

MATH 431 - REAL ANALYSIS I
HOMEWORK DUE OCTOBER 8

Question 1. Recall that any set M can be given the discrete metric d_d given by

$$d_d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

For the below, let M be any set with the discrete metric.

- (a) Show that *any* subset S of M is an open set.
- (b) Use (a) to show that any subset of M is closed.
- (c) Show that any subset S of M is discrete (hence the name ‘discrete metric’).
- (d) Show that a subset S of M is compact if and only if it is finite.

Solution 1.

- (a) Let $S \subset M$ and let $x \in S$. We will show that x is interior to S . Consider $\varepsilon = 1/2$. Notice that for all points $y \neq x$, $d_d(x, y) = 1 > 1/2$. Thus,

$$B(x; 1/2) = \{x\}.$$

Thus,

$$\{x\} = B(x; 1/2) \subset S.$$

Thus, x is interior to S . Since this is true for all $x \in S$, S is open.

- (b) Consider the complement \overline{S} . By (a), every subset of is open; thus, \overline{S} is open. So, S is closed.
- (c) Let $x \in S$. We will show that x is isolated. As above, consider $\varepsilon = 1/2$. As noted in (a), $B(x; 1/2) = \{x\}$. Thus, x is isolated and S is discrete.
- (d) This proof is identical to the one from last week’s HW. See those solutions. In particular, one can just choose the open cover to be $\mathcal{F} = \{\{x\} \mid x \in S\}$. Since every set is open, $\{x\}$ is open, so this is an open cover for S .

Question 2. Use the $\varepsilon - N$ definition of the convergence of a sequence to show that following sequences converge to the indicated limits. Unless otherwise stated, all sequences are valid for $n \geq 1$.

- (a) $\frac{1}{n^2} \rightarrow 0$
- (b) $\frac{2}{\sqrt{n}} + 1 \rightarrow 1$
- (c) $e^{-n} \rightarrow 0$

Solution 2.

(a) Given $\varepsilon > 0$, let $N = 1/\sqrt{\varepsilon}$. For all $n > N = 1/\sqrt{\varepsilon}$, we have that

$$\frac{1}{\sqrt{\varepsilon}} < n.$$

Squaring both sides preserves the inequality and we get

$$\frac{1}{\varepsilon} < n^2.$$

Since $n^2, \varepsilon > 0$, we can cross divide to get

$$\frac{1}{n^2} < \varepsilon.$$

Since $1/n^2 > 0$, this is equivalent to

$$\left| \frac{1}{n^2} - 0 \right| < \varepsilon.$$

Thus, $1/n^2 \rightarrow 0$. □

(b) Given $\varepsilon > 0$, let $N = 4/\varepsilon^2$. For all $n > N = 4/\varepsilon^2$, we have that

$$\frac{4}{\varepsilon^2} < n.$$

Taking the square root of both sides preserves the inequality. Thus, we have that

$$\frac{2}{\varepsilon} < \sqrt{n}.$$

Since $\varepsilon, \sqrt{n} > 0$, we can cross-divide to obtain

$$\frac{2}{\sqrt{n}} < \varepsilon.$$

Since $4/\varepsilon^2 > 0$, this is equivalent to

$$\left| \frac{2}{\sqrt{n}} \right| < \varepsilon.$$

Adding and subtracting 1 on the inside of the absolute values, we get

$$\left| \left(\frac{2}{\sqrt{n}} + 1 \right) - 1 \right| < \varepsilon.$$

Thus, $2/\sqrt{n} + 1 \rightarrow 1$. □

(c) Given $\varepsilon > 0$, let $N = \ln(1/\varepsilon)$. Then, for all $n > N = \ln(1/\varepsilon)$, we have

$$\ln(1/\varepsilon) < n.$$

Exponentiating both sides preserves the inequality and we get that

$$e^{\ln(1/\varepsilon)} < e^n,$$

which is equivalent to

$$\frac{1}{\varepsilon} < e^n.$$

Since $\varepsilon, e^n > 0$, we can cross-divide to get that

$$e^{-n} = \frac{1}{e^n} < \varepsilon.$$

Since $e^{-n} > 0$, this is equivalent to

$$|e^{-n} - 0| < \varepsilon.$$

Thus, $e^{-n} \rightarrow 0$. □

Question 3. In what follows, let M be a metric space with metric d .

- (a) A sequence $\{x_n\}$ in a metric space is called *eventually constant* if there exists some N such that for all $n > N$, $x_n = p$ for some $p \in M$. Show that any eventually constant sequence converges.
- (b) Let $k \in \mathbb{R}$ and let $\{x_n\}$ be a real sequence. Show that if $x_n \rightarrow a$, then the sequence $\{k \cdot x_n\}$ converges to $k \cdot a$.

Solution 3.

- (a) Let $\varepsilon > 0$. Since x_n is eventually constant, there exists some N such that for all $n > N$, $x_n = p$. Thus, for this N , we have that for all $n > N$, $d(x_n, p) = d(p, p) = 0 < \varepsilon$. Thus, $x_n \rightarrow p$.
- (b) We will prove our proposition using 2 cases: if $k = 0$ or if $k \neq 0$. If $k = 0$, then $k \cdot x_n = 0$ is the constant 0 sequence. Thus, it converges to $0 = k \cdot a$, as desired.

Next, we assume that $k \neq 0$. Given $\varepsilon > 0$, since $x_n \rightarrow a$, there exists an N such that for all $n > N$, $|x_n - a| < \varepsilon/|k|$. So, for all $n > N$,

$$|k \cdot x_n - k \cdot a| = |k(x_n - a)| = |k| \cdot |x_n - a| < \varepsilon.$$

So $k \cdot x_n \rightarrow k \cdot a$.

