Math 431 - Real Analysis I
Homework due October 31

In class, we learned that a function $f : S \to T$ between metric spaces (S, d_S) and (T, d_T) is continuous if and only if the pre-image of every open set in T is open in S. In other words, f is continuous if for all open $U \subset T$, the pre-image $f^{-1}(U) \subset S$ is open in S.

Question 1. Let S, T, and R be metric spaces and let $f : S \to T$ and $g : T \to R$. We can define the composition function $g \circ f : S \to R$ by

$$g \circ f(s) = g(f(s)).$$

(a) Let $U \subset R$. Show that $(g \circ f)^{-1}(U) = f^{-1}\left(g^{-1}(U)\right)$

(b) Use (a) to show that if f and g are continuous, then the composition $g \circ f$ is also continuous

Solution 1.

(a) We will show that $(g \circ f)^{-1}(U) = f^{-1}\left(g^{-1}(U)\right)$ by showing that $(g \circ f)^{-1}(U) \subset f^{-1}\left(g^{-1}(U)\right)$ and $f^{-1}\left(g^{-1}(U)\right) \subset (g \circ f)^{-1}(U)$. For the first direction, let $x \in (g \circ f)^{-1}(U)$. Then, $g \circ f(x) \in U$. Thus, $g(f(x)) \in U$. Since $g(f(x)) \in U$, then $f(x) \in g^{-1}(U)$. Continuing we get that $x \in f^{-1}(g^{-1}(U)$. Thus, $(g \circ f)^{-1}(U) \subset f^{-1}\left(g^{-1}(U)\right)$.

Conversely, assume that $x \in f^{-1}\left(g^{-1}(U)\right)$. Then, $f(x) \in g^{-1}(U)$. Furthermore, $g(f(x)) \in U$. Thus, $g \circ f(x) \in U$. So, $x \in (g \circ f)^{-1}(U)$. So, $f^{-1}\left(g^{-1}(U)\right) \subset (g \circ f)^{-1}(U)$.

Thus, $(g \circ f)^{-1}(U) = f^{-1}\left(g^{-1}(U)\right)$.

(b) Let U be open in R. Since g is continuous, then $g^{-1}(U) \subset T$ is open. Since f is continuous, $f^{-1}(g^{-1}(U))$ is open. Thus, by (a), $(g \circ f)^{-1}(U)$ is open. So, $g \circ f$ is continuous.

Question 2. Let (S, d_S) and (T, d_T) be metric spaces and let $f : S \to T$.

(a) A function is called *constant* if $f(s) = t_0$ for all $s \in S$. Show that any constant function is continuous.

(b) Show that if d_S is the discrete metric, then any function f is continuous.

Solution 2.

(a) Let U be an open set in T. We will show that $f^{-1}(U)$ is open. We do so in two cases: $t_0 \in U$ and $t_0 \notin U$. If $t_0 \in U$, then since $f(s) = t_0$ for all $s \in S$, $f^{-1}(U) = S$, which is always open in S. If $t_0 \notin U$, then $f^{-1}(U) = \varnothing$, which is open. In either case, the pre-image of every open set is open. So the constant function f is continuous.

(b) Recall that in a discrete metric space, every subset is open. Thus, given any open $U \subset T$, $f^{-1}(U) \subset S$ is automatically open. Thus, f is continuous.

Question 3. The *floor function* $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = \lfloor x \rfloor$, where $\lfloor x \rfloor x$ is the largest integer less than or equal to x.

(a) Let $a \notin \mathbb{Z}$. Use an $\varepsilon-\delta$ proof to show that $f(x) = \lfloor x \rfloor$ is continuous at a.

(b) Let $a \in \mathbb{Z}$. Show that $f(x) = \lfloor x \rfloor$ is not continuous at a. To do so, find an $\varepsilon > 0$ such that for any $\delta > 0$, there exists an x with $|x - a| < \delta$ such that $|f(x) - f(a)| \geq \varepsilon$.

Solution 3.

(a) Let $a \notin \mathbb{Z}$. Given $\varepsilon > 0$, let $\delta = \min\{a - \lfloor a \rfloor, \lfloor a+1 \rfloor - a\}$. Since $a \notin \mathbb{Z}$, then $a \neq \lfloor a \rfloor$ and $\lfloor a+1 \rfloor \neq a$. Tus, $\delta > 0$. Notice that for all x satisfying $|x - a| < \delta$, we have that $f(x) = \lfloor x \rfloor = \lfloor a \rfloor$. Thus, $|f(x) - f(a)| = |f(a) - f(a)| = 0 < \varepsilon$. Thus, f is continuous at a.

(b) Let $a \in \mathbb{Z}$. Then, $f(a) = \lfloor a \rfloor = a$. Let $\varepsilon = 1/2$. Let $\delta > 0$ and consider $a - \delta/2$. Since $a \in \mathbb{Z}$, then $f(a - \delta/2) < a$. In particular, since f only takes on integral values, $f(a) - f(a - \delta/2) \geq 1$. Thus,

$$|f(a - \delta/2) - f(a)| \geq 1 > \varepsilon.$$

Thus, f is discontinuous at a.

Question 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function.

(a) Assume that $f(x) \geq 0$ for all $x \in [0,1]$. Show that if $f(c) > 0$ for some $c \in (0,1)$, then

$$\int_0^1 f(x)\,dx > 0.$$

(b) Show that the above is no longer true if the term "continuous" is dropped. That is, given an example of a (necessarily discontinuous) function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \geq 0$ and $f(c) > 0$ for some $c \in (0,1)$, yet

$$\int_0^1 f(x)\,dx = 0.$$

Solution 4.

