

$$\varepsilon = \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n} \le \frac{1}{m} - \frac{1}{n} = \left\| \frac{1}{m} - \frac{1}{n} \right\|.$$

Thus, $1/m \notin B(1/n;\varepsilon)$.

Thus, the only element of S that is also in $B(1/n;\varepsilon)$ is $1/n$ itself. So, $1/n$ is isolated. Since every point is isolated, S is discrete.

(d) We will show that 0 is not an isolated point. Let $\varepsilon > 0$. By the Archimedean principle, there exists an $n \in \mathbb{Z}_+$ such that $1 < n\varepsilon$ and thus

$$0 < \frac{1}{n} < \varepsilon.$$

Thus, $\frac{1}{n} \in B(0;\varepsilon) \cap T$. So, it is not true that $B(0;\varepsilon) \cap T = \{0\}$. Thus, 0 is not isolated and therefore T is not discrete.

Question 5. Let $U, V \in \mathbb{R}$ be open sets. Consider the product set

$$U \times V = \{(x,y) \mid x \in U, y \in V\} \subset \mathbb{R}^2.$$

Show that $U \times V$ is open by showing that each $(x,y) \in U \times V$ is an interior point.

Solution 5. Let $(x,y) \in U \times V$. Thus, $x \in U$ and $y \in V$. We will show that (x,y) is interior to $U \times V$. Since U is open, there exists an $\varepsilon_1 > 0$ such that $B(x;\varepsilon_1) \subset U$. Similarly, since V is open, there exists an $\varepsilon_2 > 0$ such that $B(y;\varepsilon_2) \subset V$. Since $B(x;\varepsilon_1) \subset U$ and $B(y;\varepsilon_2) \subset V$, then

$$B(x;\varepsilon_1) \times B(y;\varepsilon_2) \subset U \times V.$$

Consider $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$; thus $\varepsilon \le \varepsilon_1$ and $\varepsilon \le \varepsilon_2$. We will show that (x,y) is an interior point of $U \times V$ by showing that

$$B((x,y);\varepsilon) \subset B(x;\varepsilon_1) \times B(y;\varepsilon_2) \subset U \times V.$$

Since the second inclusion is already established, we focus on the first inclusion. Let $(a,b) \in B((x,y);\varepsilon)$. Thus,

$$\|(a,b) - (x,y)\| = \sqrt{(a-x)^2 + (b-y)^2} < \varepsilon.$$

We wish to show that $(a,b) \in B(x;\varepsilon_1) \times B(y;\varepsilon_2)$ by showing that $|a-x| < \varepsilon_1$ and $|b-y| < \varepsilon_2$. Assume, to the contrary, that this is not true. Then, $|a-x| \ge \varepsilon_1$ or $|b-y| \ge \varepsilon_2$. If $|a-x| \ge \varepsilon_1$, then

$$\|(a,b) - (x,y)\| = \sqrt{(a-x)^2 + (b-y)^2} \ge \sqrt{(a-x)^2} = |a-x| \ge \varepsilon_1 \ge \varepsilon.$$

This contradicts the fact that

$$\|(a,b) - (x,y)\| < \varepsilon.$$

A similar computation gives the same contradiction for the case when $|b - y| \geq \varepsilon_2$. Thus, we conclude that $|a - x| < \varepsilon_1$ and $|b - y| < \varepsilon_2$. Thus,

$$B((x, y); \varepsilon) \subset B(x; \varepsilon_1) \times B(y; \varepsilon_2) \subset U \times V.$$

So, (x, y) is interior and $U \times V$ is open.

Question 6. Consider the set

$$T = \{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| < 1\}.$$

Geometrically, this set is just an "open disk" of radius 1 about the origin. Consider

$$\mathbb{S}^1 = \{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| = 1\}.$$

Geometrically, $\mathbb{S}^1$ is the circle of radius 1 about the origin. We will show that every point in $\mathbb{S}^1$ is an accumulation point of T (and therefore an adherent point of T).

As a hint, you may want to follow something similar to the below outline:

Let $\mathbf{x} \in \mathbb{S}^1$. We will show that for all $\varepsilon > 0$, $B(\mathbf{x}; \varepsilon) \cap (T - \{\mathbf{x}\}) \neq \varnothing$. First, note that $T - \{x\} = T$ since $\mathbf{x} \notin T$. Thus, we wish to show that $B(\mathbf{x}; \varepsilon) \cap T \neq \varnothing$. Then, consider the 2 cases: $\varepsilon > 1$ or $0 < \varepsilon \leq 1$. In the last case, it might be wise to consider $\left(1 - \dfrac{\varepsilon}{2}\right)\mathbf{x}$.

Solution 6. Let $\mathbf{x} \in \mathbb{S}^1$. Thus, $\|\mathbf{x}\| = 1$. We will show that for all $\varepsilon > 0$, $B(\mathbf{x}; \varepsilon) \cap (T - \{\mathbf{x}\}) \neq \varnothing$. First, note that $T - \{x\} = T$ since $\mathbf{x} \notin T$. Thus, we wish to show that $B(\mathbf{x}; \varepsilon) \cap T \neq \varnothing$. Then, consider the 2 cases: $\varepsilon > 1$ or $0 < \varepsilon \leq 1$.

If $\varepsilon > 1$, consider the point $(0, 0) \in T$. Notice that

$$\|(0, 0) - \mathbf{x}\| = \|\mathbf{x}\| = 1 < \varepsilon.$$

Thus, $(0, 0) \in B(\mathbf{x}; \varepsilon) \cap T$, which is thus non-empty.

If $0 < \varepsilon \leq 1$. Consider the point $(1 - \varepsilon/2)\mathbf{x}$. Since $0 < \varepsilon \leq 1$, then $0 < \varepsilon/2 \leq 1/2 < 1$. Thus, $0 < 1 - \varepsilon/2 < 1$.

Note that $(1 - \varepsilon/2)\mathbf{x} \in T$ since

$$\|(1 - \varepsilon/2)\mathbf{x}\| = |1 - \varepsilon/2| \, \|\mathbf{x}\| = (1 - \varepsilon/2) \cdot 1 = 1 - \varepsilon/2 < 1.$$

Next, we will show that $(1 - \varepsilon/2)\mathbf{x} \in B(\mathbf{x}; \varepsilon)$. To see this, note that

$$\|(1 - \varepsilon/2)\mathbf{x} - \mathbf{x}\| = \| - \varepsilon/2\mathbf{x}\| = \varepsilon/2 \, \|\mathbf{x}\| = \varepsilon/2 < \varepsilon.$$

Thus, $(1 - \varepsilon/2)\mathbf{x} \in B(\mathbf{x}; \varepsilon) \cap T$, which is thus non-empty. So, every $\mathbf{x} \in \mathbb{S}^1$ is an accumulation point (and thus an adherent point).