Question 4. In this question, we will investigate specific examples of convergence in the metric space

$$C([0, 1]) = \{f : [0, 1] \rightarrow \mathbb{R} \mid f \text{ is continuous}\}.$$

In other words, $C([0, 1])$ is the set of all continuous real-valued functions whose domain is $[0, 1]$. We will consider $C([0, 1])$ with its L^1 metric given by

$$d(f, g) = \int_0^1 |f(x) - g(x)| dx.$$

Below, you will be given a sequence of these functions. You should

- (i) Draw/graph these functions for $n = 2, 3, 5$, and 10 .
 - (ii) Show that f_n converges to the indicated function.
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- (a) Let $\{f_n\}$ be the sequence given by $f_n(x) = x^n$. Show that f_n converges to the constant 0 function.
 - (b) Let $\{f_n\}$ be the sequence given by

$$f_n(x) = \begin{cases} nx & \text{if } 0 \leq x \leq 1/n \\ 1 & \text{if } 1/n < x \leq 1 \end{cases}$$

Show that f_n converges to the constant function 1.

Solution 4.

- (a) We will show that $x^n \rightarrow 0$ by showing that $d(x^n, 0) \rightarrow 0$. Since $x^n \geq 0$ on $[0, 1]$, we have that $|x^n - 0| = x^n$. Thus, computing the distance, we get

$$d(x^n, 0) = \int_0^1 |x^n - 0| dx = \int_0^1 x^n = \frac{1}{n+1} x^{n+1} \Big|_0^1 = \frac{1}{n+1}.$$

Notice that

$$0 \leq \frac{1}{n+1} \leq \frac{1}{n}.$$

Since $0 \rightarrow 0$ and $1/n \rightarrow 0$, by the Squeeze Theorem, we have that

$$d(x^n, 0) = \frac{1}{n+1} \rightarrow 0.$$

Thus, $x^n \rightarrow 0$.

- (b) We will show that $f_n \rightarrow 1$ by showing that $d(f_n, 1) \rightarrow 0$. Since $1 \geq f_n(x)$ on $[0, 1]$, we know that $|1 - f_n(x)| = 1 - f_n(x)$. Thus, computing the distance, we have

$$\begin{aligned} \int_0^1 |1 - f_n(x)| dx &= \int_0^1 1 - f_n(x) dx = \int_0^{1/n} 1 - nx dx + \int_{1/n}^1 1 - 1 dx = \\ &= x - \frac{n}{2} x^2 \Big|_0^{1/n} = \frac{1}{n} - \frac{n}{2} \cdot \frac{1}{n^2} = \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n}. \end{aligned}$$

Since $1/n \rightarrow 0$, then $\frac{1}{2n} \rightarrow \frac{1}{2} \cdot \frac{1}{n} \rightarrow \frac{1}{2} \cdot 0 = 0$. Thus, $d(1, f_n) \rightarrow 0$ and so $f_n \rightarrow 1$.

Question 5. In Question 3a, we showed that in any metric space, every eventually constant sequence converges. In this question, we will show that in a metric space with the discrete metric, the converse is true. In other words, let M be a metric space with the discrete metric d_d (defined in Question 1). Show that if x_n converges, then $\{x_n\}$ is eventually constant.

Solution 5. Assume $x_n \rightarrow p$. Then, there exists some N such that for all $n > N$, $d_d(x_n, p) < 1/2$. In this discrete metric d_d , this means that $d_d(x_n, p) = 0$ and thus $x_n = p$. Thus, for all $n > N$, $x_n = p$; so x_n is eventually constant.

Question 6. Let $\{x_n\}$ be a real sequence. We will show that $x_n \rightarrow 0$ if and only if $|x_n| \rightarrow 0$.

- (a) Show that

$$-|x_n| \leq x_n \leq |x_n|$$

for all n .

- (b) Use (a) to show that $|x_n| \rightarrow 0$, then $x_n \rightarrow 0$.
(c) Use an ε - N proof to show that if $x_n \rightarrow 0$, then $|x_n| \rightarrow 0$.
(d) Conclude that $x_n \rightarrow 0$ if and only if $|x_n| \rightarrow 0$.

Solution 6.

- (a) Note that trivially $|x_n| \leq |x_n|$. Thus, using properties of the absolute value, we have that

$$-|x_n| \leq x_n \leq |x_n|.$$

- (b) If $|x_n| \rightarrow 0$, then $-|x_n| = -1 \cdot |x_n| \rightarrow -1 \cdot 0 = 0$. Since $-|x_n| \leq x_n \leq |x_n|$, by the Squeeze Theorem we have that $x_n \rightarrow 0$.
- (c) Let $\varepsilon > 0$. Since $x_n \rightarrow 0$, there exists some N such that for all $n > N$, $|x_n - 0| < \varepsilon$. This is equivalent to $|x_n| < \varepsilon$. This, of course, is also equivalent to $||x_n| - 0| < \varepsilon$. Thus, for all $n > N$, $||x_n| - 0| < \varepsilon$. So, $|x_n| \rightarrow 0$.
- (d) Summarizing the above, by (b) we know that if $|x_n| \rightarrow 0$, then $x_n \rightarrow 0$. By (c), we know that if $x_n \rightarrow 0$, then $|x_n| \rightarrow 0$. Thus, $|x_n| \rightarrow 0$ if and only if $x_n \rightarrow 0$.