(a) Since f is continuous, there exists a $\delta > 0$ such that whenever $|x - a| < \delta$, then $|f(x) - f(c)| < f(c)/2$. Thus, for x satisfying $|x - c| < \delta$ (which is equivalent to $-\delta < x - c < \delta$, we have that

$$-\frac{f(c)}{2} < f(x) - f(c) < \frac{f(c)}{2}.$$

Using the first inequality and adding $f(c)$ to both sides, we get that

$$\frac{f(c)}{2} < f(x)$$

for all x satisfying $-\delta < x - c < \delta$. Since this last pair of inequalities is equivalent to $c - \delta < x < c + \delta$, for these x, we have that $\frac{f(c)}{2} < f(x)$. Thus,

$$0 < \frac{f(c)}{2} \cdot 2\delta = \int_{c-\delta}^{c+\delta} \frac{f(c)}{2}\,dx \leq \int_{c-\delta}^{c+\delta} f(x)\,dx \leq \int_0^1 f(x)\,dx.$$

(b) Consider the piecewise function given by

$$f(x) = \begin{cases} 0 & \text{if } x \neq 1/2 \\ 1 & \text{if } x = 1/2 \end{cases}$$

Then $f(x) \geq 0$ and $f(1/2) > 0$, but $\int_0^1 f(x) = 0$.

Question 5. Recall that we can equip $C([0,1])$, the space of all continuous functions on $[0,1]$, with its L^1 metric, which is given by

$$d(f,g) = \int_0^1 |f(x) - g(x)|\, dx.$$

Consider the function $\varphi : C([0,1]) \to \mathbb{R}$ given by

$$\varphi(f) = \int_0^1 f(x)\, dx.$$

In this question, we will show that φ is a continuous function.

(a) Show that

$$\left| \int_0^1 h(x)\, dx \right| \leq \int_0^1 |h(x)|\, dx.$$

Hint: We previously proved that $-|a| \leq a \leq |a|$ for all $a \in \mathbb{R}$.

(b) Use the above to give an $\varepsilon - \delta$ proof that φ is continuous.

Solution 5.

(a) Notice that for all x, $-|h(x)| \leq h(x) \leq |h(x)|$. Integrating each side, we get that

$$-\int_0^1 |h(x)|\, dx \leq \int_0^1 h(x)\, dx \leq \int_0^1 |h(x)|\, dx.$$

This is equivalent to

$$\left| \int_0^1 h(x)\, dx \right| \leq \int_0^1 |h(x)|\, dx.$$

(b) We will show that φ is continuous at any $f \in C([0,1])$. Given $\varepsilon > 0$, let $\delta = \varepsilon > 0$. Then, for all $g \in C([0,1])$ satisfying

$$\int_0^1 |g(x) - f(x)|\, dx < \delta = \varepsilon,$$

we can use the above fact to get that

$$|\varphi(g) - \varphi(f)| = \left| \int_0^1 g(x)\, dx - \int_0^1 f(x)\, dx \right| = \left| \int_0^1 g(x) - f(x)\, dx \right| \leq$$

$$\int_0^1 |g(x) - f(x)|\, dx < \varepsilon.$$

Thus, $|\varphi(g) - \varphi(f)| < \varepsilon$, as desired. So, φ is continuous at any $f \in C([0,1])$ and thus φ is a continuous function.

Question 7. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

We will show that f is continuous only at $a = 0$.

(a) Use an $\varepsilon - \delta$ proof to show that $f(x)$ is continuous at $a = 0$.

(b) Use the theorem relating convergent sequences to continuous functions to show that if $a \neq 0$, then $f(x)$ is not continuous at a.

Solution 7.

(a) Given $\varepsilon > 0$, let $\delta = \varepsilon$. We will show that for any x satisfying $|x - 0| < \delta$, then $|f(x) - f(0)| < \varepsilon$. So, let x satisfy $|x| = |x - 0| < \delta = \varepsilon$. We take two cases: $x \in \mathbb{Q}$ or $x \notin \mathbb{Q}$. If $x \in \mathbb{Q}$, then $f(x) = x$. Thus,

$$|f(x) - f(0)| = |x - 0| < \varepsilon = \delta.$$

In the second case, if $x \notin \mathbb{Q}$, then $f(x) = 0$, so $|f(x) - f(0)| = |0 - 0| < \varepsilon$. In either case, we have that if $|x - 0| < \delta$, then $|f(x) - f(0)| < \varepsilon$. Thus, f is continuous at $a = 0$.

(b) Let $a \neq 0$. We will consider the two cases: $a \in \mathbb{Q}$ or $a \notin \mathbb{Q}$. If $a \in \mathbb{Q}$, then, let x_n be a sequence of irrational numbers converging to a. If f were continuous at a, then $f(x_n) \to f(a)$. However, for all n, $f(x_n) = 0$, which converges to 0. However, since $a \in \mathbb{Q}$, $f(a) = a \neq 0$. Thus, $f(x_n) \not\to f(a)$. So, f is discontinuous at a. For the second case, assume that $a \notin \mathbb{Q}$. Then, there exists a sequence of rational numbers x_n such that $x_n \to a$. If f were continuous at a, then $f(x_n) \to f(a)$. But $f(x_n) = x_n$ since $x_n \in \mathbb{Q}$. Thus, $f(x_n) = x_n \to a$. However, since $a \notin \mathbb{Q}$, $f(a) = 0 \neq a$. Thus, $f(x_n) \not\to f(a)$. So, f is discontinuous at a. So, at any $a \neq 0$, f is discontinuous at a.